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A NEW HILBERT TYPE INEQUALITY FOR DOUBLE SERIES

Xiaozhou Shang[1], Mingzhe Gao[2 §]
[1,2]Department of Mathematics and Computer Science
Normal College
Jishou University
Jishou, Hunan, 416000, P.R. CHINA
[1]e-mail: jdsysxz003@163.com
[2]e-mail: mingzhegao@163.com

Abstract: A new Hilbert type inequality for double series can be established by introducing a proper logarithm function. The weight function is estimated by means of the Euler-Maclaurin summation formula. And the constant factor $\pi^{2r+1}E_r$ is proved to be the best possible, where $E_0 = 1$and $E_{r's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, etc. As applications, some equivalent inequalities are considered.

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1. Introduction and Lemmas

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} \leq \pi\Big\{\sum_{n=0}^{\infty}a_n^2\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}b_n^2\Big\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. This is the famous Hilbert theorem for double series (see [1]). Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, some improvements and

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§Correspondence author

extensions of (1.1) appear in a great deal of papers (see [2]-[8], etc.). Especially, Gao and Hsu enumerated the research articles more than 40 in the paper [3]. The aim of the present paper is to build some new Hilbert type inequalities for double series by introducing a proper logarithm function and by using the Euler-Maclaurin summation formula to estimate the weight function and by applying the technique of analysis, and to prove the constant factor to be the best possible, and then to study some equivalent forms of them.

For convenience, throughout the paper we define $\left(\ln \frac{2n+1}{2m+1}\right)^0 = 1$, if $m = n$.

In order to prove our assertion, we need the following lemmas.

Lemma 1.1. *Let r be a nonnegative integer and $x > 0$. Define two functions by*

$$A(x) = \frac{x+2}{2(x+1)} + \frac{2r}{\ln(2x+1)},$$

and

$$B(x) = \frac{2x+1}{3(x+1)}\Big(\frac{r}{\ln(2x+1)} + \frac{1}{4} + \frac{1}{2(x+1)}\Big). \tag{1.2}$$

Then $A(x) - B(x) > 0$.

Proof. It is easy to deduce that

$$\begin{aligned} &A(x) - B(x) \\ &= \frac{x+2}{2(x+1)} + \Big(\frac{2r}{\ln(2x+1)} - \frac{(2x+1)r}{3(x+1)\ln(2x+1)}\Big) - \frac{2x+1}{3(x+1)}\Big(\frac{1}{4} + \frac{1}{2(x+1)}\Big) \\ &\geq \frac{x+2}{2(x+1)} - \frac{2x+1}{3(x+1)}\Big(\frac{1}{4} + \frac{1}{2(x+1)}\Big) = \frac{4x^2+11x+9}{12(x+1)^2} > 0 \end{aligned}$$

Lemma 1.2. *Let r be a nonnegative integer. Define a function by*

$$\varphi(x) = \begin{cases} \varphi_1(0) & \text{if } x = 0, \\ \varphi_1(x) - \varphi_2(x) - \varphi_3(x) & \text{if } x > 0, \end{cases} \tag{1.3}$$

where $\varphi_1(x) = \int\limits_0^{1/(2x+1)} \frac{(\ln\frac{1}{u})^{2r}}{1+u}(\frac{1}{u})^{\frac{1}{2}}du$, $\quad \varphi_2(x) = \frac{\sqrt{2x+1}(\ln(2x+1))^{2r}}{2(x+1)}$,

$$\varphi_3(x) = \frac{\sqrt{2x+1}(\ln(2x+1))^{2r-1}\theta}{12(x+1)}\Big(4r + \ln(2x+1) + \frac{\ln(2x+1)}{x+1}\Big),\ 0 < \theta < 1,$$

then $\varphi(x) > 0$, and when $x > 0$, $\varphi(x) \downarrow 0$.

Proof. When $x = 0$, it is obvious that $\varphi(0) = \int\limits_0^1 \frac{\left(\ln\frac{1}{u}\right)^{2r}}{(1+u)}\left(\frac{1}{u}\right)^{\frac{1}{2}}du > 0$.

When $x > 0$, let us consider the derivative of $\varphi(x)$. We firstly write the derivative of $\varphi_3(x)$ in the following form:

$$\varphi_3'(x) = u(x) + v(x),$$

where $u(x)$ is the sum of all the positive terms of $\varphi_3'(x)$ and $v(x)$ is the sum of all the negative terms of $\varphi_3'(x)$. So we have

$$\varphi_3'(x) \geq v(x) = -\frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}\theta}{(x+1)}B(x) \quad (0 < \theta < 1), \tag{1.4}$$

where $B(x)$ is a function defined by (1.2).

Next it is easy to deduce that

$$\varphi_1'(x) = -\frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}}{x+1},$$

$$\varphi_2'(x) = \frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}}{(x+1)}\Big\{\frac{1}{2} + \frac{2r}{\ln(2x+1)} - \frac{2x+1}{2(x+1)}\Big\}.$$

Hence

$$\varphi_1'(x) - \varphi_2'(x) = -\frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}}{(x+1)}A(x), \tag{1.5}$$

where $A(x)$ is a function defined by (1.2).

Based on (1.4), (1.5) and Lemma 1.1, we have

$$\begin{aligned}
\varphi'(x) &= \varphi_1'(x) - \varphi_2'(x) - \varphi_3'(x) \leq \Big(\varphi_1'(x) - \varphi_2'(x)\Big) - v(x) \\
&= -\frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}}{(x+1)}A(x) \\
&\quad + \frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}\theta}{(x+1)}B(x) \quad (0 < \theta < 1) \\
&< -\frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}}{(x+1)}A(x) + \frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}}{(x+1)}B(x) \\
&= -\frac{(2x+1)^{-\frac{1}{2}}\Big(\ln(2x+1)\Big)^{2r}}{(x+1)}\Big(A(x) - B(x)\Big) < 0.
\end{aligned}$$

Hence $\varphi(x)$ is monotonously decreasing in $(0, \infty)$. Notice that $\lim_{x\to\infty} \varphi(x) = 0$, so we have $\varphi(x) \downarrow 0$. The proof of the lemma is finished. □

Lemma 1.3. *Let $0 < \alpha < 1$ and r be a nonnegative integer. Then*

$$\int_0^1 u^{\alpha-1}\Big(\ln\frac{1}{u}\Big)^r \frac{1}{1+u}du = r!\sum_{k=0}^{\infty}\frac{(-1)^k}{(\alpha\ +\ k)^{r+1}}. \tag{1.6}$$

This result is given in the paper [9] (p. 247).

Lemma 1.4. *With the assumptions as in Lemma 1.3, we have*

$$\int_0^\infty u^{\alpha-1}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du$$
$$= (2r)!\Big\{\sum_{k=0}^{\infty}(-1)^k\Big(\frac{1}{(\alpha+k)^{2r+1}} + \frac{1}{(1-\alpha+k)^{2r+1}}\Big)\Big\}. \tag{1.7}$$

Proof. It is easy to deduce that

$$\begin{aligned}
&\int_0^\infty u^{\alpha-1}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du\\
&= \int_0^1 u^{\alpha-1}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du + \int_1^\infty u^{\alpha-1}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du\\
&= \int_0^1 u^{\alpha-1}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du + \int_0^1 v^{-\alpha}\Big(\ln v\Big)^{2r}\frac{1}{1+v}dv\\
&= \int_0^1 u^{\alpha-1}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du + \int_0^1 v^{(1-\alpha)-1}\Big(\ln\frac{1}{v}\Big)^{2r}\frac{1}{1+v}dv.
\end{aligned}$$

By using Lemma 1.3, the relation (1.7) is obtained at once. □

Lemma 1.5. *Let r be a nonnegative integer. Then*

$$\int_0^\infty u^{-\frac{1}{2}}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du = \pi^{2r+1}E_r, \tag{1.8}$$

where $E_0 = 1$ and $E_{r's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, etc.

Proof. When $\alpha = 1/2$, it follows from (1.7) that

$$\int_0^\infty u^{-\frac{1}{2}}\Big(\ln\frac{1}{u}\Big)^{2r}\frac{1}{1+u}du = (2r)!2\Big\{\sum_{k=0}^\infty(-1)^k\Big(\frac{1}{\big(\frac{1}{2}+k\big)^{2r+1}}\Big)\Big\}$$

$$= (2r)!2^{2r+2}\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{2r+1}}. \quad (1.9)$$

It is known from the paper [10] (p. 231) that

$$\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{2r+1}} = \frac{\pi^{2r+1}}{2^{2r+2}(2r)!}E_r, \quad (1.10)$$

where $E_{r's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, etc. Notice that

$$\sum_{k=0}^\infty\frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Hence we can define $E_0 = 1$, It follows from (1.9) and (1.10) that the equation (1.8) is true. □

2. Main Results

We are ready now to formulate our main results.

Theorem 2.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers, and r be a nonnegative integer. If $\sum_{n=0}^\infty a_n^2 < +\infty$ and $\sum_{n=0}^\infty b_n^2 < +\infty$, then*

$$\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^{2r}a_mb_n}{m+n+1} \le \Big(\pi^{2r+1}E_r\Big)\Big\{\sum_{n=0}^\infty a_n^2\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^\infty b_n^2\Big\}^{\frac{1}{2}}, \quad (2.1)$$

where the constant factor $\pi^{2r+1}E_r$ is the best possible, and that $E_0 = 1$ and $E_{r's}$ is the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, etc.

Proof. We may apply the Cauchy inequality to estimate the left-hand side of (2.1) as follows

$$\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^{2r}a_mb_n}{m+n+1}$$

$$= \sum_{m=0}^\infty\sum_{n=0}^\infty\Big\{\Big(\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^{2r}}{m+n+1}\Big)^{\frac{1}{2}}\Big(\frac{2m+1}{2n+1}\Big)^{\frac{1}{4}}a_m\Big\}\Big\{\Big(\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^{2r}}{m+n+1}\Big)^{\frac{1}{2}}\Big(\frac{2n+1}{2m+1}\Big)^{\frac{1}{4}}b_n\Big\}$$

$$\leq \Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^{2r}}{m+n+1}\Big(\frac{2m+1}{2n+1}\Big)^{\frac{1}{2}}a_m^2\Big\}^{\frac{1}{2}}\Big\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^{2r}}{m+n+1}\Big(\frac{2n+1}{2m+1}\Big)^{\frac{1}{2}}b_n^2\Big\}^{\frac{1}{2}}$$

$$= \Big\{\sum_{n=0}^{\infty}\omega(n)a_n^2\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}\omega(n)b_n^2\Big\}^{\frac{1}{2}}, \tag{2.2}$$

where

$$\omega(n) = \sum_{m=0}^{\infty}\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^{2r}}{m+n+1}\Big(\frac{2n+1}{2m+1}\Big)^{\frac{1}{2}}. \tag{2.3}$$

Let

$$f(x) = \frac{\Big(\ln\frac{2n+1}{2x+1}\Big)^{2r}}{n+x+1}\Big(\frac{2n+1}{2x+1}\Big)^{\frac{1}{2}}.$$

It is easy to deduce that

$$f(+\infty) = 0, \quad f'(+\infty) = 0, \quad f(0) = \frac{\sqrt{2n+1}\Big(\ln(2n+1)\Big)^{2r}}{n+1},$$

$$f'(0) = -\frac{\sqrt{2n+1}\Big(\ln(2n+1)\Big)^{2r-1}}{n+1}\Big(4r+\ln(2n+1)+\frac{\ln(2n+1)}{n+1}\Big),$$

$$\int_0^{\infty} f(x)dx = \int_0^{\infty}\frac{2\Big(\ln\frac{2n+1}{2x+1}\Big)^{2r}}{(2n+1)\Big(1+\frac{2x+1}{2n+1}\Big)}\Big(\frac{2n+1}{2x+1}\Big)^{\frac{1}{2}}dx$$

$$= \int_{\frac{1}{2n+1}}^{\infty}\frac{\Big(\ln\frac{1}{u}\Big)^{2r}}{1+u}\Big(\frac{1}{u}\Big)^{\frac{1}{2}}du = \int_0^{\infty}\frac{\Big(\ln\frac{1}{u}\Big)^{2r}}{1+u}\Big(\frac{1}{u}\Big)^{\frac{1}{2}}du - \varphi_1(n),$$

where $\varphi_1(n)$ is defined by (1.2).

Applying the Euler-Maclaurin summation formula (see [11]) to $\omega(n)$, we have

$$\omega(n) = \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{\theta}{12}f'(0) \quad (0<\theta<1)$$

$$= \int_0^{\infty}\frac{\Big(\ln\frac{1}{u}\Big)^{2r}}{1+u}\Big(\frac{1}{u}\Big)^{\frac{1}{2}}du - \varphi(n), \tag{2.4}$$

where $\varphi(n)$ is a function defined by (1.2).

Based on Lemmas 1.5, we can write (2.4) in the following form:

$$\omega(n) = \pi^{2r+1}E_r - \varphi(n), \tag{2.5}$$

where $E_0 = 1$ and $E_{r's}$ is the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, etc. It is known from Lemma 1.2 that $\varphi(n) \geq 0$. Thereby we have

$$\omega(n) \leq \pi^{2r+1}E_r. \tag{2.6}$$

It follows from (2.2) and (2.6) that the inequality (2.1) is valid.

We need only to show that the constant factor $\pi^{2r+1}E_r$ is the best possible in (2.1).

$\forall\varepsilon :\ 0 < \varepsilon < 1$. Define two sequences by

$$\tilde{a}_m = (2m+1)^{-\frac{1}{2} - \frac{\varepsilon}{4}} \quad \text{and} \quad \tilde{b}_n = (2n+1)^{-\frac{1}{2} - \frac{\varepsilon}{4}} \quad (m, n = 0, 1, 2, \cdots).$$

Since the sequence $\left\{\tilde{a}_m^2\right\}$ is monotonously decreasing, we have

$$\begin{aligned}\frac{1}{\varepsilon} &= \int_0^\infty (2x+1)^{-1-\frac{\varepsilon}{2}}dx < \sum_{m=0}^\infty (2m+1)^{-1-\frac{\varepsilon}{2}} \\ &= \sum_{m=0}^\infty \tilde{a}_m^2 = \tilde{a}_0^2 + \sum_{m=1}^\infty \tilde{a}_m^2 \\ &< 1 + \int_0^\infty (2x+1)^{-1-\frac{\varepsilon}{2}}dx = 1 + \frac{1}{\varepsilon}\end{aligned}$$

Hence $\sum\limits_{m=1}^\infty \tilde{a}_m^2 = \frac{1}{\varepsilon} + o(1)\ (\varepsilon \to 0)$. Similarly: $\sum\limits_{n=1}^\infty \tilde{b}_n^2 = \frac{1}{\varepsilon} + o(1)\ (\varepsilon \to 0)$.

If the constant factor $\pi^{2r+1}E_r$ in (2.1) is not the best possible then there exists a constant $C > 0$ such that $C < \pi^{2r+1}E_r$ and

$$\begin{aligned}S(\tilde{a}, \tilde{b}) &= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\left(\ln \frac{2n+1}{2m+1}\right)^{2r} \tilde{a}_m \tilde{b}_n}{m+n+1} \\ &\leq\ C\left(\sum_{m=1}^\infty \tilde{a}_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty \tilde{b}_n^2\right)^{\frac{1}{2}} = \frac{C}{\varepsilon}\left\{1 + \circ(1)\right\} \quad (\varepsilon \to 0).\end{aligned} \tag{2.7}$$

On the other hand we have

$$S(\tilde{a}, \tilde{b}) = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\left(\ln \frac{2n+1}{2m+1}\right)^{2r} (2m+1)^{-\frac{1}{2}-\frac{\varepsilon}{4}}(2n+1)^{-\frac{1}{2}-\frac{\varepsilon}{4}}}{m+n+1}$$

$$= \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{\infty} \frac{\left(\ln \frac{2n+1}{2m+1}\right)^{2r}}{m+n+1} \Big(\frac{2n+1}{2m+1} \Big)^{-\frac{1}{2}-\frac{\varepsilon}{4}} \Big) \Big(2m+1\Big)^{-1-\frac{\varepsilon}{2}}. \tag{2.8}$$

When ε is sufficiently small, it is known from (2.3) and (2.5) that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\left(\ln \frac{2n+1}{2m+1}\right)^{2s}}{m+n+1} \Big(\frac{2n+1}{2m+1} \Big)^{-\frac{1}{2}-\frac{\varepsilon}{4}} \\ &= \sum_{n=0}^{\infty} \frac{\left(\ln \frac{2m+1}{2n+1}\right)^{2s}}{m+n+1} \Big(\frac{2m+1}{2n+1} \Big)^{\frac{1}{2}+\frac{\varepsilon}{4}} \\ &= \omega(m) + \tilde{o}(1) = \Big(\pi^{2r+1}E_r - \varphi(m)\Big) + \tilde{o}(1) \quad (\varepsilon \to 0). \end{aligned} \tag{2.9}$$

Since the sequence $\{(2m+1)^{-1-\frac{\varepsilon}{2}}\}$ is monotonously decreasing, it follows from (2.8) and (2.9) that

$$\begin{aligned} S(\tilde{a},\tilde{b}) &= \sum_{m=0}^{\infty} \Big\{ \pi^{2r+1}E_r - \Big(\varphi(m) - \tilde{o}(1)\Big) \Big\} (2m+1)^{-1-\frac{\varepsilon}{2}} \\ &= \Big(\pi^{2r+1}E_r\Big) \sum_{m=0}^{\infty} (2m+1)^{-1-\frac{\varepsilon}{2}} - \sum_{m=0}^{\infty} \Big(\varphi(m) - \tilde{o}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} \\ &> \Big(\pi^{2r+1}E_r\Big) \int_0^{\infty} (2x+1)^{-1-\frac{\varepsilon}{2}} dx - \sum_{m=0}^{\infty} \Big(\varphi(m) - \tilde{o}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} \\ &= \frac{\pi^{2r+1}E_r}{\varepsilon} - \sum_{m=0}^{\infty} \Big(\varphi(m) - \tilde{o}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} \quad (\varepsilon \to 0). \end{aligned} \tag{2.10}$$

We will prove that the series $\sum_{m=1}^{\infty} \Big(\varphi(m) - \tilde{o}(1)\Big) m^{-1-\frac{\varepsilon}{2}}$ is bounded. In fact, it is known from Lemma 1.2 that $\varphi(m) \downarrow 0 \ \ (m \to \infty)$. Hereby there exists a positive integer m_0 such that $\Big|\varphi(m) - \tilde{o}(1)\Big| \le \varphi(m) + \Big|\tilde{o}(1)\Big| < \varepsilon$ when $m > m_0$. So we have

$$\begin{aligned} &\sum_{m=0}^{\infty} \Big(\varphi(m) - \tilde{o}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} \\ &= \sum_{m=0}^{m_0} \Big(\varphi(m) - \tilde{o}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} + \sum_{m=m_0+1}^{\infty} \Big(\varphi(m) - \tilde{o}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} \end{aligned}$$

$$< \sum_{m=0}^{m_0} \Big(\varphi(m) - \tilde{\circ}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} + \sum_{m=m_0+1}^{\infty} \varepsilon(2m+1)^{-1-\frac{\varepsilon}{2}}$$

$$< \sum_{m=0}^{m_0} \Big(\varphi(m) - \tilde{\circ}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} + \varepsilon \int_{m_0}^{\infty} (2x+1)^{-1-\frac{\varepsilon}{2}} dx$$

$$= \sum_{m=0}^{m_0} \Big(\varphi(m) - \tilde{\circ}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}} + \frac{1}{(2m_0+1)^{\varepsilon/2}} \quad (\varepsilon \to 0). \tag{2.11}$$

It shows that the series $\sum_{m=0}^{\infty} \Big(\varphi(m) - \tilde{\circ}(1)\Big)(2m+1)^{-1-\frac{\varepsilon}{2}}$ is bounded. Consequently, we can write (2.10) in the following form

$$S(\tilde{a}, \tilde{b}) > \frac{\pi^{2r+1} E_r}{\varepsilon} - O(1) \quad (\varepsilon \to 0). \tag{2.12}$$

When m is sufficiently small, we obtain from (2.12) that

$$(\tilde{a}, \tilde{b}) > \frac{\pi^{2r+1} E_r}{\varepsilon}\Big(1 - \circ(1)\Big). \qquad (\varepsilon \to 0). \tag{2.13}$$

It is obvious that the inequality (2.13) is in contradiction with the inequality (2.7). It shows that the constant factor $\pi^{2r+1} E_r$ in (2.1) is the best possible. The proof of theorem is completed. □

Corollary 2.2. *Let r be a nonnegative integer. If $\sum\limits_{n=0}^{\infty} a_n^2 < +\infty$, then*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Big(\ln \frac{2n+1}{2m+1}\Big)^{2r} a_m a_n}{m+n+1} \leq \Big(\pi^{2r+1} E_r\Big) \sum_{n=0}^{\infty} a_n^2, \tag{2.14}$$

where the constant factor $\pi^{2r+1} E_r$ is the best possible, $E_0 = 1$ and E_r is the Euler number.

When $r = 1$, based on Theorem 2.1 we have the following result.

Corollary 2.3. *If $\sum\limits_{n=0}^{\infty} a_n^2 < +\infty$ and $\sum\limits_{n=0}^{\infty} b_n^2 < +\infty$, then*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Big(\ln \frac{2n+1}{2m+1}\Big)^{2} a_m b_n}{m+n+1} \leq \pi^3 \Big\{\sum_{n=0}^{\infty} a_n^2\Big\}^{\frac{1}{2}} \Big\{\sum_{n=0}^{\infty} b_n^2\Big\}^{\frac{1}{2}}, \tag{2.15}$$

where the constant factor π^3 is the best possible.

In particular, when $r = 0$, based on (2.1) the inequality (1.1) can be obtained. Therefore Theorem 2.1 is an extension of (1.1).

3. Some Applications

As applications, we shall build some equivalent inequalities each other.

Theorem 3.1. *Let r be a nonnegative integer. If $\sum_{m=0}^{\infty} a_m^2 < +\infty$, then*

$$\sum_{n=0}^{\infty}\Big\{\sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m}{m+n+1}\Big\}^2 \le \left(\pi^{2r+1}E_r\right)^2\sum_{m=0}^{\infty}a_m^2, \tag{3.1}$$

where the constant factor $\left(\pi^{2r+1}E_r\right)^2$ is the best possible, and that $E_0 = 1$ and $E_{r's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, etc. And the inequality (3.1) is equivalent to (2.1).

Proof. We assume firstly that the inequality (2.1) holds. Set

$$b_n = \sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m}{m+n+1}.$$

By using (2.1), we have

$$\sum_{n=0}^{\infty}\Big\{\sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m}{m+n+1}\Big\}^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m}{m+n+1}\Big(\sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m}{m+n+1}\Big)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m b_n}{m+n+1}$$

$$\le \left(\pi^{2r+1}E_r\right)\Big\{\sum_{m=0}^{\infty}a_m^2\Big\}^{\frac12}\Big\{\sum_{n=0}^{\infty}b_n^2\Big\}^{\frac12}$$

$$= \left(\pi^{2r+1}E_r\right)\Big\{\sum_{m=0}^{\infty}a_m^2\Big\}^{\frac12}\Big\{\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}\Big(\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m}{m+n+1}\Big)\Big)^2\Big\}^{\frac12}, \tag{3.2}$$

where $E_0 = 1$, and $E_{r's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, etc. The inequality (3.1) is obtained from (3.2) after some simplifications.

Next, let us assume that the inequality (3.1) is valid. By applying in turn the Cauchy inequality and (3.1), we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m b_n}{m+n+1} = \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}\frac{\left(\ln\frac{2n+1}{2m+1}\right)^{2r}a_m}{m+n}\Big)b_n$$

$$\le \Big\{\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty}\frac{(\ln\frac{2n+1}{2m+1})^{2r}a_m}{m+n+1}\Big)^2\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}b_n^2\Big\}^{\frac{1}{2}}$$

$$\le \Big\{\Big(\pi^{2r+1}E_r\Big)^2\sum_{m=0}^{\infty}a_m^2\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}b_n^2\Big\}^{\frac{1}{2}}$$

$$= (\pi^{2r+1}E_r)\Big\{\sum_{m=0}^{\infty}a_m^2\Big\}^{\frac{1}{2}}\Big\{\sum_{n=0}^{\infty}b_n^2\Big\}^{\frac{1}{2}}, \tag{3.3}$$

where $E_0 = 1$ and $E_{r's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, etc. Hence the inequality (2.1) is valid.

If the constant factor $\Big(\pi^{2r+1}E_r\Big)^2$ of (3.1) is not possible, then it is known from (3.3) that the constant factor $\pi^{2r+1}E_r$ of (2.1) is also not the best possible. This is a contradiction. Therefore theorem is proved. □

When $r = 1$, based on Theorem 3.1, we have the following result.

Corollary 3.2. *If* $\sum\limits_{m=0}^{\infty} a_m^2 < +\infty$, *then*

$$\sum_{n=0}^{\infty}\Big\{\sum_{m=0}^{\infty}\frac{\Big(\ln\frac{2n+1}{2m+1}\Big)^2 a_m}{m+n+1}\Big\}^2 \le \pi^6\sum_{m=0}^{\infty}a_m^2, \tag{3.4}$$

where the constant factor π^6 *is the best possible. And the inequality (3.4) is equivalent to (2.15).*

In particular, for case $r = 0$, based on Theorem 3.1, a new inequality is obtained.

Corollary 3.3. *If* $\sum\limits_{m=1}^{\infty} a_m^2 < +\infty$, *then*

$$\sum_{n=0}^{\infty}\Big\{\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\Big\}^2 \le \pi^2\sum_{m=0}^{\infty}a_m^2. \tag{3.5}$$

where the constant factor π^2 *is the best possible. And the inequality (3.5) is equivalent to (1.1).*

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