International Journal of Pure and Applied Mathematics

Volume 57 No. 4 2009, 503-508

OPTIMIZATION APPROACH TO THE VALUATION MODEL

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AMS Subject Classification: 12J20
Key Words: valuation equation, present value and value of the asset, cash flow

1. The Basic Valuation Model

The value of any asset is the present value of all future cash flows which is expected to provide over the relevant time period. The value of an asset is therefore determined by discounting the expected cash flows back to their present value, using the required return commensurate with the asset's risk as the appropriate discount rate.

We can express the value of any asset at time zero, V_0 is the net present value, as the valuation equation

$$V_0 = \sum_{k=1}^{n} \frac{a_k}{(1+i)^k}, \tag{1}$$

where V_0 – net present value and value of the asset at time zero, a_k – cash flow expected at the end of year k, i – appropriate required return (discount rate), n – relevant time period.

Received: November 6, 2009

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If we know a_k as a function $a_k = \varphi(k)$ of k then formula (1) becomes

$$V_0 = \sum_{k=1}^{n} \frac{\varphi(k)}{(1+i)^k}. \tag{2}$$

If we replace (2) with an integral sum then we have

$$V_0 = \int_1^n \frac{\varphi(x)}{(1+i)^x} dx.$$

When $a_k = a, k = \overline{1;n}$ then formula (2) has the form

$$V_0 = \left(\frac{a}{1+i}\right) \left(\frac{\left[1 - \frac{1}{(1+i)^n}\right](1+i)}{i} \right).$$

We can find n as

$$n = \frac{\ln\left(1 - \frac{V_0 \cdot i}{a}\right)}{\ln\left(\frac{1}{1+i}\right)}$$

for given a_i and V_0 and compute i from the equation:

$$V_0 = \sum_{k=1}^{n} \frac{a_k}{(1+i)^k}, \qquad a_k > 0. \tag{3}$$

Introduce the function

$$\varphi_k(i) = \frac{a_k}{(1+i)^k}.$$

Lemma 1.1. *The equation (3) has a unique solution.*

Proof. Since $\varphi_k^{'} = -k \cdot a_k \cdot \frac{1}{(1+i)^k}$ the function $f(i) = \sum_{k=1}^n \varphi_k(i)$ is a monotonically decreasing function. On the other hand, the function $f(i)$ is positive

$$\lim_{i \to \infty} f(i) = 0$$

and bonded below.

Then by the mean value theorem the equation

$$F(i) = \sum_{k=1}^{n} \varphi_k(i) - V_0 = 0$$

has a unique solution i^* which can be computed by the Newton method as

$$i_{m+1} = i_m - \frac{F(i_m)}{F'(i_m)}, \qquad m = 0, 1, \ldots,$$

where i_0 is an initial approximation and

$$\lim_{m\to\infty} i_m = i^*.$$

The proof is complete. □

2. Optimization Approach

Now consider the following minimization problem:

$$\psi(i,n) \to \min\,, \tag{4}$$

subject to

$$\sum_{k=1}^{n} \frac{a_k}{(1+i)^k} = V_0,$$

where $\psi(i,n)$ is a risk function.

For $a_k = a$, $k = 1, 2, \ldots$, problem (4) can be reduced to the one dimensional minimization problem.

$$\psi(i,n) \to \min, \quad i \in [i_A, i_B], \tag{5}$$

$$n = \frac{\ln\left(1 - \frac{NPV\cdot i}{a}\right)}{\ln\left(\frac{1}{1+i}\right)}.$$

For example, if

$$\psi(i,n) = (i-i_0)^2 + (n-n_0)^2,$$

then the problem is a one dimensional global minimization problem

$$f(i) = (i-i_0)^2 + \left(\frac{\ln\left(1 - \frac{V_0\cdot i}{a}\right)}{\ln\left(\frac{1}{1+i}\right)} - n_0\right)^2 \to \min_{i_A \le i \le i_B}. \tag{6}$$

In general, problem

$$\min_{i_A \le i \le i_B} f(i)$$

is a global optimization problem.

In order to solve globally this problem, we can use the Piyavskii algorithm, see [8].

We present the algorithm of the method. Introduce function $P(i,y)$ for any fixed $y \in [i_A, i_B]$ in the following way:

$$P(i,y) = f(i) - L[i-y], \qquad i \in [i_A, i_B],$$

where L is the Lipschitz constant.

Choose arbitrary point $t^0 \in [i_A, i_B]$. Set $y^0 = i_0$ and define piecewise linear function:

$$L_1(i) = P(i, y^0).$$

Let y^1 be the solution of the problem:

$$L_1(i) \to \min, \quad i \in [i_A, i_B].$$

Find point i^1 such that

$$f(i^1) = \min\{f(i^0), f(y^1)\}, \qquad i^1 = i^0 \cup y^1.$$

Construct a piecewise linear function: $L_2(i) = \max\{L_1(i); P(i, y^1)\}$.

Let y^2 be the solution of the problem: $L_2(i) \to \min, \quad i \in [i_A, i_B]$.

Define a point i^2 so as to satisfy

$$f(i^k) = \min\{f(i^{k-1}); f(y^k)\},$$

where $y^k : \quad L_k(i) \to \min, \quad i \in [i_A, i_B]$

$$L_k(i) = \max\{L_{k-1}(i); P(i, y^{k-1})\}.$$

Consequently, the algorithm generates two sequences of point y^k and i^k. It is well known [7] that the sequences $\{i^k\}$ an $\{y^k\}$ converge to the global minimum point of the original problem, i.e.

$$\lim_{k\to\infty} f(i^k) = \lim_{k\to\infty} L_k(y^k) = \min_{i\in[i_A, i_B]} f(i).$$

Let us consider a continuous version of problem (4):

$$\psi(i, n) \to \min, \tag{7}$$

$$\int_1^n \frac{\varphi(x)dx}{(1+i)^x} - V_0 = 0.$$

Considering n as a continuous variable. We can solve this problem by the Lagrange method.

$$L(i, n, \lambda) = \psi(i, n) + \lambda \left[\int_1^n \frac{\varphi(x)dx}{(1+i)^x} - V_0 \right].$$

Write down optimality condition:

$$\begin{cases} \dfrac{\partial L}{\partial i} = \dfrac{\partial \psi(i,n)}{\partial i} + \lambda \dfrac{\partial \Phi(i,n)}{\partial i} = 0, \\ \dfrac{\partial L}{\partial n} = \dfrac{\partial \psi(i,n)}{\partial n} + \lambda \dfrac{\varphi(n)}{(1+i)^n} = 0, \\ \dfrac{\partial L}{\partial \lambda} = \displaystyle\int_1^n \frac{\varphi(x)dx}{(1+i)^x} - V_0 = 0. \end{cases}$$

Here $\Phi(i,n)$ is an integral:

$$\Phi(i,n)=\int_1^n \frac{\varphi(x)dx}{(1+i)^x}.$$

We can use the same arguments to calculate this integral:

a. $\dfrac{\partial\Phi(i,n)}{\partial i}=\dfrac{i-i_0}{n-n_0}\cdot\dfrac{\varphi(n)}{(1+i)^n}\Rightarrow$

$$\Phi(i,n)=\frac{\varphi(n)}{(n-n_0)(1+i)^(n-1)}\left[\frac{1+i}{2-n}-\frac{1+i_0}{1-n}\right].$$

b. If $\varphi(n)$ is an arithmetic progression $\varphi(n)=a+bn$ then the integral has the following form:

$$\Phi(i,n)=\frac{(a+b)\ln(1+i)+b}{(1+i)(\ln(1+i))^2}-\frac{(a+bn)\ln(1+i)+b}{(1+i)^n(\ln(1+i))^2}$$

and we computed problem (7) in *MATLAB* for given data.

3. Numerical Computation

The following problems have been solved numerically on *MATLAB*.

I. We used a data for efficiency of investment V_0 and cash flow of Mongolian National Company APU. We consider that the cash flow is constant since 2010 years. This company invested 8275,2 million tugrugs in 2006. Using a cash flow for 2007-2015 and V_0, we can find "i^*" from the equation:

$$\frac{339,3}{1+i}+\frac{526}{(1+i)^2}+\frac{1520,7}{(1+i)^3}+\sum_{k=4}^{9}\frac{2967}{(1+i)^k}=8275,2.$$

The optimal solution found by by the Newton method has

$$i^*=0.1281.$$

II. We write equation (5) in the following from:

$$f(i)=(i-i_0)^2+\left(\frac{\ln\left(1-\dfrac{V_0\cdot i}{a}\right)}{\ln(1+i)}+n_0\right)^2\to\min_{i_A\le i\le i_B}. \qquad (8)$$

The condition $\frac{a}{V_0}>i$ has an efficient condition of financial investment. Solutions are found for different data in the following table:

Inputs		Solutions	
i_0	n_0	i^*	n^*
0.04	5	0.0412	4.8806
0.06	10	0.0593	7.1982
0.08	15	0.0819	12.5661
0.1	20	0.0984	16.0002
0.12	25	0.1209	18.4395
0.14	5	0.1387	6.3412
0.16	10	0.1621	8.1497
0.18	15	0.1846	13.2106
0.2	20	0.2113	17,4974
0.22	25	0.2351	19.1312

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