International Journal of Pure and Applied Mathematics
Volume 57 No. 4 2009, 525-532

UTILITY MAXIMIZATION WITH PARTIAL INFORMATION FOR ORNSTEIN-UHLENBECK MODEL

Fangfang Liu[1], Chengxin Luo[2 §]
[1,2]College of Mathematics and System Science
Shenyang Normal University
Shenyang, 110034, P.R. CHINA
[1]e-mail: liufangfang0302@163.com
[2]e-mail: luochengxin@163.com

Abstract: This paper deals with a class of stochastic optimization and consumption models for the exponential utility, which is maximizing the expected utility of the terminal wealth and intermediate consumption. The stock price is modelled as a stochastic differential equation with instantaneous rates of return modelled as an Ornstein-Uhlenbeck process. Only the stock price and interest rate can be observable for an investor. It is reduced to a partially observed stochastic control problem. Combining the filtering theory with the dynamic programming approach, corresponding optimal strategies are derived.

AMS Subject Classification: 93E20, 91B28, 93E11
Key Words: Hamilton-Jacobi-Bellman (HJB) equation, utility maximization, filtering theory

1. Introduction

This paper investigates an optimal investment and consumption model for a financial market when the investor consists of two stocks. The stock price is modelled as a diffusion process. The fundamental assumption is that the coefficient of the second stock depends on another stochastic differential process, to be referred as an Ornstein-Uhlenbeck process. The individual preferences are

Received: November 11, 2009

§Correspondence author

modelled in terms of a constant relation risk aversion (CRRA) function. The objective of the individual is to maximize his expected utility of intermediate consumption and terminal wealth and to specify the optimal investment and consumption strategies.

The fundamental stochastic model of this problem was introduced by Merton [2], in which the strategies have been found. In practice, the investor can only observe the stock price with partial information and the drift process, but the Brownian motion appearing in the stochastic differential equation for the stock price cannot be observed. Clearly, it is more realistic to assume that the investor has only partial information since price is published and available to the public, but the drift and the path of Brownian motion which are not observable are mere mathematical tools for modeling.

In the recent years, portfolio optimization problems under partial information have been studied widely via the dynamic programming method corbelled with the filtering theory. Rishel [3]applies the dynamic programming method for a finite-horizon linear Gaussian model with one unobserved factor that is independent of the risky asset. Bai and Guo [1] consider the exponential and logarithmic utility cases. Yang, Liu and Shen [4] consider the optimal portfolio for Stein-Stein model. But in these works, they all considered the investor choose one bank and one risky asset. In this paper, we mainly use dynamic programming approach to deal with the utility maximization considering two stocks.

The paper is organized as follows: In Section 2, the investment and consumption model is formulated. In Section 3, the main result and the optimal policies are provided. Section 4 concludes the paper.

2. The Investment and Consumption Model

Suppose the investor has the choice of two stocks, whose prices $S_1(t)$ and $S_2(t)$ satisfy the following stochastic differential equations respectively

$$dS_1(t) = \mu_1(t)S_1(t)dt + \sigma_1(t)S_1(t)dW_1(t), \quad S_1(0) = s_1, \tag{2.1}$$

$$dS_2(t) = \mu_2(t)S_2(t)dt + \sigma_2(t)S_2(t)dW_2(t), \quad S_2(0) = s_2, \tag{2.2}$$

where $\mu_1(t) < \mu_2(t) : [0,T] \to R$ represent the instantaneous expected rates of return of two stocks respectively, $\sigma_1(t), \sigma_2(t) : [0,T] \to R$ represent the instantaneous volatility of two stocks respectively, $\mu_1(t), \sigma_1(t), \sigma_2(t)$ are assumed to be deterministic and bounded uniformly in $t \in [0,T]$.

The process $\mu_2(t)$ is assumed to satisfy the stochastic differential equation

$$d\mu_2(t) = \alpha\mu_2(t)dt + \beta dW_3(t). \tag{2.3}$$

Let $(\Omega,\ \mathcal{F},\ \{\mathcal{F}\},\ \mathbf{P})$ be a complete filtered probability space for $0 \le t \le T$ with a fixed terminal time $T > 0$. α and β are known real numbers. The process $W_1(t), W_2(t)$ and $W_3(t)$ are standard Brownian motions defined on $(\Omega,\ \mathcal{F},\ \{\mathcal{F}\},\ \mathbf{P})$, and $W_3(t)$ is independent of $W_2(t)$. We assume that $\mu_2(0)$ follows a normal distribution with the mean m_0 and the variance γ_0.

The investor chooses a strategy at any times s, for $s \in [t,T]$. π_s represents the proportion of money invested in the second stock, then $X_s\pi_s$ represents the amount of money invested in the second stock, $X_s(1-\pi_s)$ represents the amount of money invested in the first stock. Intermediate consumption is allowed at a nonnegative rate C_s. Due to (2.1), (2.2) and (2.3), the total wealth satisfies the stochastic differential equation

$$dX_s = [X_s(\mu_1 + \pi_s(\mu_2 - \mu_1)) - C_s]ds + X_s(1-\pi_s)\sigma_1 dW_1 + X_s\pi_s\sigma_2 dW_2. \tag{2.4}$$

The pair of control (π_s, C_s) is said to be admissible if it is $\mathcal{F}_s$-progressively measurable, where $\mathcal{F}_s = \mathcal{F}_t^s; t \le T$ is the augmented filtration generated by the price process $S_2(t)$, and $E\{\int_t^T C_s ds\} < +\infty$ and $E\{\int_t^T \sigma_i^2(Y_s, s)\pi_s^2 ds\} < +\infty$ $(i = 1, 2)$. The set of all admissible controls is denoted by $\mathcal{A}$.

The investor wants to maximize his expected utility of terminal wealth and intermediate consumption

$$J(x,y,t;\pi,c) = E[\int_t^T U_1(C_s)ds + U_2(X_T)], \tag{2.5}$$

where U_1 is the instantaneous utility from consumption. The functions U_1 and U_2 are called utility functions if $U : (0,+\infty) \to R$ is C^1 function and satisfies: (1) $U(\cdot)$ is strictly increasing and strictly concave; (2) The derivative $U'(0) = \lim_{x\downarrow 0} U'(x) > 0$, and $U'(z) = 0$ for a unique value $z \in (0,\infty]$, where U can take U_1 or U_2.

The value function is defined by

$$v(x,y,t) = \sup_{(\pi,c)\in\mathcal{A}} J(x,y,t;\pi,c). \tag{2.6}$$

For given the initial time and the initial state, maximize (2.5) subject to (2.4) over the admissible controls set $\mathcal{A}$, the above problem can be formulated as the following optimization problem

$$v(x,y,t) = \sup_{(\pi,c)\in\mathcal{A}} J(x,y,t;\pi,c),$$

$$\begin{cases} dX_s = [X_s(\mu_1(s) + \pi_s(\mu_2 - \mu_1)) - C_s]ds \\ \qquad + X_s(1-\pi_s)\sigma_1 dW_1 + X_s\pi_s\sigma_2 dW_2, \\ (\pi, c) \in \mathcal{A}. \end{cases} \tag{2.7}$$

3. The Solution of the Model

In order to derive the corresponding complete information control problem, first the conditional mean and variance of $\mu_2(t)$ would be introduced according to Bai and Guo [1],

$$m_t = E[\mu_2(t) \mid \mathcal{F}_t], \gamma_t = E[(\mu_2(t) - m_t)^2 \mid \mathcal{F}_t]. \tag{3.1}$$

Then, m_t satisfies the following equation

$$dm_t = \alpha m_t dt + \frac{\gamma_t}{\sigma_2} d\bar{W}_t, \tag{3.2}$$

where γ_t is the solution of the Riccati equation

$$\frac{d\gamma_t}{dt} = 2\alpha\gamma_t - \frac{\gamma_t^2}{\sigma_2^2} + \beta^2. \tag{3.3}$$

The process $\bar{W}_t$ is called the innovations process and it is a Brownian motion with respect to $\mathcal{F}_t$. The wealth process of (2.4) under the strategy (π_s, C_s) can be rewritten in terms of innovation Brownian motion as

$$dX_s = [X_s(\mu_1 + \pi_s(m_s - \mu_1)) - C_s]ds + X_s(1-\pi_s)\sigma_1 dW_1 + X_s\pi_s\sigma_2 d\bar{W}. \tag{3.4}$$

Thus the original problem is reduced to the following problem with complete information:

$$v(x, m, t) = \sup_{(\pi,c)\in\mathcal{A}} J(x, m, t; \pi, c),$$

$$\begin{cases} dX_s = [X_s(\mu_1(s) + \pi_s(m_s - \mu_1)) - C_s]ds \\ \qquad + X_s(1-\pi_s)\sigma_1 dW_1 + X_s\pi_s\sigma_2 d\bar{W}_s, \\ dm_t = \alpha m_t dt + \frac{\gamma_t}{\sigma_2} d\bar{W}_t, \end{cases} \tag{3.5}$$

where

$$\gamma_t = \sqrt{A}\sigma_2 \frac{A_1 \exp\{2(\sqrt{A}/\sigma_2)t\} + A_2}{A_1 \exp\{2(\sqrt{A}/\sigma_2)t\} - A_2} + \alpha\sigma_2^2 \tag{3.6}$$

is the solution of (3.3) with

$$A = \alpha^2\sigma_2^2 + \beta^2, A_1 = \sqrt{A}\sigma^2 + \gamma_0 - \alpha\sigma_2^2,$$
$$A_2 = -\sqrt{A}\sigma^2 + \gamma_0 - \alpha\sigma_2^2. \tag{3.7}$$

Here, the goal is to analyze the value function and to determine the optimal investment strategy when the utility function is CRRA type

$$U_1(c) = -\frac{\rho}{\eta}e^{-\eta c}, \quad U_2(x) = -\frac{\rho}{\eta}e^{-\eta x}, \tag{3.8}$$

where $\rho,\ \eta > 0$ are constants. The following theorem shows the main result of this section.

Theorem 1. *The value function has the following form*

$$v(x,y,t) = -\frac{\rho}{\eta}\exp\{-\eta x e^{\mu_1(T-t)} + G(t,m)\}, \tag{3.9}$$

where

$$G(t,m) = K(t)m^2 + J(t)m + H(t), \tag{3.10}$$

with

$$K(t) = \frac{\exp\{\frac{t\sqrt{Z^2-4MN}}{M}\}(Z+\sqrt{Z^2-4MN}) - Z + \sqrt{Z^2-4MN}}{2M(1-\exp\{\frac{t\sqrt{Z^2-4MN}}{M}\})}, \tag{3.11}$$

$$J(t) = e^{\int_t^T Pds}\int_t^T P_1 e^{-\int_u^T Pds}du, \tag{3.12}$$

$$H(t) = \int_t^T Qds, \tag{3.13}$$

where

$$Z = -2(\alpha - \frac{\gamma_t}{\sigma_1^2+\sigma_2^2}), \quad M = -2\gamma_t^2(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2+\sigma_2^2}),$$

$$N = \frac{1}{2(\sigma_1^2+\sigma_2^2)}, \quad P = (\alpha + \frac{2\gamma_t^2}{\sigma_2^2}K(t) - \frac{2\gamma_t^2 K(t) + \gamma_t}{\sigma_1^2+\sigma_2^2}),$$

$$P_1 = \frac{\gamma_t\mu_1 K(t) + \mu_1 - \eta x\sigma_1^2 e^{\mu_1(T-t)}(2\gamma_t K(t)+1)}{\sigma_1^2+\sigma_2^2},$$

$$Q = \frac{\gamma_t^2(J^2(t)+2K(t))}{2\sigma_2^2} - \frac{\gamma_t^2 J^2(t) - 2\gamma_t\mu_1 J(t) + \mu_1^2}{2(\sigma_1^2+\sigma_2^2)} + \frac{x^2\sigma_1^2\sigma_2^2\eta^2 e^{2\mu_1(T-t)}}{2(\sigma_1^2+\sigma_2^2)}$$
$$- \frac{\eta x\sigma_1^2 e^{\mu_1(T-t)}}{\sigma_1^2+\sigma_2^2}(\gamma_t J(t) - \mu_1) + e^{\mu_1(T-t)}[\eta x e^{\mu_1(T-t)} - \mu_1(T-t) + 1],$$

and γ_t is given by (3.6). The optimal portfolio polity is given by

$$\pi^*(x,m,t) = \frac{m - \mu_1 + \gamma_t(2K(t)m + J(t))}{\eta x(\sigma_1^2+\sigma_2^2)e^{\mu_1(T-t)}} + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}, \tag{3.14}$$

$$c^*(x,m,t) = xe^{\mu_1(T-t)} - \frac{K(t)m^2 + J(t)m + H(t) + \mu_1(T-t)}{\eta}. \tag{3.15}$$

Proof. A classical approach in stochastic control theory is to examine Hamilton-Jacobi-Bellman (HJB) equation which the value function is satisfied. For the problem (3.5), using dynamic programming principle and stochastic analysis, the HJB equation is:

$$v_t + \mu_1 x v_x + \alpha m v_m + \frac{\gamma_t^2}{2\sigma_2^2} v_{mm} + \frac{1}{2} x^2 \sigma_1^2 v_{xx}$$
$$+ \max_{\pi} \{\frac{1}{2}\pi^2 x^2 (\sigma_1^2 + \sigma_2^2) v_{xx} + \pi x \gamma_t v_{xm} + \pi x (m - \mu_1) v_x - \pi x^2 \sigma_1^2 v_{xx}\}$$
$$+ \max_{c} \{-c v_x + U_1(c)\} = 0, \quad (3.16)$$

where v satisfies the terminal condition

$$v(x, m, T) = U_2(X_T), \quad (3.17)$$

for $(x, m, t) \in \bar{D} = \{(x, m, t) : x \geq 0, m \in R, 0 \leq t \leq T\}$.

Suppose the value function has the following form

$$v(x, m, t) = -\frac{\rho}{\eta} \exp\{-\eta x e^{\mu_1 (T-t)} + G(t, m)\},$$

where $v(x, m, t) \in C^{1,2}(\bar{D}), G(t, m)$ is a suitable function. The boundary condition implies that $G(t, m) = 0$. Direct substitution in the HJB equation leads to

$$G_t + \alpha m G_m + \frac{\gamma_t^2}{2\sigma_2^2}(G_m^2 + G_{mm}) + \frac{1}{2} x^2 \sigma_1^2 \eta^2 e^{2\mu_1 (T-t)}$$
$$+ \max_{\pi} \{\frac{1}{2}(\sigma_1^2 + \sigma_2^2)\pi^2 x^2 \eta^2 e^{2\mu_1 (T-t)} - \pi x (m - \mu_1 + \gamma_t G_m) \eta e^{\mu_1 (T-t)}$$
$$-\pi x^2 \sigma_1^2 \eta^2 e^{2\mu_1 (T-t)}\}$$
$$+ \max_{c} \{c \eta e^{\mu_1 (T-t)} + \exp\{\eta x e^{\mu_1 (T-t)} - \eta c - G(t, m)\}\} = 0. \quad (3.18)$$

Now apply the first order conditions in (3.18). We have that the maximum is achieved at

$$\pi^*(x, m, t) = \frac{m - \mu_1 + \gamma_t G_m}{\eta x (\sigma_1^2 + \sigma_2^2) e^{\mu_1 (T-t)}} + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2},$$
$$c^*(x, m, t) = x e^{\mu_1 (T-t)} - \frac{G + \mu_1 (T - t)}{\eta}.$$

Using the form of π^*, c^*, (3.18) reduces to

$$G_t + \alpha m G_m + \frac{\gamma_t^2}{2\sigma_2^2}(G_m^2 + G_{mm}) + \frac{1}{2} x^2 \sigma_1^2 \eta^2 e^{2\mu_1 (T-t)} + \eta x e^{2\mu_1 (T-t)}$$

$$-\frac{(m-\mu_1+\gamma_t G_m+\eta x\sigma_1^2 e^{\mu_1(T-t)})^2}{2(\sigma_1^2+\sigma_2^2)}+e^{\mu_1(T-t)}[1-G-\mu_1(T-t)]=0.$$

Let $G(t,m) = K(t)m^2 + J(t)m + H(t)$, then, $K(t)$, $J(t)$ and $H(t)$ are solutions of the following ordinary differential equations:

$$K'(t) - ZK(t) - M^2K^2(t) - N = 0,$$

$$J'(t) + PJ(t) + P_1 = 0,$$

$$H'(t) + Q = 0.$$

Solving the above three equations, we get $K(t)$, $J(t)$, $H(t)$ as (3.11), (3.12) and (3.13).

Then the optimal portfolio is as (3.14) and (3.15). □

4. Summary

In this paper we analyzed an optimal investment and consumption model with stochastic factor in markets. An important extension is to allow for intermediate consumption and discount factor. For the specific class of separable CRRA utilities, we derived the optimal policies in terms of the dynamic programming approach combining the filtering theory. The conclusions of this paper no doubt have a certain guiding role for the actual risk investment behavior. The method of discussing the stochastic control model and the HJB equation is also very instructive.

References

[1] Lihua Bai, Junyi Guo, Utility maximization with partial information: Hamilton-Jacobi-Bellman equation approach, *Front. Math*, P.R. China, **2**, No. 4 (2007), 527-537.

[2] R.C. Merton, Optimum consumption and portfolio rules in a continuous time model, *J. Econ. Theory*, **3** (1971), 373-413.

[3] R. Rishel, Optimal portfolios management with partial observation and power utility function, In: *Stochastic Analysis, Control, Optimization and Applications: Volume in Honor of W.H. Fleming* (Ed-s: W. McEneany, G. Yin, Q. Zhang), Boston, Birkhäuser (1999), 605-620.

[4] Jide Yang, Limin Liu, Yanlin Shen, The optimal portfolio for Stein-Stein model, *Journal of Yang Zhou University*, National Science Edition, **9**, No. 1 (2006), 22-24.