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FINITELY GENERATED SUBALGEBRAS OF COX RINGS (CURVES OF HIGHER GENUS AND EASY ABELIAN SURFACES)

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Abstract: Let $G \cong \mathbb{Z}^{\oplus s}$, $s \geq 2$, be a free and finitely generated subgroup of $\mathrm{Pic}(X)$ with $\dim(\mathrm{Pic}(X)) > 0$. Here in two cases ($X$ a curve of genus > 0 and X an Abelian surface with $\mathrm{Num}(X) \cong \mathbb{Z}$) we get an exponential measure of non-finitely generation of the commutative ring $\oplus_{L \in G} H^0(X, L)$.

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Let X be an integral projective variety defined over an algebraically closed field $\mathbb{K}$. Set

$$\mathcal{R}(X) := \oplus_{L \in \mathrm{Pic}(X)} H^0(X, L).$$

If $\mathrm{Pic}(X)$ is finitely generated, then the choice of a set of generators of it was sufficient to define a multiplication map in $\mathcal{R}(X)$ (see [2], Section 3, and [5] for the non-locally free case; the trouble in the non-finitely generated case is that $\mathrm{Pic}(X)$ only parametrizes line bundles up to isomorphisms). In this case the commutative ring $\mathcal{R}(X)$ is called the Cox ring of X and it is extensively studied ([1], [2], [4], [5], [6], [7]). Now assume $\dim(\mathrm{Pic}^0(X)) > 0$. For any finitely generated subgroup Σ of $\mathrm{Pic}(X)$ set

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$$\mathcal{R}(X)[\Sigma] := \oplus_{L\in\Sigma} H^0(X,L).$$

Then $\mathcal{R}(X)[\Sigma]$ is a graded ring with the Abelian group Σ as its grading (use the proofs in [2], Section 3). For any $A \subseteq \mathcal{R}(X)[G]$ let $\langle A\rangle$ denote the $\mathbb{K}$-subalgebra of $\mathcal{R}(X)[G]$ generated by A. Fix a basis $\mathbb{L} = a_1,\dots,a_s$ of G. Set $G_{\mathbb{L},k} := \{\sum_{i=1}^{s}\sum_{j=-k}^{k} n_j a_i\}$ and $\mathcal{R}(X)[G]_{\mathbb{L},k} := \oplus_{L\in G_{\mathbb{L},k}} H^0(X,L)$. We use the increasing families $G_{\mathbb{L},k}$, $k \in \mathbb{N}\backslash\{0\}$, and the linear subspaces $\mathcal{R}(X)[G]_{\mathbb{L},k}$ to give two examples in which $\mathcal{R}(X)[G]$ requires an exponential number of generators as a $\mathbb{K}$-algebra when $k \gg 0$. In both cases we give the measure both in terms of k and of s, i.e. we also see what happens when we take a tower $G_s \subset G_{s+1} \subset \cdots$ of subgroups of $\mathrm{Pic}(X)$ such that $G_i \cong \mathbb{Z}^{\oplus i}$ for all i and take $s \gg 0$ (for a fixed k) or increase both s and k. For any $a_1,\dots,a_z \in \mathcal{R}(X)[G]$ let $\langle a_1,\dots,a_z\rangle$ denote the $\mathbb{K}$-subalgebra of $\mathcal{R}(X)[G]$ generated by $a_1,\dots,a_z$.

Theorem 1. *Let X be an integral projective curve defined over an algebraically closed field $\mathbb{K}$ such that $g := p_a(X) > 0$. Let $G \subset Pic(X)$ be a subgroup isomorphic to $\mathbb{Z}^{\oplus s}$ for some $s \geq 2$. Assume $\deg(L) \neq 0$ for some $L \in G$ and let e be the minimal positive integer in the set $\{\deg(L)\}_{L\in G}$. Assume $e \geq g$. There is a basis $\mathbb{L} = L_1,\dots,L_s$ of G such that $\deg(L_1) = e$ and $\deg(L_i) = 0$ for all $i \geq 2$. Then for every integer $k \geq 1$ the minimal integer z such $\mathcal{R}(X)[G]_{\mathbb{L},k} \subset \langle R_1,\dots,R_z\rangle$, $R_j \in G$, is at least $(e+1-g)(s-1)^{2k+1}$. If $e \geq 2g+2$ and X is smooth, then equality holds.*

Theorem 2. *Let X be an Abelian surface such that $Num(X) \cong \mathbb{Z}$. For any $M \in Pic(X)$ let $\beta(M)$ denote the numerical class of M, with the convention that $\beta(M) > 0$ if and only if M is ample and that $\beta : Pic(X) \to \mathbb{Z}$ is surjective. Set $\alpha := R^2$ for any $R \in Pic(X)$ such that $\beta(R) = 1$. Fix a subgroup $G \cong \mathbb{Z}^{\oplus s} \subset Pic(X)$, $s \geq 2$, such that $\beta(G) \neq 0$. Let $e > 0$ be the positive generator of the subgroup $\beta(G) \subseteq \mathbb{Z}$. There is a basis $\mathbb{L} = L_1,\dots,L_s$ of G such that $\beta(L_1) = e$ and $\beta(L_i) = 0$ for all $i \geq 2$. Then for every integer $k \geq 1$ the minimal integer z such $\mathcal{R}(X)[G]_{\mathbb{L},k} \subset \langle R_1,\dots,R_z\rangle$, $R_j \in G$, is at least $\alpha e^2(s-1)^k/2$. If $char(\mathbb{K}) = 0$ and $e \geq 3$, then equality holds.*

Remark 1. In the statement of Theorems 1 and 2 we always take $s \geq 2$, because if $s = 1$ the Cox ring $\mathcal{R}(X)[G]$ is obviously finitely generated. In the set-up of Theorem 1 this assumption shows that if $\mathbb{K}$ is the algebraic closure of a finite field, then X must be singular (use that for any Abelian variety A defined over $\mathbb{F}_q$ every point of $A(\overline{\mathbb{F}}_q)$ is torsion).

Proof of Theorem 1. By assumption $h^0(X,L) > 0$ for all $L \in G$ such that $\deg(L) > 0$. For any integer $k > 0$ set $E_k := \cap G_{\mathbb{L},k}$ and $A_k := \oplus_{L\in G_{\mathbb{L},k}} H^0(X,L)$. Notice that $E_k = \{L_1 \otimes L_2^{\otimes a_2} \otimes \cdots \otimes L_s^{\otimes a_s} : -k \leq a_i \leq k$ for all $i \in$

$\{2,\dots,s\}\}$. Since $h^0(X, L_1 \otimes L_2^{\otimes a_2} \otimes \cdots \otimes L_s^{\otimes a_s}) \geq e+1-g > 0$ for all $a_i \in \mathbb{Z}$, we get that any subset M of $\mathcal{R}(G)$ whose span $\langle M \rangle$ contains $\mathcal{R}(X)[G]_{\mathbb{L},k}$ contain a set of basis of the $\mathbb{K}$-vector space A_k which as dimension at least $(e-g+1)(s-1)^{2k+1)}$. For the last assertion use that the multiplication map $H^0(X,L) \otimes H^0(X,R) \to H^0(X, L\otimes R)$ is surjective for all line bundles R, L if X is smooth, $\deg(R) \geq 2g+2$ and $\deg(R) \geq 2g+2$ (Castelnuovo (see the introduction of [3])). Hence the vector space $\oplus_{L \in G \cap \mathrm{Pic}^e(X)} H^0(X,L)$ generates $\mathcal{R}(X)[G]$ as a $\mathbb{K}$-algebra. If $\deg(L) = e \geq 2g-1$, then $h^0(X,L) = e+1-g$. □

Remark 2. Let X be an Abelian surface such that $\mathrm{Num}(X) \cong \mathbb{Z}$. For any $M \in \mathrm{Pic}(X)$ let $\beta(M)$ denote the numerical class of M, with the convention that $\beta(M) > 0$ if and only if M is ample and that $\beta : \mathrm{Pic}(X) \to \mathbb{Z}$ is surjective. Hence a line bundle L on X is ample (resp. numerically trivial) if and only if $\beta(L) > 0$ (resp. $\beta(L) = 0$). Fix $R, L \in \mathrm{Pic}(X)$ with L ample. Since $\omega_X \cong \mathcal{O}_X$, Serre duality and Kodaira's vanishing (see [9], p. 150, for the case of an ample line bundle on any Abelian variety in arbitrary characteristic) give $h^1(X,L) = h^1(X,L^*) = 0$. Set $\alpha := R^2$, where R is any line bundle on X such that $\beta(R) = 1$. Fix any integer $t > 0$ and any $L \in \mathrm{Pic}(X)$ such that $\beta(L) = t$. Since $\omega_X \cong \mathcal{O}_X$, Riemann-Roch and Serre duality gives $h^0(X,L) = \alpha t^2/2$ and $h^i(X,L) = 0$ for $i = 1,2$. Hence α is a positive even integer.

Proof of Theorem 2. Set $E_k := \beta^{-1}(e) \cap G_{\mathbb{K},e}$ and $A_k := \oplus_{L \in E_k} H^0(X,L)$. Notice that $E_k = \{L_1 \otimes L_2^{\otimes a_2} \otimes \cdots \otimes L_s^{\otimes a_s} : -k \leq a_i \leq k$ for all $i \in \{2,\dots,s\}\}$. Since $h^0(X, L_1 \otimes L_2^{\otimes a_2} \otimes \cdots \otimes L_s^{\otimes a_s}) = \alpha e^2/2$ for all a_i (Remark 2), E_k is a vector space of dimension $\alpha e^2(s-1)^{2k+1}/2$ for all integers $k > 0$. Since β is additive and $H^0(X,L) = 0$ for every line bundle L such that $\beta(L) \leq 0$ and $L \neq \mathcal{O}_X$, any family generating a $\mathbb{K}$-subalgebra containing $\mathcal{R}(X)[G]_{\mathbb{L}.k}$ must contain a system of generators of A_k as a $\mathbb{K}$-vector space. The last assertion follows from [8], Proposition 7.3.4. □

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