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ON HILBERT TYPE INTEGRAL INEQUALITY AND APPLICATIONS

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Abstract: In this paper it is shown that a new Hilbert type integral inequality can be established by introducing an integral kernel function of the form $\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}$, where $\alpha \geq 0$. And the constant factor expressed by the Euler number and π is proved to be the best possible. As applications, some equivalent forms are given.

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1. Introduction and Lemmas

Let $\alpha \geq 0$, and $f(x),\ g(x) \in L^2(\alpha, +\infty)$. Then

$$\int_\alpha^\infty\int_\alpha^\infty \frac{f\left(x\right)g\left(y\right)}{x+y-2\alpha}dxdy \leq \pi\left\{\int_\alpha^\infty f^2\left(x\right)dx\right\}^{\frac{1}{2}}\left\{\int_\alpha^\infty g^2\left(x\right)dx\right\}^{\frac{1}{2}}. \tag{1.1}$$

This is the famous Hilbert integral inequality, where the constant factor π is the best possible. This inequality was generalized in the paper [1].

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In the papers [2]-[3], the following inequality of the form

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)f\left(x\right)g\left(y\right)}{x-y}dxdy \le \pi^2\left\{\int_0^\infty f^2\left(x\right)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2\left(x\right)dx\right\}^{\frac{1}{2}} \tag{1.2}$$

was established, and the constant factor π^2 is also the best possible. This inequality is also extended in the paper [4].

Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) and (1.2) appear in a great deal of papers (see [5]-[9], etc.). Specially, Gao and Hsu enumerated the research articles more than 40 in the paper [5]. The aim of the present paper is to build some new Hilbert type integral inequalities by introducing a proper integral kernel function and by using the technique of analysis, and to discuss the constant factor of which is related to Euler number, and then to study some equivalent forms of them.

In the sake of convenience, we introduce some notations and define some functions.

Let $0 < s < 1$ and n be a nonnegative integer. Define a function F by

$$F(n,\ s) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(s+k)^{n+1}}. \tag{1.3}$$

And further define the function G by

$$G(n) = (2n)!\left\{2F(2n,\ \frac{1}{2})\right\}, \tag{1.4}$$

In order to prove our main results, we need the following lemmas.

Lemma 1.1. *Let $0 < s < 1$ and n be a nonnegative integer. Then*

$$\int_0^1 u^{s-1}\left(\ln\frac{1}{u}\right)^n\frac{1}{1+u}du = n!F(n,\ s). \tag{1.5}$$

This result has been given in the paper [10]. Hence its proof is omitted here.

Lemma 1.2. *With the assumptions as Lemma 1.1, then*

$$\int_0^\infty u^{s-1}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du = (2n)!\left\{F\left(2n,\ s\right)+F\left(2n,\ 1-s\right)\right\}, \tag{1.6}$$

where the function F is defined by (1.3).

In particular, when $s = \frac{1}{2}$, $\int\limits_0^\infty u^{-\frac{1}{2}}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du = G(n)$, where $G(n)$ is defined by (1.4).

Proof. We show only that the equality (1.6) holds. It is easy to deduce that

$$\begin{aligned}
&\int\limits_0^\infty u^{s-1}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du \\
&\quad= \int\limits_0^1 u^{s-1}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du + \int\limits_1^\infty u^{s-1}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du \\
&\quad= \int\limits_0^1 u^{s-1}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du + \int\limits_0^1 v^{-s}\left(\ln v\right)^{2n}\frac{1}{1+v}dv \\
&\quad= \int\limits_0^1 u^{s-1}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du + \int\limits_0^1 v^{(1-s)-1}\left(\ln\frac{1}{v}\right)^{2n}\frac{1}{1+v}dv. \qquad (1.7)
\end{aligned}$$

By using Lemma 1.1, the equality (1.6) from (1.7) follows. □

2. Main Results

In this section, we will prove our assertions by using the above lemmas.

Theorem 2.1. *Let $\alpha \geq 0$, f and g be two real functions, and n be a nonnegative integer. If $0 < \int_\alpha^\infty f^2(x)dx < +\infty$ and $0 < \int_\alpha^\infty g^2(x)dx < +\infty$, then*

$$\begin{aligned}
&\int\limits_\alpha^\infty\int\limits_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n} f(x)g(y)}{x+y-2\alpha}dxdy \\
&\qquad< \left(\pi^{2n+1}E_n\right)\left\{\int\limits_\alpha^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int\limits_\alpha^\infty g^2(x)dx\right\}^{\frac{1}{2}}, \qquad (2.1)
\end{aligned}$$

where the constant factor $\pi^{2n+1}E_n$ is the best possible, and $E_0 = 1$ and that $E_{n's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

Proof. We may apply the Cauchy inequality to estimate the left-hand side of (2.1) as follows:

$$\int_\alpha^\infty\int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n} f(x)g(y)}{x+y-2\alpha}dxdy$$

$$= \int_\alpha^\infty\int_\alpha^\infty \Big(\frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}\Big)^{\frac12}\Big(\frac{x-\alpha}{y-\alpha}\Big)^{\frac14} f(x)\Big(\frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}\Big)^{\frac12}\Big(\frac{y-\alpha}{x-\alpha}\Big)^{\frac14} g(y)dxdy$$

$$\le \Big\{\int_\alpha^\infty\int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}\Big(\frac{x-\alpha}{y-\alpha}\Big)^{\frac12} f^2(x)dxdy\Big\}^{\frac12}$$

$$\times\Big\{\int_\alpha^\infty\int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}\Big(\frac{y-\alpha}{x-\alpha}\Big)^{\frac12} g^2(y)dxdy\Big\}^{\frac12}$$

$$= \Big(\int_\alpha^\infty \omega(x)f^2(x)dx\Big)^{\frac12}\Big(\int_\alpha^\infty \omega(x)g^2(x)dx\Big)^{\frac12}, \tag{2.2}$$

where $\omega(x) = \int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac12}dy.$

By using Lemma 1.2, it is easy to deduce that

$$\omega(x) = \int_\alpha^\infty \frac{(\ln\frac{x-\alpha}{y-\alpha})^{2n}}{(x-\alpha)\left(1+\frac{y-\alpha}{x-\alpha}\right)}\Big(\frac{x-\alpha}{y-\alpha}\Big)^{\frac12}dy$$

$$= \int_0^\infty u^{-\frac12}(\ln\frac1u)^{2n}\frac{1}{1+u}du = G(n),$$

where $G(n)$ is defined by (1.4). Based on (1.3), we have

$$G(n) = (2n)!\,2\sum_{k=0}^\infty \frac{(-1)^k}{(\frac12+k)^{2n+1}} = (2n)!\,2^{2n+2}\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{2n+1}}. \tag{2.3}$$

It is known from the paper [11] that

$$\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!}E_n, \tag{2.4}$$

where $E_{n's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 =$

1385, $E_5 = 50521$, etc.

Since $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$, we can define $E_0 = 1$. Thus, the relation (2.4) is also valid when $n = 0$. So, we get from (2.3) and (2.4) that

$$\begin{aligned}\omega(x) &= G(n)\\ &= \int_0^{\infty} u^{-\frac{1}{2}} (\ln \frac{1}{u})^{2n} \frac{1}{1+u} du\\ &= \pi^{2n+1} E_n, \end{aligned} \tag{2.5}$$

It follows from (2.2) and (2.5) that

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n} f(x)g(y)}{x+y-2\alpha} dxdy$$

$$\leq \left(\pi^{2n+1}E_n\right)\left\{\int_{\alpha}^{\infty} f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_{\alpha}^{\infty} g^2(x)dx\right\}^{\frac{1}{2}}. \tag{2.6}$$

If (2.6) takes the form of the equality, then there exist a pair of non-zero constants c_1 and c_2 such that

$$c_1 \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha} f^2(x)\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{1}{2}} = c_2 \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha} g^2(y)\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{1}{2}}$$

$$\text{a.e. on } (0,+\infty)\times(0,+\infty).$$

Then we have

$c_1(x-\alpha)\ f^2(x) = c_2(y-\alpha)\ g^2(y) = C_0$ (constant), a.e. on $(0,+\infty)\times(0,+\infty)$.

Without losing the generality, we suppose that $c_1 \neq 0$, then

$$\int_{\alpha}^{\infty} f^2(x)dx = \frac{C_0}{c_1}\int_{\alpha}^{\infty}(x-\alpha)^{-1}dx.$$

This contradicts that $0 < \int_{\alpha}^{\infty} f^2(x)dx < +\infty$. Hence it is impossible to take the equality in (2.6). So the inequality (2.1) is valid.

It remains only to show that $\pi^{2n+1}E_n$ in (2.1) is the best possible.

$\forall\varepsilon:\ 0 < \varepsilon < 1$. Define two functions by

$$\tilde{f}(x) = \begin{cases} 0 & x \in (\alpha,\ \alpha+1),\\ (x-\alpha)^{-\frac{1+\varepsilon}{2}} & x \in [\alpha+1,\ \infty), \end{cases}$$

and

$$\tilde{g}(y) = \begin{cases} 0 & y \in (\alpha,\ \alpha+1), \\ (y-\alpha)^{-\frac{1+\varepsilon}{2}} & y \in [\alpha+1,\ \infty). \end{cases}$$

It is easy to deduce that

$$\int_{\alpha}^{+\infty} \tilde{f}^2(x)dx = \int_{\alpha}^{+\infty} \tilde{g}^2(y)dy = \frac{1}{\varepsilon}\ .$$

If $\pi^{2n+1}E_n$ is not the best possible, then there exists $C > 0$, such that

$$\begin{aligned} H(\alpha,\varepsilon) &= \int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n} \tilde{f}(x)\tilde{g}(y)}{x+y-2\alpha} dxdy \\ &\le C\Big(\int_{\alpha}^{\infty} \tilde{f}^2(x)dx\Big)^{\frac{1}{2}}\Big(\int_{\alpha}^{\infty} \tilde{g}^2(y)dy\Big)^{\frac{1}{2}} = \frac{C}{\varepsilon} < \frac{\pi^{2n+1}E_n}{\varepsilon}. \end{aligned} \tag{2.7}$$

On the other hand, we have

$$\begin{aligned} H(\alpha,\varepsilon) &= \int_{\alpha}^{\infty}\int_{\alpha}^{\infty} \frac{\Big\{(x-\alpha)^{-\frac{1+\varepsilon}{2}}\Big\}\Big\{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}(y-\alpha)^{-\frac{1+\varepsilon}{2}}\Big\}}{x\ +\ y-2\alpha} dxdy \\ &= \int_{\alpha}^{\infty}\Big\{\int_{\alpha}^{\infty} \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}(y-\alpha)^{-\frac{1+\varepsilon}{2}}}{(x\ -\alpha)\left(1+\ \frac{y-\alpha}{x-\alpha}\right)} dy\Big\}\Big\{(x-\alpha)^{-\frac{1+\varepsilon}{2}}\Big\}dx \\ &= \int_{\alpha}^{\infty}\Big\{\int_{0}^{\infty} \frac{(\ln \frac{1}{u})^{2n}\ u^{-\frac{1+\varepsilon}{2}}}{1\ +\ u} du\Big\}\Big\{(x-\alpha)^{-1-\varepsilon}\Big\}dx \\ &= \frac{1}{\varepsilon}\int_{0}^{\infty} \frac{(\ln \frac{1}{u})^{2n}\ u^{-\frac{1+\varepsilon}{2}}}{1\ +\ u} du. \end{aligned} \tag{2.8}$$

When ε is sufficiently small, we obtain from (2.5) that

$$\int_{0}^{\infty} \frac{(\ln \frac{1}{u})^{2n}\ u^{-\frac{1+\varepsilon}{2}}}{1\ +\ u} du = \pi^{2n+1}E_n + \circ(1) \quad (\varepsilon \to 0). \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$H(\alpha,\varepsilon) = \frac{1}{\varepsilon}\Big\{(\pi^{2n+1}E_n\ ) +\ \circ(1)\Big\} \quad (\varepsilon \to 0). \tag{2.10}$$

Clearly, the inequality (2.7) is in contradiction with (2.10). Therefore, the constant factor $\pi^{2n+1}E_n$ in (2.1) is the best possible. Thus the proof of the theorem is completed. □

In particular, when $n = 0$, the inequality (2.1) is reduced to (1.1).

Notice that the constant factor $\pi^{2n+1}E_n$ in (2.1) can be reduced to π^3, if $n = 1$. Hence we have the following result.

Corollary 2.2. *With the assumptions as Theorem 2.1, if $\alpha = 0$, then*

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^2 f(x)g(y)}{x+y} dxdy \;<\; \pi^3 \Big\{ \int_0^\infty f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_0^\infty g^2(x)dx \Big\}^{\frac{1}{2}}, \qquad (2.11)$$

where the constant factor π^3 is the best possible.

3. Some Applications

As applications, we will build the following inequalities.

Theorem 3.1. *Let n be a nonnegative integer. If $0 < \int_\alpha^\infty f^2(x)dx < +\infty$, then*

$$\int_\alpha^\infty \Big\{ \int_\alpha^\infty \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}}{x \,+\, y - 2\alpha} f(x)dx \Big\}^2 dy < \; (\pi^{2n+1}E_n)^2 \int_\alpha^\infty f^2(x)dx, \qquad (3.1)$$

where $(\pi^{2n+1}E_n)^2$ in (3.1) is the best possible, and $E_0 = 1$ and that $E_{n's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc. And the inequality (3.1) is equivalent to (2.1).

Proof. First, we show that the inequality (3.1) is equivalent to (2.1). Setting a real function $g(y)$ as

$$g(y) = \int_\alpha^\infty \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}}{x \,+\, y - 2\alpha} f(x)dx, \quad y \in (\alpha, \, +\infty).$$

By using (2.1), we have

$$\int_\alpha^\infty \Big\{ \int_\alpha^\infty \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}}{x \,+\, y - 2\alpha} f(x)dx \Big\}^2 dy = \int_\alpha^\infty \int_\alpha^\infty \frac{\left(\ln \frac{x-\alpha}{y-\alpha}\right)^{2n}}{x \,+\, y - 2\alpha} f(x)g(y)dxdy$$

$$< (\pi^{2n+1}E_n) \Big\{ \int_\alpha^\infty f^2(x)dx \Big\}^{\frac{1}{2}} \Big\{ \int_\alpha^\infty g^2(y)dy \Big\}^{\frac{1}{2}}$$

$$= (\pi^{2n+1}E_n)\Big\{\int_0^\infty f^2(x)dx\Big\}^{\frac{1}{2}}\Big\{\int_\alpha^\infty\Big(\int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}f(x)dx\Big)^2dy\Big\}^{\frac{1}{2}}, \tag{3.2}$$

where $E_0 = 1$ and $E_{n's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

It follows from (3.2) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid, by applying in turn Cauchy's inequality and (3.1), we have

$$\int_\alpha^\infty\int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}f(x)g(y)dxdy = \int_\alpha^\infty\Big\{\int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}f(x)dx\Big\}g(y)dy$$

$$\le \Big\{\int_\alpha^\infty\Big(\int_\alpha^\infty \frac{\left(\ln\frac{x-\alpha}{y-\alpha}\right)^{2n}}{x+y-2\alpha}f(x)dx\Big)^2dy\Big\}^{\frac{1}{2}}\Big\{\int_\alpha^\infty g^2(y)dy\Big\}^{\frac{1}{2}}$$

$$< \Big\{(\pi^{2n+1}E_n)^2\int_\alpha^\infty f^2(x)dx\Big\}^{\frac{1}{2}}\Big\{\int_\alpha^\infty g^2(y)dy\Big\}^{\frac{1}{2}}$$

$$= (\pi^{2n+1}E_n)\Big\{\int_\alpha^\infty f^2(x)dx\Big\}^{\frac{1}{2}}\Big\{\int_\alpha^\infty g^2(y)dy\Big\}^{\frac{1}{2}}, \tag{3.3}$$

where $E_0 = 1$ and $E_{n's}$ are the Euler numbers, viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc.

If the constant factor $(\pi^{2n+1}E_n)^2$ in (3.1) is not the best possible, then it is known from (3.3) that the constant factor $\pi^{2n+1}E_n$ in (2.1) is also not the best possible. This is a contradiction. The theorem is proved. □

Corollary 3.2. *With the assumptions as Theorem 3.1, if $n = 1$ and $\alpha = 0$, then*

$$\int_0^\infty\Big\{\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^2}{x+y}f(x)dx\Big\}^2dy < \pi^6\int_0^\infty f^2(x)dx, \tag{3.4}$$

where the constant factor π^6 is the best possible. Inequality (3.4) is equivalent to (2.11).

Its proof is similar to the one of Theorem 3.1. Hence it is omitted here.

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