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GENERALIZED EXPO-RATIONAL B-SPLINES

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Abstract: The concept of *expo-rational B-spline (ERBS)* was introduced in [6] where the 'superproperties' of ERBS were studied for the first time. In 2006 the first author of the present paper proposed a framework for the generalization of ERBS, the so-called generalized ERBS (GERBS), and considered one important instance of GERBS, the so-called Euler Beta-function B-splines (BFBS) which offered a good trade-off between preservation of the important 'superproperties' of ERBS (with some reductions) and easy computability (BFBS being explicitly and exactly computable piecewise polynomials, while ERBS were computed by a very rapidly converging, yet approximate, numerical integration algorithm). The practical performance of BFBS was tested against classical polynomial B-splines and ERBS in some MSc Diploma Thesis projects at the R&D Group for Mathematical Modelling, Numerical Simulation and Computer Visualization at Narvik University College, resulting in the development of a related software application in [19]. The first detailed exposition of the concepts and theory related to GERBS and BFBS was in [5] at the Seventh International Conference on Mathematical Methods for Curves and Surfaces in Tønsberg, Norway, in 2008.

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The purpose of the present paper is to provide a detailed systematic exposition of the definitions, basic properties and advanced features and 'superproperties' of GERBS and BFBS, to trace the evolution of these properties as the consideration is being gradually focused from the most general concept of GERBS as reparametrizations of the piecewise-affine B-splines, with bounded Jordan variation, through the justification of the introduction of BFBS as C^m-smooth GERBS, $m \in \mathbb{N}$, to the ultimate construction of ERBS as C^∞-smooth GERBS. In the course of the exposition we keep track of the trade-off between the computability of the GERBS versus the extension of the range of its 'superproperties'. At the same time, we compare the features of GERBS with those of classical polynomial Schoenberg B-splines. The main new mathematical concepts in the study are being geometrically elucidated via visualization plots generated by our own in-house software applications.

Several new special relevant topics are included in the study, as follows.

- comparison between the computational complexity of interpolation, approximation and numerical linear algebraic problems when solved by classical polynomial B-splines and by (G)ERBS, in sections 2.1.3 and 2.2.1;
- heuristic motivation for introducing GERBS, complementary to the heuristics for introducing ERBS [6], in section 4;
- several new model constructions of GERBS, including, but not limited to, BFBS, in section 5;
- a concluding discussion about the considerable advantages of replacing NURBS (Non Uniform Rational B-splines) by rational forms of GERBS as universal tools of Isogeometric Analysis, in section 6.

This is the first in a sequence of papers dedicated to the design and properties of GERBS, as functions of one and several variables, in one and several dimensions.

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1. Introduction

In [6] was introduced a new type of B-spline - the expo-rational B-spline (ERBS). One of the main results of [6] was the derivation of an Edgeworth and a steepest-descent/saddlepoint asymptotic expansion which showed that ERBS are the asymptotic limits of polynomial B-splines when the degree of the latter (or, equivalently, the number of the knots of the latter) tends to infinity. Further in [6] it was shown that, as a consequence of their nature as asymptotic limits, the new B-splines exhibit 'superproperties' by outperforming usual B-splines in a number of important aspects: for example, in constructing a minimally supported C^∞-smooth partition of unity over triangulated polygonal domains of any dimension (a detailed definition of ERBS over n-dimensional triangulations (simplectifications) was announced in [5] and will appear in a separate publication). In [6] it was also noted that from computational point of view there is 'a price to pay' for the 'superproperties' of these new 'hyperspline' special functions, part of which price is that their evaluation involves integrals which cannot be computed in terms of classical elementary functions. To solve this problem, in [6] was proposed a numerical quadrature process based on Romberg integration which is highly efficient because of the infinite smoothness of the integrand. Nevertheless, it is of considerable interest to study more general classes of special functions which contain ERBS together with other 'non-trivial non-ERBS' subclasses of functions that retain as much as possible of the 'superproperties' of ERBS but, unlike ERBS, can be evaluated in closed form in terms of classical elementary functions, or are simpler than ERBS in other important aspects. Research in this direction has lead to [**?**] and to the present study of a general class of *special functions of bounded variation* which provide *minimally supported convex partition of unity* and which contain as particular subclasses both ERBS and piecewise linear (piecewise affine) B-splines. In correspondence with their properties, we shall call the functions in this class Generalized Expo-rational B-splines, or GERBS, for short.

The organization of this paper is, as follows. The next Section 2 deals with a preliminary comparison between classical polynomial B-splines and ERBS. The purpose of this section is to reveal the set of desired essential properties which GERBS should share with ERBS. Section 3 contains the definition of GERBS in its maximal generality. In Section 4 are discussed some important heuristic arguments in favour of introducing GERBS which are additional and complementary to the heuristic motives for introducing ERBS (as described in Section 3 of [6]). After this, in Section 5 we consider several particular subclasses of GERBS each of which contains ERBS but is also of interest 'per

se', for specific theoretical and/or practical applications. This consideration is made in order of decreasing generality of the class and, respectively, increasing range of the set of 'superproperties' which the functions of this class share with ERBS. In particular, in this section we consider for the first time the new *Euler Beta function B-splines*, previously mentioned in relevance to parametric curves and tensor-product surfaces in [8, 5] and in relevance to smooth convex partition of unity and Hermite interpolation on scattered point sets on multidimensional domains in [5]. Finally, Section 6 contains some concluding remarks concerning the potential for using GERBS in isogeometric analysis.

The graphical images appearing in this exposition are selected from the image gallery generated during the work of the students Ivana P. Gancheva and Nedyalka D. Delistoyanova on their M. Sc. Diploma thesis [19] at Narvik University College.

2. Classical Polynomial B-Splines Versus Expo-Rational B-Splines (ERBS)

2.1. Polynomial B-Splines

Let T be a set of $n+1$ non-decreasing real numbers $t_0 \leq t_1 \leq \cdots \leq t_n$ for $t_i \in [a,b)$, $-\infty < a < b < +\infty$. Traditionally in spline-related literature, the t_i's are called *knots*, the semi-open interval $[t_i, t_{i+1})$ - the i-th *knot-interval span*, and the set T is called a *knot vector*.

Definition 1. The i-th B-spline basis function of degree p, written as $N_{i,p}(t)$, is defined recursively, as follows:

$$N_{i,0}(t) = \begin{cases} 1, & \text{if} \quad t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise}, \end{cases}$$

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t).$$

These two equations are usually referred to as the *Cox/de Boor recursion formula*.

If the knot vector is strictly increasing, i.e., $t_0 < t_1 < \cdots < t_n$, the knots are said to be *simple knots* or, otherwise, to have *simple multiplicity*, or multiplicity equal to 1. If this knot vector is monotonously (not necessarily strictly) increasing, i.e., if equality of some of the consecutive t_i's is allowed, the respec-

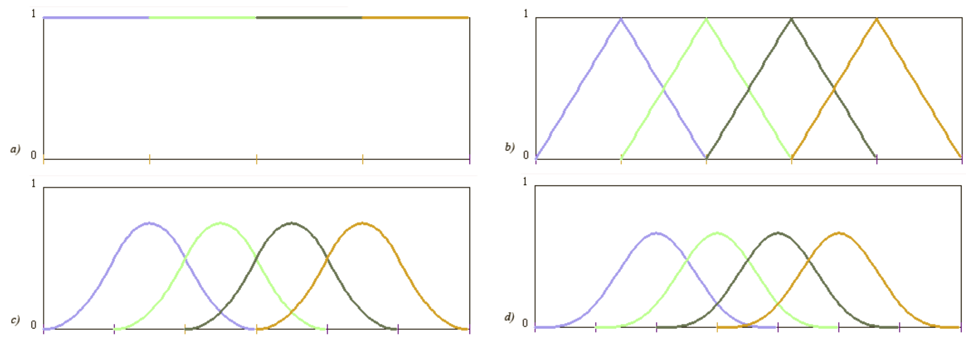

Figure 1: Piecewise constant (a), piecewise affine (b), piecewise quadratic (c), and piecewise cubic (d) B-spline basis functions for the simple knot case. Notice that increase of the order of smoothness, resp. the degree of the polynomial B-spline, leads to increase of the number of intervals between knots, over which the support of the B-spline spans.

tive repeated knots are called *multiple knots*, and the *multiplicity* of the knot t_i is the number of consecutive times the knot t_i appears in the knot vector.

Recall some of the properties of a system of polynomial B-spline basis functions of a given degree over a given knot vector.

- $N_{i,p}$ is a p-th degree polynomial in t.
- Non-negativity: $N_{i,p} \geq 0$, $\forall\ i$, p and t.
- Local support: $N_{i,p}$ is a piecewise non-zero polynomial *only* on $[t_i, t_{i+p+1})$.
- Convex partition of unity: The set of basis functions forms a convex (non-negative barycentric) partition of unity on $[t_0, t_n)$.
- At a knot of multiplicity k, basis function $N_{i,p}(t)$, which has the knot in its support, is C^{p-k}-smooth.

2.1.1. Simple Knots

Consider a strictly increasing knot vector $\{t_k\}_{k=0}^{n+1}$. Figure 1 shows examples of general polynomial B-spline basis functions of different degree, i.e., piecewise constant, piecewise linear (piecewise affine), and smoother basis functions. The

knot vectors chosen for the presented figures are equidistant, but our observations will be irrelevant to this assumption and will hold true for both uniform and non-uniform knot vector of the B-spline system.

In terms of continuity, knot differentiability and support size, the examples show the following (see Figure 1):

- piecewise constant B-splines: discontinuous at the knots, support - one knot-interval span – case (a);
- piecewise linear (piecewise affine) B-splines: continuous at the knots, support - two knot-interval spans – case (b);
- quadratic B-splines: continuous and continuously differentiable at the knots, support - three knot-interval spans – case (c);
- cubic B-splines: continuous and two times continuously differentiable on the knots, support - four knot-interval spans – case (d);
- and so on.

It is easy to see that for general polynomial B-spline basis functions, raising of the degree leads to raising the number of knot spans included in the support of each B-spline. If the degree is raised, visible improvement of the smoothness is achieved, but the price for that is the so-called 'blow-up' of the support of the B-spline which shares parts of its support with an ever increasing number of neighbours.

2.1.2. Multiple Knots

In the above situation of simple knots only, consider now 'gluing' (coalescing, joining together) adjacent simple knots, thus getting (fewer in number) multiple knots.

First, the support is obviously getting smaller because the knot-interval span of previously non-zero length, becomes of length zero. If the process of joining knots together be continued, then, upon joining together more simple knots than is the initial degree of the B-spline system with simple knots, two adjacent B-splines containing the multiple knot in its support will become discontinuous at this knot.

Figures 2, 3 illustrate the possible variety of B-splines with multiple knots for the case of quadratic, cubic and quartic B-spline systems.

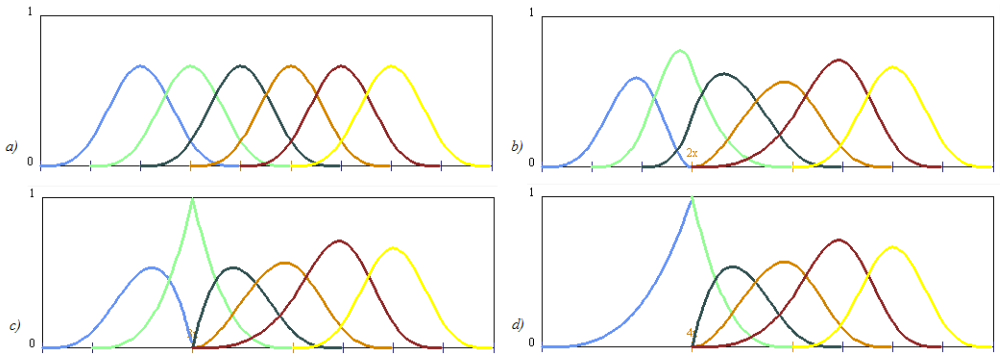

Figure 2: Illustration of the so-called "gluing"/coalescing of knots for a set of cubic B-spline basis functions. a) simple knots; b) knot t_4 moved to t_3; c) knot t_2 moved to t_3; d) knot t_1 moved to t_3. Notice that increase of the multiplicity of a knot leads to decrease of the smoothness of the respective B-spline basis function, and, ultimately (see (d)) to discontinuity. (This is because certain divided differences (in the definition of all B-splines containing in their support the knots subjected to 'gluing') must converge to respective derivatives upon the 'gluing'.) The 'cusp' in (c) is the graph of a single B-spline which is continuous, but not smooth, at the knot corresponding to its maximum (which is 1, because the B-splines sum up to 1 and all other B-splines are 0 at this knot). The 'cusp' in (d) is the union of the graphs of two B-splines with adjacent supports, both B-splines being discontinuous at the knot where the cusp is situated. Both B-splines are right-continuous, the left-hand one being 0 at the 'cusp knot', while the right-hand one is 1 at the same knot. Note that the right-hand B-spline has a maximum at the 'cusp knot', while the left-hand one has only its unattainable supremum there. In this way, at the expense of losing their continuity, the two B-splines still sum up to 1 at the 'cusp knot' (all other B-splines are zero at this knot).

2.1.3. Interpolation, Approximation and Numerical Linear Algebra

Piecewise affine B-splines are only C^0-continuous in the knots, hence, they can only be used for Lagrange interpolation but not for Hermite interpolation. However, from the above description of the classical construction of polynomial B-splines it is clear that, compared to the use of polynomial B splines which

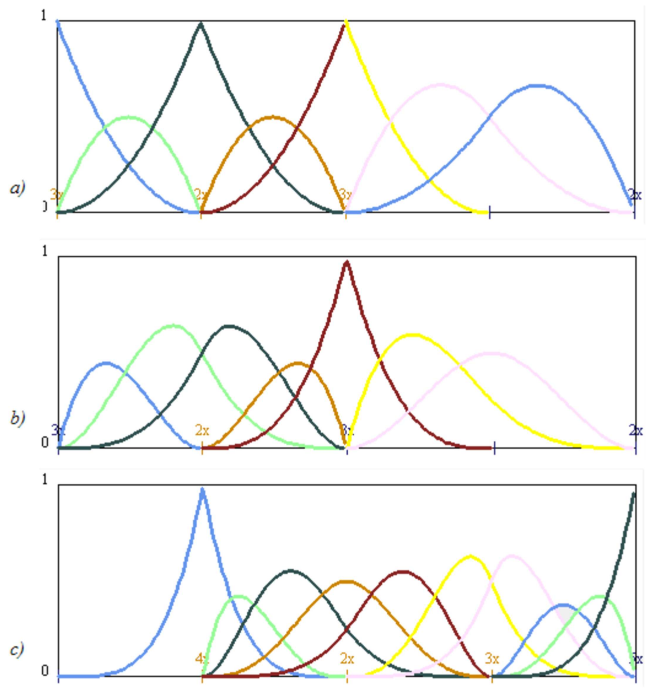

Figure 3: Polynomial B-spline basis functions of second (a), third (b) and fourth (c) degree, with various multiplicity of knots.

are smooth at the knots, the case of piecewise affine B-splines has the substantial advantage of being minimally supported, since each B-spline of the system has only the minimal possible number of neighbours with which to share parts of its support. In the univariate case considered in detail here, this means that a B-spline shares support with only one neighbour on each side. In the multivariate case, this advantage *drastically increases with the increase of the dimension*. This is true for the relatively uniform multivariate constructions like tensor-product and Boolean-sum manifolds, as well as for manifolds obtained by subdivision, and just as well for more complex interpolation and approximation constructions based on scattered data (say, for triangulated (simplectified) multivariate parametric manifolds).

For example, for the multivariate triangulation of M scattered knots (so that the vertices of the triangulation are the given scattered knots) a piecewise affine B-spline 'peaking' at a given vertex of the triangulation shares support

with only the B-splines which 'peak' at a vertex from the *star-1 neighbourhood (see [22]) of the given vertex*.

In comparison, B-splines which are more than C^0-continuous at the vertices and edges of the triangulation, e.g., C^k-smooth, $k = 1, 2, \dots,$ must be quadratic, cubic, etc., and, hence, their supports must *spread over the star-2, star-3, etc. neighbourhoods of the given vertex of the triangulation* (see, e.g., [22, 23, 24]). Thus, *the generalized Vandermonde matrix of Lagrange interpolation with piecewise affine B-splines* will be *diagonal for interpolation problems over domains of any dimension*, while the generalized Vandermonde matrix for Hermite interpolation in the vertices of the triangulation with smoother classical polynomial B-splines will be only *band-limited*, with the bandwidth drastically expanding. This is not only due to the number of neighbours in the star-1 neighbourhood (as with piecewise affine B-splines) but, to a much greater extent, also due to the increased smoothness. Hermite interpolation in the vertices up to total smoothness order k will require the bandwidth m of the generalized Vandermonde matrix to expand drastically.

For simplicity, let us consider here only the best case of Delaunay triangulation of uniformly scattered knots (which leads to uniform triangulation). Even in this best case, the bandwidth of the generalized Vandermonde matrix will expand proportionally to $(n!)n^k$. This is because it must be proportional to n^k equations per vertex (for Hermite interpolation of total order k in n dimensions), with number of neighbours in the uniform star-1 neighbourhood proportional to $n!$. This estimate provides only a lower asymptotic bound, because the absence of uniformity in the triangulation, on its part, increases the bandwidth even more drastically (including, in the worst possible case, a *full* generalized Vandermonde matrix).

Note that the above asymptotics for bandwidth is only valid for *essential* Hermite interpolation, i.e., $k \geq 1$; recall that for Lagrange interpolation (Hermite interpolation with $k = 0$) the generalized Vandermonde matrix is diagonal, i.e., the bandwidth is 1 (not $O(n!)$).

On the other hand, the total number N of equations in the *square* linear system will be proportional to $M.n^k$ (the number M of vertices in the triangulation, times the number of Hermite interpolation conditions per vertex)

$$N \sim M.n^k.$$

This means that the generalized Vandermonde matrix will have dimensions $N \times N$, where N is as estimated above. This implies that the complexity of the inversion of the generalized Vandermonde matrix for this Hermite interpolation problem via fast sequential algorithms for band-limited sparse matrices would

be (again, very roughly) proportional (see, e.g., [27]) to the square of the order of the band-limited matrix, times its bandwidth, i.e., proportional to

$$N^2.m \sim (M.n^k)^2.(n!)n^k = (n!)n^{3k}.M^2,$$

which means that the complexity would roughly satisfy

$$Complexity(polyBspl, sequential,\ Hermite\ interpolation,\ n,\ k,\ M) \sim (n!)n^{3k}.M^2.$$

Using, e.g., Stirling's formula, this estimate can be further simplified.

We shall show that by using expo-rational B-splines, considered in the next subsection, the complexity of high-order Hermite interpolation of scattered data in multidimensional domains can be very considerably improved (see item 2.2.1 below).

In conclusion of this discussion, let us note that here was considered only the simple situation of interpolating a parametric manifold which is a *patch*, i.e., its parametric domain, as well as the manifold itself, are both simply connected, with topological genus 0. This is, of course, a very restrictive assumption in dimensions higher than 1 (for example, for trimmed surfaces and volume deformations the parametric domain has higher topological genus; also, a great diversity of surfaces and solid shapes arising in applications are *not* single patches). Unfortunately, if this very restrictive assumption is not fulfilled, additional great difficulties and complications arise with the use of classical polynomial B-splines on triangulations for Hermite interpolation. For example, while in the case of surfaces with complex topology the star-1 neighbourhood of every vertex in the triangulation retains the property of simple connectedness, in general, the star-j neighbourhood of this vertex may not be simply connected for $j = 2, 3\dots,$ which, depending on the concrete problem, may sometimes cause nearly unsurmountable additional algorithmic and computational complications.

2.2. Expo-Rational B-Splines (ERBS)

Let $t_k \in \mathbb{R}$ and $t_k < t_{k+1}$ for $k = 0, 1, 2, ..., n+1,$ i.e., consider a strictly increasing knot vector $\{t_k\}_{k=0}^{n+1}$.

Definition 2. The so-called 'default' version of an Expo-rational B-spline (ERBS), associated with the (strictly increasing) knots t_{k-1}, t_k and t_{k+1}, is

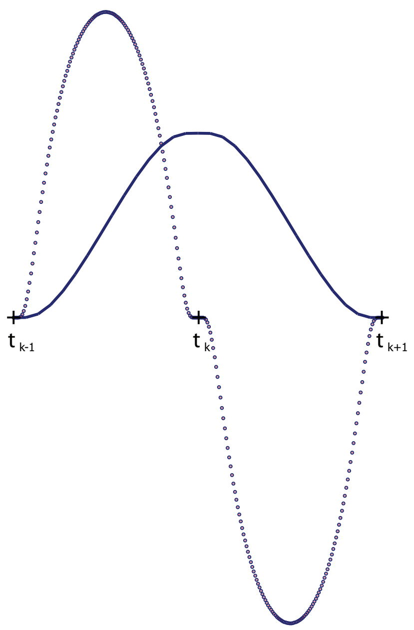

Figure 4: A graph of an expo-rational B-spline (ERBS) basis function $B_k(t)$ (solid blue) and its first derivative (dotted blue). The knots t_{k-1}, t_k and t_{k+1} are also marked on the plot. Notice that the graphs of the ERBS and its first derivative are very flat around the knots.

defined as follows:

$$B_k(t) = \begin{cases} S_{k-1} \displaystyle\int_{t_{k-1}}^{t} \psi_{k-1}(s)ds, & \text{if} \quad t \in (t_{k-1}, t_k], \\ S_k \displaystyle\int_{t}^{t_{k+1}} \psi_k(s)ds, & \text{if} \quad t \in (t_k, t_{k+1}), \\ 0, & \text{otherwise}, \end{cases}$$

with

$$S_k = \left[\int_{t_k}^{t_{k+1}} \psi_k(t)dt \right]^{-1},$$

and

$$\psi_k(t) = e^{-\frac{(t-\frac{t_k+t_{k+1}}{2})^2}{(t-t_k)(t_{k+1}-t)}}.$$

A plot of $B_k(t)$, $\psi_{k-1}(t)$ and $-\psi_k(t)$ is illustrated in Figure 4. In the figure $\psi_{k-1}(t)$ and $-\psi_k(t)$ are scaled so that the integral on their respective knot interval has value 1 for each of them.

Some basic properties of ERBS

- For $k = 1, 2, ..., n$, $B_k(t_k) = 1$ holds, while $B_k(t_j) = 0$ for all $j \neq k$ (which is also a property of the Lagrange polynomials and the piecewise affine B-splines).
- Every basis function $B_k(t)$ is positive on (t_{k-1}, t_{k+1}) and zero otherwise (which is also a property of the piecewise affine B-splines).
- The set of basis functions $B_k(t)$ for $k = 1, 2, ..., n$ forms a partition of unity on $(t_1, t_n]$. It follows that if $t_k < t \leq t_{k+1}$, then $B_k(t) + B_{k+1}(t) = 1$ (the same is true for piecewise affine B-splines).
- Every basis function $B_k(t)$ is C^∞-smooth on $\mathbb{R}$ (a new property, not shared by piecewise affine B-splines).
- All derivatives of all basis functions are zero at all knots t_i, $i = 0, \ldots, n$ (again a new property not shared by piecewise affine B-splines).

Note that $B_k(t)$ is defined on $\mathbb{R}$, and its support is (t_{k-1}, t_{k+1}), which is also the support of the piecewise affine B-spline, and actually the minimal possible support for continuous B-splines over the given knot vector to form partition of unity.

There is a more general definition of ERBS which is given in [6] and [25], but for this presentation it is enough to consider the default case given here.

2.2.1. Interpolation, Approximation and Numerical Linear Algebra Revisited

Let us consider the meaning of the term *multiplicity of a knot* in the case of ERBS, analogously to the consideration made for polynomial B-splines.

Raising by 1 the multiplicity of a knot in the case of polynomial B-splines of fixed degree leads to decreasing the support of the B-splines (since an interval of nonzero length becomes an interval of zero length), while the smoothness of the B-splines decreases, as well. Additionally, in order for the B-spline to be continuous, the smallest possible support is again the support of a linear B-spline. Smaller support is only possible in the case of discontinuity. Moreover, the multiplicity of the knot is equal to the order of Hermite interpolation plus 1.

In the case of ERBS, Hermite interpolation is achieved by the simple and elegant formula [6, (50)] which combines in a very lucid way the property of the Taylor polynomial to interpolate derivatives with the blending role of the ERBS-based infinitely smooth convex partition of unity. It is remarkable that formula [6, (50)] is valid, *mutatis mutandis*, for all types of data interpolation mentioned above, for any dimension and any order of the Hermite interpolation. Since in this case all the ERBS basis functions are always infinitely smooth and have the minimal compact support corresponding to the support of a piecewise affine B-spline, formula [6, (50)] shows that raising the order of the ERBS-based Hermite interpolation at a knot is obtained by multiplying the ERBS 'peaking'at this knot with a Taylor polynomial of higher degree.

The deep connection between the term 'multiplicity of a knot' in the polynomial B-spline case and the ERBS case can be revealed only after observing that (see [6, Section 4]) the derivative of ERBS (an expo-rational function) corresponds to the main term in the asymptotic expansion of a polynomial B-spline *with simple knots* when the number of the knots of the latter tends to infinity. This result was obtained in [6]) on the basis of upgrading the classical technique of Daniels [3, 6]. We shall return on the topic about multiplicity of the interpolation knots for GERBS after the definition of GERBS is given in the next section. Here it remains to estimate the complexity of ERBS-based Hermite interpolation and to compare it to the case of polynomial B-splines (see item 2.1.3 above).

Thanks to formula [6, (50)], ***the generalized Vandermonde matrix of ERBS-based Hermite interpolation is always in block diagonal form which is, moreover, a Jordan normal (canonical) form.*** This is true for all cases of interpolation problems mentioned in item 2.1.3, including Hermite interpolation on scattered data sets for triangulated high-dimensional domains. It is true also *regardless of the topology of the parametric domain or the parametric manifold itself.*

Similar to item 2.1.3, let us consider in detail the most interesting case of

Hermite interpolation in the vertices of a triangulation in n dimensions. The number M of blocks on the main diagonal is equal to the number of different vertices in the triangulation (recall that they are exactly the ones forming the original scattered point data set). The size of each diagonal block depends only on the geometric dimension n of the problem and the (possibly different for different vertices) multiplicity μ_i of the Hermite interpolation problem at the $i-th$ vertex, $i = 1, \ldots, M$. Because the generalized Vandermonde matrix is block diagonal, its inversion is reduced to M completely decoupled problems ($i = 1, \ldots, M$) for inverting a diagonal block. Because these blocks are Jordan cells, they are upper/lower triangular, and every analytic function of this block (in particular, its resolvent) can be be computed in closed form as an upper/lower triangular matrix. Hence, the complexity of the inversion of the $i - th$ Jordan block is proportional to

$$\sum_{j=1}^{\mu_i} j = \frac{\mu_i(\mu_i + 1)}{2}.$$

(Note also that this ERBS-based matrix inversion algorithm is readily parallellized and very easy to implement on a parallel computer, much unlike the case of polynomial B-splines.)

Let us estimate the complexity of Hermite interpolation in the same problem about a uniformly triangulated n-dimensional domain which was considered in item 2.1.3. Under the same assumptions, and using the same notation as in item 2.1.3, the following estimates are obtained.

The overall size N of the square generalized Vandermonde matrix is the same as earlier, $N \sim M.n^k$ (see item 2.1.3).

Due to the uniformity of the triangulation and the uniformity of the total order k of Hermite interpolation, all multiplicities μ_i, $i = 1, \ldots, M$, are the same, more precisely,

$$\mu_i = \mu \sim n^k, \;\; i = 1, \ldots, M.$$

Since the generalized Vandermonde matrix is in Jordan normal form, its inversion will be obtained by the inversion of M Jordan blocks, each with dimensions proportional to $n^k \times n^k$. Thus, we get for ERBS the following:

$$Complexity(ERBS, sequential,\ Hermite\, interpolation,\ n,\ k,\ M) \sim$$

$$\sim M.\frac{n^k(n^k + 1)}{2} \sim n^{2k}.M.$$

.

Comparing this estimate to the respective estimate of the complexity of

Hermite interpolation using classical polynomial B-splines (see item 2.1.3) yields the following ratio:

$$\frac{Complexity(ERBS, sequential,\ Hermite\ interpolation,\ n,\ k,\ M)}{Complexity(polyBspl, sequential,\ Hermite\ interpolation,\ n,\ k,\ M)} \sim$$
$$\sim \frac{n^{2k}.M}{(n!)n^{3k}.M^2} \sim \frac{1}{(n!)n^k.M}\ ,$$

which shows that for the considered problem ERBS tremendously outperforms classical B-splines, and the advantage of using ERBS increases in a 'monstrously drastic way' with the increase of each of the key parameters: number M of interpolation knots, number n of geometric dimension, and number k of total order of Hermite interpolation.

Note that the above rough comparison of the complexity for the polynomial and expo-rational class of B-splines holds only for the case of *essential Hermite interpolation*, i.e., $k \geq 1$ (see also the relevant comments about this in item 2.1.3).

Using Stirling's formula and refinements of it via taking more terms in the Euler-Maclaurin asymptotic expansion, it is possible to relate this tremendous superiority of ERBS to the fact (again!) that ERBS are asymptotic limits of polynomial B-splines with simple knots, as the number of knots / spline degree of the latter tends to infinity. Thus, we have just observed one more instance of a 'superproperty' exhibited by ERBS in comparison with classical polynomial B-splines.

3. Generalized Expo-Rational B-Splines (GERBS)

Consider again a strictly increasing knot vector $T = T_{n+1} = \{t_k\}_{k=0}^{n+1}$ where $-\infty \leq t_0 < t_1 < \cdots < t_{n+1} \leq +\infty$, $n \in \mathbb{N} \cup \{+\infty\}$.

We propose the following general definition.

Definition 3. Consider the system $\{F_i\}_{i=1}^{n+1}$ of *cumulative distribution functions* (see, e.g., [16, 17] and the references therein) (CDF, for short) such that F_i is supported on the interval span $[t_{i-1}, t_i]$, i.e.,

1. the left-hand limit $F(t_{i-1}+) = F(t_{i-1}) = 0$,
2. the left-hand limit $F(t_i+) = F(t_i) = 1$,

3. $F(t) = 0$ for $t \in (-\infty, t_{i-1}]$,

4. $F(t) = 1$ for $t \in [t_i, +\infty)$, and $F(t)$ is monotonously increasing, possibly discontinuous, but left-continuous for $t \in [t_{i-1}, t_i]$.

5. The j-th (g)eneralized (e)xpo-(r)ational (B)-(s)pline (GERBS), is defined, as follows.

$$G_j(t) = \begin{cases} F_j(t), & \text{if} \quad t \in (t_{j-1}, t_j], \\ 1 - F_{j+1}(t), & \text{if} \quad t \in (t_j, t_{j+1}), \\ 0, & \text{if} \quad t \in (-\infty, t_{j-1}] \cup [t_{j+1}, +\infty), \end{cases} \tag{1}$$

$j = 1, \dots, n.$

Note that both piecewise affine B-splines and ERBS satisfy the conditions of this definitions, hence, GERBS is, indeed, a simultaneous generalization of both piecewise affine B-splines and ERBS.

The following properties of GERBS are immediate consequences of Definition 3.

Some basic properties of GERBS

For $k = 1, 2, ..., n$,

1. $0 \leq G_k(t) \leq 1$, for every $t \in \mathbb{R}$.

2. $G_k(t)$ is non-negative on (t_{k-1}, t_{k+1}) and zero otherwise.

3. G_k is left-continuous at every knot t_j, $j = 0, \dots, n+1$ (continuous for $j \neq k$, $j \neq k+1$).

4. For $j = 1, 2, ..., n$, $B_k(t_j) = \delta_{jk}$ holds, where δ_{jk} is the Kronecker's delta.

5. $G_k(t)$ is strictly positive on a closed segment contained in the interior of (t_{k-1}, t_{k+1}), with positive length, and containing t_k (possibly as its left endpoint), because it is non-negative on (t_{k-1}, t_{k+1}) and left-continuous and equal to 1 in the point t_k which is interior point for (t_{k-1}, t_{k+1}).

6. G_k has 1st derivative almost everywhere on $\mathbb{R}$, because it is piecewise monotone. Where it exists, the derivative is non-negative in (t_{k-1}, t_k), non-positive in (t_k, t_{k+1}), and zero elsewhere.

7. The functional system $G_j(t)$ for $j = 1, 2, ..., n$ forms a convex partition of unity on $[t_1, t_n]$. Moreover, if $t_j \leq t \leq t_{j+1}$, then $G_j(t) + G_{j+1}(t) = 1$. This property follows easily from the definition of GERBS and some of the the previous properties on this list.

8. G_k is a function with bounded (Jordan) variation on $\mathbb{R}$, because it is bounded, piecewise monotone, with only finite number of intervals of monotonicity (4 intervals), and it is constant on those of its intervals of monotonicity which are not compact.

9. Let P_k, $k = 1, 2, ..., n$, be points in $\mathbb{R}^d$, $d = 2, 3, \ldots$, where $\mathbb{R}^d$ is being considered as an affine space (e.g., by defining barycentric coordinates with respect to the $d+1$ simplex vertices Q_0 (coinciding with the origin) and Q_i, $i = 1, \ldots, d$ (coinciding with the endpoints of the unit orts with common initial point at Q_0)). In the so-defined d-dimensional affine space, consider the following curve with parametrization

$$P(t) = \sum_{k=1}^{n} P_k G_k(t), \quad t \in \mathbb{R}. \tag{2}$$

Then,

(i) G_k, $k = 1, 2, ..., n$, form a convex partition of unity for $t \in [t_1, t_n]$;

(ii) the right-hand side of (2) defines a GERBS-based interpolation operator which Lagrange–interpolates at the knots t_j, $j = 1, \ldots, n$;

(iii) the graph of the curve is the closed piecewise affine polygon beginning and ending at the origin Q_0, and having as consecutive vertices the coefficient points P_k, $k = 1, 2, ..., n$, of the convex barycentric combination in (2);

(iv) this graph is independent of the concrete choice of the GERBS convex partition of unity for $t \in [t_1, t_n]$ in (2).

Note that $G_k(t)$ is defined on $\mathbb{R}$, and its support is (t_{k-1}, t_{k+1}), which is also the support of the piecewise affine B-spline, and actually the minimal possible support for continuous B-splines over the given knot vector to form partition of unity.

Now Definition 3 can be reformulated, briefly but rigorously, as follows.

GERBS ***is any piecewise monotone reparametrization of a piecewise affine B-spline which preserves the intervals of monotonicity of the latter, as well as the range of the latter in each of these intervals.***

4. Additional Heuristic Motivation for Introducing GERBS

In [6] it was explained that the heuristic motivation for the introduction of ERBS comes from important similarities in several celebrated mathematical constructions originating in approximation theory, differential geometry and operator theory. In Section 3 of [6] four general heuristic aspects were listed, as follows:

I. Density of C^∞ and C_0^∞ in distribution spaces.

II. Differentiable manifolds.

III. The Riesz-Dunford integral representation.

IV. Carleman inequalities for operator resolvents.

Recent progress in the study of ERBS suggested that for certain new theoretical and practical problems (about interpolation and approximation of functions and operators which do not demand that the basis functions of the approximating finite-dimensional space have very high order of smoothness) it is of interest to extend the class of ERBS to GERBS, as defined in the previous section. Here are the most important instances of such newly considered problems.

1. **Tradeoff between reducing the range of validity of the ERBS 'superproperties' and computability in closed form.** ERBS provide the possibility to use Taylor series as local functions, which results in transfinite Hermite interpolation. Most of the time we do not need that much, however. It is, therefore, of interest to study special functions which retain the Hermite interpolation properties only up to some finite order, but are otherwise simpler than the expo-rational ones. For example, it is of special interest to find out if there can be designed polynomial classes of such functions. The use of integration in the definition of ERBS suggests that if there is an appropriate polynomial class of such functions, it will be possible to compute the elements of this class explicitly, in closed

form, without using numerical integration. (The Euler Beta function B-splines, considered in Section 5.4.2 below, provide a positive answer to this question.)

2. **Numerical methods for solving operator equations.** In many cases when solving integral equations by finite elements/volumes, the functions from the finite element/volume basis do not have to be very smooth; in fact, often they can be even discontinuous. At the same time, the possibility for easy explicit integration and the preservation of properties such as positivity, monotonicity and other order- and shape-preserving constraints is of considerable interest. These properties are possessed by GERBS of any regularity, and GERBS with low regularity would be used most of the time.

3. **Subdivision, multilevel splines and wavelets.** For subdivision, when adding new knots to the knot vector, it is necessary to be able to keep the interpolant unchanged, before starting to modify the coefficients corresponding to the new, additional, degrees of freedom. This is possible to do with ERBS *only if* the local functions used as coefficients of the ERBS basis functions are Taylor *series* (which form an infinite-dimensional space of analytic functions). This is certainly a construction of considerable theoretical interest, but for practical computations using computers we need to limit our considerations only to finite-dimensional spaces of local functions. (The Euler Beta function B-splines considered in Section 5.4.2 below provide an answer to this question, too.) Certainly, also other bases which are integrable in quadratures can be used in the place of algebraic polynomials: trigonometric polynomials, exponential functions, certain special functions, etc.

4. **Constrained deterministic approximation versus constrained statistical (indeterministic) estimation**

   (a) **Asymptotic rates.** The classical Korovkin theory [21] of approximation by positive operators postulates some fundamental limitations on the maximal order of approximation which can be attained by approximations preserving positivity. This early result of Korovkin has been followed by many studies on approximation rates for various types of constrained approximation (see, e.g., [2, 26], as well as [20] and the references therein). All of these very diverse results can be united together by some important common observations, one of which is that, under other equal conditions, the

rates of constrained approximation are worse (often – much worse) than the rates of free, unconstrained, approximation. Intuitively, the reason for this deterioration of the rates is quite clear: best unconstrained approximation is achieved via *oscillation* around the exact function, while constraints such as positivity, onesidedness, monotonicity, convexity, etc., make oscillating approximations inadmissible. In relevance to this, the following important observation can be made. While in approximation, which is deterministic in nature, the introduction of shape and/or order constraints of some of the types mentioned above is crucial for the deterioration of the approximation rates, in non-parametric statistical estimation, which is indeterministic in nature, the situation is very different and, in a way, much more beneficial for the introduction of order and/or shape constraints of the afore mentioned types. The reason for this is that deterministic approximation theory is responsible only for one part of the results in statistical estimation - the estimation of the *bias* of the statistical estimator. There is a second, equally important part of the statistical estimate: the estimation of the *variance* of the statistical estimator. A typical situation in statistical estimation is that estimators which have very small bias (or are even *unbiased*) tend to have large variance, and vice versa. The problem is solved by finding respective estimators which provide a good tradeoff between bias and variance. It turns out that *the statistical risk estimation rates which offer optimal tradeoff between variance and bias are well within the limitations imposed by the classical Korovkin theory of order-preserving approximation*. This means that in statistical estimation it is of much greater interest and higher priority, compared to deterministic approximation, to design new sufficiently simple order-preserving estimators whose smoothness does not exceed the Korovkin constraints. With this in mind, in the last 15 years the first author of the present work conducted several studies of order and shape preserving estimators, the results of which are summarized in [16, 17, 29, 9, 13, 10, 14, 18, 11, 12, 4, 15]. Based on the results obtained in these papers, the following general results can be formulated. ***For any, generally unconstrained, statistical estimator (of a function satisfying certain order/shape constraints) which achieves the asymptotic-minimax optimal rates of risk estimation, it is possible to construct a new estimator which obeys all prescribed constraints and also***

attains the same asymptotic-minimax optimal rates (possibly, with a worse constant factor in the O-estimate). The great importance of order/shape-preserving statistical estimation for the applications of statistics is by now beginning to be universally acknowledged. For example, in [28], where [16] was cited, Daniel McFadden explicitly emphasized the importance of the application of shape-preserving estimation for statistics-based economical studies. It remains to note that the estimator from [16] which [28] was referring to, is a low-regularity GERBS, according to our present Definition 3. We agree with [28] that the estimator from [16] needs to be upgraded in the multivariate case. In fact, this recommendation is valid for the theory of GERBS in general, and the main results announced in [6, 8, 5] serve exactly to this purpose.

(b) **Main term of the asymptotics of polynomial B-splines.** In [6, Section 4] was discussed the early paper [3] dedicated to shape-preserving asymptotic methods of approximation of the joint distribution density of a large sample of random variables. In particular, it was explained how, thanks to cooperation with Charles A. Micchelli in 1997–1999, the important relevance of this paper for the asymptotic theory of polynomial B-splines (when the number of knots of the latter tends to infinity) was discovered. This discovery was even more surprising due to the fact that in a sequence of papers [30, 31, 32, 33, 34] the founder of spline theory I. J. Schoenberg has also addressed, with considerable emphasis, the topic of the asymptotic limits of polynomial B-splines as the number of their knots tends to infinity. Comparing the results of Daniels and Schoenberg shows that the main term in the asymptotic expansion of the kind developed by Daniels involves an infinitely smooth function of expo-rational type. The reason for this is that this type of asymptotics has been studied so far (see Subsections 4.2 and 4.3 in [6]) only for B-splines *with simple knots.* It is possible, in principle, to extend this approach also for the case of coalescing knots of higher multiplicity while still obtaining infinitely smooth function in the main term of the asymptotic expansion, but *only if the maximal knot multiplicity is finite and remains uniformly bounded as the number of knots (this time also counting their multiplicities) tends to infinity.* The approach of Schoenberg, on the other hand, is much more general from the very beginning (see [30]): he studies also the case of coalescing knots; moreover, his approach does not exclude the case when the

knot multiplicity may tend to infinity. Under these very general assumptions, the main term in the asymptotics may be any (possibly non-smooth, or even discontinuous) *probabilistic measure* characterized by a respective *cumulative distribution function* appearing in Riemann-Stieltjes integration (or, if an even more general weak distributional asymptotic theory is considered, Lebesgue-Stieltjes integration and the various upgrades of the duality bilinear form in distribution spaces). It seems, however, that this general approach of Schoenberg has not found notable response in the splinist community so far, perhaps because, at first glance, these results seem too general to be informative from any useful practical point of view. To the best of the author's knowledge, it is now and here, that *the introduction of GERBS makes a first attempt for practical use of the general approach of Schoenberg in [30, 31, 32, 33, 34] to the study of asymptotic limits of polynomial B-splines when the number of knots of the latter tends to infinity.* Indeed, another reformulation of Definition 3 can (loosely) be given, as follows: ***GERBS are all possible limit functions with bounded variation appearing in the main term of the asymptotics of polynomial B-splines, as the number of their knots tends to infinity, counting also multiplicities, and not excluding the case when multiplicities may also tend to infinity.*** This presentation of the definition of GERBS shows that the extension of ERBS to GERBS is heuristically consistent: ERBS were introduced as asymptotic limits of polynomial B-splines with simple knots with arbitrary distribution, as an application of the early results of Daniels; GERBS, on their part, are now being introduced as asymptotic limits of polynomial B-splines with possibly multiple knots with arbitrary distribution, as an application of the early results of Schoenberg. The fact that both of these asymptotics have certain order-/shape-preserving properties, ensures that GERBS is an order- and shape-preserving extension of ERBS which is consistent in the sense of asymptotic theory of polynomial B-splines.

(c) **Main term of the asymptotics when estimating CDFs.** In [17] we considered the following problem: *given a sample of independent, identically distributed (i.i.d) random variables with unknown CDF F; the task is to estimate F from the sample; the problem is to estimate the risk.* As we showed in [17, Theorem 2.1.1], the main term in the asymptotics for the risk always contains as multiplicative

factor the following invariant of F: F(1-F). The quadratic functional $F \mapsto F(1-F)$ has very interesting properties which deserve a lot of attention. For *continuous* CDFs it makes sense to 'pre-normalize' it as follows: 4F(1-F), which has some interesting properties, for example,

- $0 \leq 4F(x)(1-F(x)) \leq 1, \ \forall x \in \mathbb{R}$;
- $\forall \xi \in [0,1] \ \exists x_\xi \in \mathbb{R}: \ 4F(x_\xi)(1-F(x_\xi)) = \xi$;
- $4F(1-F)$ peaks at 1 at the median(s) of F.

These properties can be useful in a number of ways, for instance, in subdivision, for generation of absolutely continuous GERBS from an arbitrary GERBS, and others.

5. Model Examples of Types of GERBS

To begin with, we note that the invariant $F(1-F)$ of a given CDF F, in the sense considered in Section 4 item 4(c), suggests the use of the following equivalent representation of G_j, as defined in (1):

$$G_j(t) = F_j(t)(1 - F_{j+1}(t)), \quad t \in \mathbb{R}. \tag{3}$$

The equivalence of (1) and (3) can be verified directly.

As we shall see, formula (3) proves to be more convenient to use most of the time and, in particular, when dealing with smooth GERBS.

5.1. Discontinuous GERBS

1. The characteristic function $\frac{1}{2a}\chi_{[\tau-a,\tau+a)}(t), \ t \in \mathbb{R}$ where $0 < a < \infty$, $\tau \in \mathbb{R}$, with numerous applications. Some examples:

   (a) piecewise constant B-splines (of degree 0 / order 1 – see Figure 1(a));

   (b) basis functions in Rademacher and Walsh bases;

   (c) scaling functions in Haar wavelet bases;

   (d) components in the representation of martingales;

   (e) Fourier images of sinc-basis functions in Shannon's sampling formula (with appropriate modification of the rule about left continuity), etc.

2. More generally, all possible discontinuous GERBS are defined by all possible families $\{G_j\}$ satisfying the assumptions of Definition 3, for which at least one F_j among the ones appearing in (3) has discontinuities in the respective interval (t_{j-1}, t_j).

Remark 1. We note that GERBS can have only *discontinuity points of the first kind* (i.e., left-hand and right-hand limits exist but may be unequal to each other and/or to the functional value) because GERBS has always bounded Jordan variation.

5.2. Continuous Non-Smooth GERBS

1. Piecewise affine B-splines (of degree 1 / order 2) with knots t_j, $j = 1, \dots, n$ (see Figure 1(b)).

2. More generally, piecewise polynomial GERBS (of possibly variable degree/order in (t_{j-1}, t_j), $j = 1, \dots, n$, without continuity at some or all of the knots t_j, $j = 1, \dots, n$.

3. More generally, piecewise polynomial GERBS (of possibly variable degree/order between adjacent knots) over a 'more detailed' knot vector over the segment $[t_1, t_n]$, containing as essential subset the knot vector $\{t_j\}$.

4. Most generally, all possible continuous non-smooth GERBS are defined by all possible families $\{G_j\}$ satisfying the assumptions of Definition 3, such that for at least one F_j among the ones appearing in (3) there is at least one $t_{0j} \in [t_{j-1}, t_j]$ such that $F_j'(t_{0j})$ does not exist. This may happen in diverse ways, e.g.,

   (a) left-hand and/or right-hand derivative at t_{0j} does not exist;

   (b) left-hand and right-hand derivative at t_{0j} exist but are not equal to each other.

5.3. Absolutely Continuous GERBS

In this case, for every F_j, $j = 1, \ldots, n$, in Definition 3 there exists Lebesgue-a.e. on $\mathbb{R}$ respective $f_j \in L_1(\mathbb{R})$ with support $\operatorname{supp} f_j \subset [t_{j-1}, t_j]$, so that

$$F_j(t) = \int_{-\infty}^{t} f_j(\tau) d\tau, \quad t \in \mathbb{R}. \tag{4}$$

In this case, formula (3) provides also the useful identity

$$G_j' = F_j'(1 - F_{j+1}) - F_j F_{j+1}' = f_j(1 - F_{j+1}) - F_j f_{j+1} \quad \text{Lebesgue-a.e. on } \mathbb{R}. \tag{5}$$

The examples, considered in items 1, 2, 3 in Section 5.2, are all instances of absolutely continuous non-smooth GERBS. These examples show, in particular, that absolute continuity of GERBS alone does not imply existence of the derivatives of GERBS in the knots t_j, $j = 1, \ldots, n$.

Another example which gives evidence of the same, but is interesting in itself and has potential applications of its own, is the following one. Let us consider again the conditions of Definition 3 in full generality, i.e., F_j in the notations of Definition 3 and formula (3) is a general CDF, not necessarily absolutely continuous. By Definition 3 and formula (3), construct the respective GERBS family $\{G_j\}$.

Next, let us construct from F_j a new CDF, Φ_j, which is absolutely continuous:

$$\Phi_j(t) = S_j \int_{-\infty}^{t} F_j(\tau)(1 - F_j(\tau)) d\tau, \quad t \in \mathbb{R}, \tag{6}$$

where

$$S_j = \left(\int_{-\infty}^{\infty} F_j(\tau)(1 - F_j(\tau)) d\tau \right)^{-1} = \left(\int_{t_{j-1}}^{t_j} F_j(\tau)(1 - F_j(\tau)) d\tau \right)^{-1}. \tag{7}$$

Now it is possible to construct a new GERBS family $\{\Gamma_j\}$ over the same knots, generated from the new family of absolutely continuous CDFs $\{\Phi_j\}$. The new family $\{\Gamma_j\}$ is generally different from $\{G_j\}$, and Γ_j, $j = 1, \ldots, n$, are all absolutely continuous.

5.4. C^m-Smooth GERBS, $m \in \mathbb{N}$

For the C^m-smooth (i.e., the m times continuously differentiable) GERBS, $m \in \mathbb{N}$, formulae (4, 5) hold with $f_j,\ f_{j+1} \in C^{m-1}(\mathbb{R})$, and these GERBS have some new remarkable properties, as follows.

1.
$$G_j^{(\mu)} = \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} F_j^{(\nu)} (1 - F_{j+1})^{(\mu-\nu)} \quad \text{everywhere on } \mathbb{R}, \tag{8}$$
where $G_j^{(\mu)} \in C^{m-\mu}(\mathbb{R})$, $\mu = 1, \ldots, m$.

2. In view of the existence and continuity of all derivatives $G_j^{(\mu)}(t)$ for every $t \in \mathbb{R}$, $\mu = 1, \ldots, m$, the following property holds true:
$$G_j^{(\mu)}(t_k) = 0, \quad k = 1, \ldots, n,\ j = 1, \ldots, n,\ \mu = 1, \ldots, m, \tag{9}$$
which follows easily from the fact that the left-hand derivatives $G_j^{(\mu)}(t_{j-1}-)$ at the left endpoint of $[t_{j-1}, t_j]$ and the right-hand derivatives $G_j^{(\mu)}(t_j+)$ at the right endpoint of $[t_{j-1}, t_j]$ of order $\mu = 1, \ldots, m$ are all equal to zero and, in view of the C^m-smoothness, all of the respective quantities $G_j^{(\mu)}(t_{j-1}+)$, $G_j^{(\mu)}(t_{j-1})$ and $G_j^{(\mu)}(t_j-)$, $G_j^{(\mu)}(t_j)$ exist and are equal to 0; for the knots t_k, $k \neq j$, the proof is much simpler.

3. More generally, assume that $F_j \in C^{m_j},\ m_j \in \mathbb{N},\ j = 1, \ldots, n$ and for every $j = 1, \ldots, n$ define $\mu_j \in \mathbb{N}$, as follows: $\mu_j = \min\{m_j, m_{j+1}\}$ (with obvious modification for $j = 1, n$). Then, (9) holds in the following more general form, which guarantees only uniform C^m-smoothness of the GERBS for
$$m = \min\{m_1, \ldots, m_n\} = \min\{\mu_1, \ldots, \mu_n\} \tag{10}$$
but locally at every knot
$$G_j^{(\mu)}(t_k) = 0, \quad j = 1, \ldots, n,\ \mu = 1, \ldots, \mu_k,\ k = 1, \ldots, n. \tag{11}$$
For this type of GERBS the Lagrange interpolation via formula (2) generalizes and upgrades to Hermite interpolation up to order $\mu_j - 1$ at the knot t_j, $j = 1, \ldots, n$, via the following.
$$P(t) = \sum_{k=1}^{n} P_k(t) G_k(t), \quad t \in \mathbb{R}, \tag{12}$$

where $P_k(t)$ is the Taylor polynomial in powers of $t - t_j$ of degree $\mu_j - 1$:

$$P_k(t) = \sum_{j=0}^{\mu_k - 1} P_{jk} \frac{(t - t_k)^j}{j!} \quad t \in \mathbb{R}, \tag{13}$$

and where

$$P_{jk} = P^{(j)}(t_k), \quad j = 0, \ldots, \mu_k - 1, \; k = 1, \ldots, n. \tag{14}$$

The graph of the resulting curve is already both C^m-smooth and G^m-smooth, where the order of smoothness is given in (10).

Remark 2. Note the remarkable fact that for GERBS which are C^1-smooth, the hierarchy of the concepts of G-smoothness and C-smoothness becomes reversed compared to their usual hierarchy in computer-aided geometric design: while, say, for polynomial B-splines G-smoothness of a certain order is a more general concept than C-smoothness of the same order, in the case of GERBS we have the remarkable fact that ***the order of level of generality between these two concepts is reversed***. This is due to the fact that C^1-smooth GERBS ***are NOT regular at the knots***.

Remark 3. Another remarkable fact is that the procedure described via (6, 7) can be reiterated. Considering the procedure, as described in (6, 7), to be the first iteration, the result on the second iteration will generate a GERBS family with a.e. existing and absolutely continuous *second* derivative, i.e., this will be a C^1-smooth GERBS family over the same knot-vector. Continuing like this, after the ν-th iteration, $\nu \in \mathbb{N}$, a $C^{\nu-1}$-smooth GERBS family with a.e. existing and absolutely continuous ν-th derivative will be generated over the same knot-vector. (In fact, the ν-th derivative will have even better properties: it will have bounded Jordan variation.)

Some examples of C^m-smooth GERBS, $m \in \mathbb{N}$, follow, including also case 3 of the last list (non-uniform index μ_j of smoothness in (t_{j-1}, t_{j+1})), see Sections 5.4.1, 5.4.2.

5.4.1. C^1-Smooth Absolutely Continuous GERBS, and Increase of the Smoothness to C^m, $m = 2, 3, \ldots$

Given is the strictly increasing knot vector $\{t_k\}_{k=0}^{n+1}$, $t_k \in \mathbb{R}$ for $k = 0, 1, 2, ..., n+1$.

Definition 4. An absolutely continuous GERBS $G_k(t)$, associated with

three strictly increasing knots t_{k-1}, t_k and t_{k+1}, is defined (see Section 5.3) by

$$G_k(t) = \begin{cases} S_k \displaystyle\int_{t_{k-1}}^{t} \psi_k(s)ds, & \text{if} \quad t \in (t_{k-1}, t_k], \\ S_{k+1} \displaystyle\int_{t}^{t_{k+1}} \psi_{k+1}(s)ds, & \text{if} \quad t \in (t_k, t_{k+1}), \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

with

$$S_k = \left[\int_{t_{k-1}}^{t_k} \psi_k(t)dt \right]^{-1}, \tag{16}$$

where ψ_k is defined Lebesgue–a.e. on $(-\infty, \infty)$ and supp $\psi_k \subset [t_{k-1}, t_k]$, with

$$\psi_k \in L_1(\mathbb{R}) \cap L_1[t_{k-1}, t_k].$$

For the absolutely continuous GERBS G_k to belong to C^1, $k = 1, 2, ..., n$, $\psi_k(t)$, $k = 1, 2, ..., n+1$, must have the following additional property:

$$\psi_k \text{ is continuous everywhere on } \mathbb{R}. \tag{17}$$

In particular, this implies,

1. ψ_k is continuous on (t_{k-1}, t_k) and $\psi_k(t) > 0$ on a subinterval of (t_{k-1}, t_k) with positive length;
2. ψ_k is continuous at t_{k-1} and t_k;
3. $\lim\limits_{t \to t_{k-1}+} \psi_k(t) = \lim\limits_{t \to t_k-} \psi_k(t) = 0$;
4. $\psi_k(t) = 0 \ \forall t \in (-\infty, t_{k-1}] \cup [t_k, +\infty)$.

Remark 4. Of main interest is the case when ψ_k is not only continuous, but also has some additional smoothness and property (17), together with the implied properties 1–4, holds not only for ψ_k but also for derivatives of ψ_k up to certain order. See Figure 5 for a C^m-smooth GERBS where m is high: this results in very flat graph of the GERBS and its first derivative near the knots; compare with the second graph (from the left) on Figure 7 which is only C^1-smooth (in the latter case, notice also the complete absence of flatness around the central knot t_k of the first derivative of the GERBS).

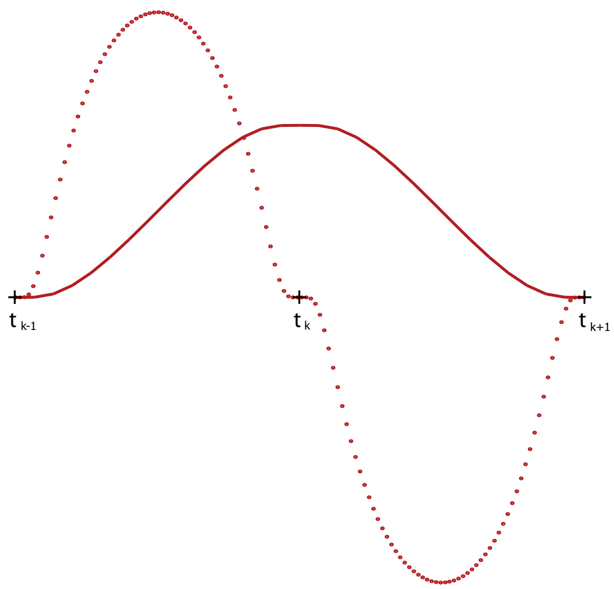

Figure 5: A graph of an example of an absolutely continuous GERBS basis function $G_k(t)$ (solid red) with considerable additional smoothness of its first derivative (dotted red). The knots t_{k-1}, t_k and t_{k+1} are also marked on the plot. The flatness of the graph and its first derivative around the knots is controlled by its smoothness.

5.4.2. Euler Beta Function B-Splines

Euler Beta-function B-splines (BFBS or EBFBS) are an important example of the GERBS considered in Section 5.4.1 and satisfying (15, 16, 17) (see Figure 6).

ERBS (see [6] and Section 2.2 here) also satisfy the minimal conditions (15, 16, 17) for smoothness but in this 'super B-spline' case $\psi_k \in C^\infty$. This is because the derivative ψ_k of an ERBS between the consecutive knots t_j, t_{j+1} : $t_j < t_{j+1}$ of a strictly increasing knot-vector is either identically zero (for $j \neq k$) or (for $j = k$) it is an expo-rational function (i.e., a function which is the exponent of a rational function taking negative values for $t :\ t_k < t < t_{k+1}$ and having poles at t_k and t_{k+1}). The computation of the integral in the definition of ERBS is by fast-converging numerical quadratures.

BFBS are an instance of GERBS where some tradeoff has been made be-

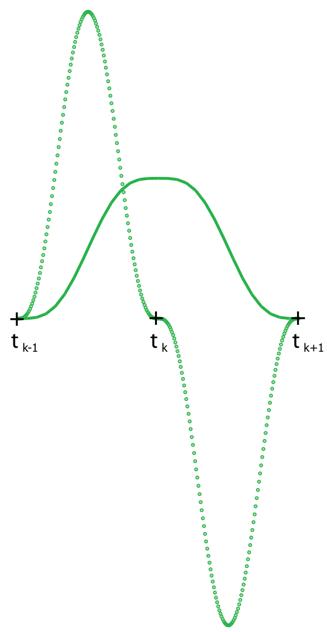

Figure 6: A graph of an Euler Beta-function B-spline basis function $G_k(t)$ (solid green) and its first derivative (dotted green). The knots t_{k-1}, t_k and t_{k+1} are also marked on the plot. This graph is not so flat around the knots as that of an ERBS (see Figure 4), relevant to the lower order of smoothness of the former.

tween the properties and the ease of computation. In the case of BFBS, the expo-rational ψ_k in the definition of the derivative of ERBS is replaced by a Bernstein polynomial rescaled to the interval $[t_k, t_{k+1}]$ (see Figure 7). Thus, between the knots a BFBS is a piecewise polynomial function which can be computed exactly without using numerical integration.

While multiplying ERBS with a Taylor series in powers of $t - t_k$ of an analytic function makes sense and has the effect of transfinite Hermite interpolation (cf. [6, formula (50)]), the same can be done with the BFBS only with a Taylor polynomial of degree not exceeding the multiplicity of t_k as a zero of the respective Bernstein polynomial (see (12, 13, 14) and Figure 7). What is lost in the range of this important property is compensated in the ease of computation of the BFBS which is piecewise polynomial.

The name of Euler Beta-function B-splines comes from the relevance of BFBS as special functions to the incomplete Euler Beta function.

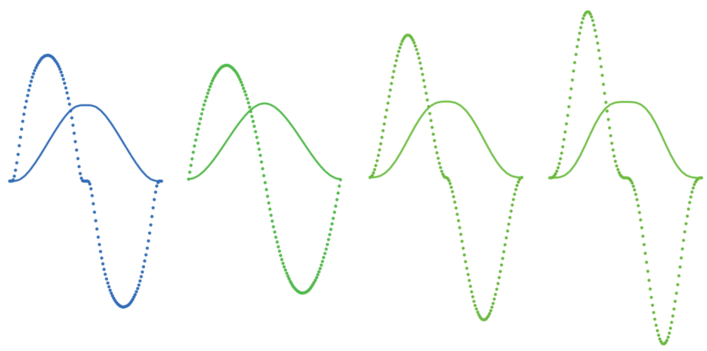

Figure 7: ERBS (leftmost, in blue) and C^1-, C^2-, C^3-smooth Euler Beta-function B-splines (EBFBS) (in green). The Euler Beta-function B-splines given in Figure 6 is the rightmost in Figure 7. It is seen how increasing of the order of smoothness of the EBFBS results in increasing the flatness around the knots of the B-spline and its derivative(s). The ERBS is C^∞-smooth and, respectively, much flatter around the knots than any of the EBFBS. At the same time, the 'peak' of the ERBS is better localized than the 'peak' of a smooth EBFBS (compare the ERBS on the leftmost picture with the EBFBS on the rightmost picture) because graphs of exponential functions can 'bend' much faster than graphs of polynomial ones.

The Hermite interpolation property of (12–14) extends from parametric curves using Taylor-polynomial local curves to parametric tensor-product surfaces using Taylor-polynomial tensor-product local patches. This makes BFBS very easy and convenient to use in geometric modelling (see example in Figure 8: the surface in the figure is an Hermite interpolation of a Klein's Bottle at 12×12 points.

A new and promising feature of the new type of B-splines is the affine transformation of local functions (curve arcs, surface patches, etc.). This provides a convenient framework of using BFBS both for geometric design using Bezier-type techniques and for simulations using finite/boundary element analysis. In Figure 9 is given an example of geometric modelling and editing. The surface is a tensor product Euler Beta-function B-spline surface, made by first interpolating a given surface with parametrization which is not algebraic polynomial (the Sin surface). Then the surface is edited to new geometric shape. For this purpose, an affine transformation of the local tensor-product patches has been

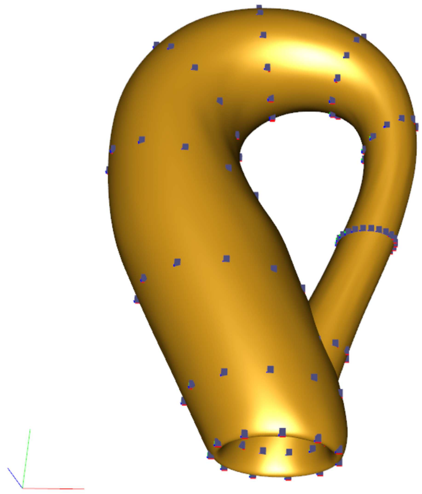

Figure 8: This tensor product Euler Beta-function B-spline surface is made by Hermite interpolation of an appropriate "Klein's Bottle" parametrization at 12×12 points. The positions of the interpolating points are seen as blue cubes.

made by converting the shifted monomial bases appearing in the local Taylor polynomial tensor-product patches to respectively shifted and scaled Bernstein polynomial bases. The small cubes, which can be seen in Figure 9, are the 3D locations of the resulting Bezier coefficients (points in 3D) of the local tensor-product patches which can be used (by 'clicking' on them on the screen while applying rigid transformations (shifts and rotations) on the whole surface) as Bezier-type editing points which are used for affine transformation of individual local tensor-product patches, thereby geometrically editing the shape of the entire surface.

5.5. Infinitely Smooth GERBS

In this case, the CDFs F_j in Definition 3, formula (3) and Section 5.4 are infinitely smooth; the same refers to the densities ψ_j: $F_j \in C^\infty(\mathbb{R})$, $\psi_j \in C^\infty(\mathbb{R})$, $\quad j = 1, \ldots, n+1$. The following property of F_j, ψ_j, $j = 1, \ldots, n+1$, needs to be noted:

Property (*): ***At the knots*** t_{j-1}, t_j ***forming the boundary of the sup-***

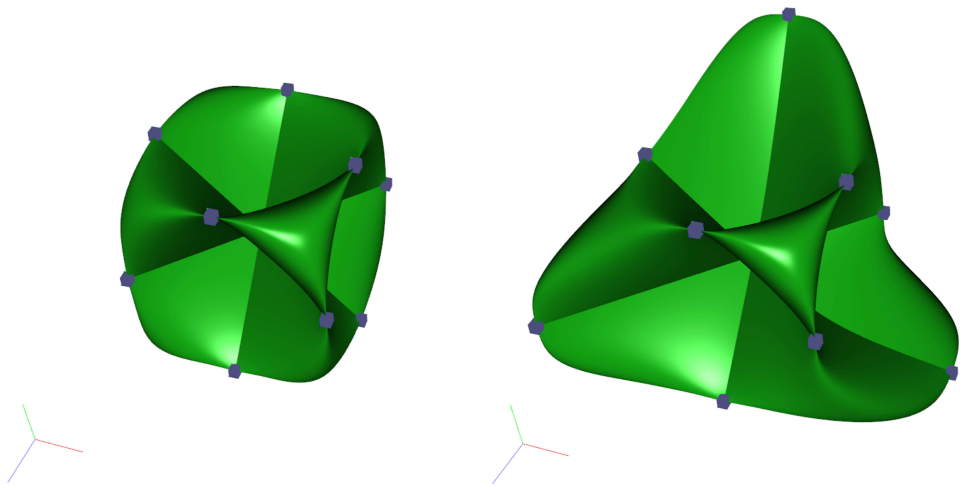

Figure 9: The tensor-product Euler Beta-function B-spline (EBFBS) surface on the left-hand side of the figure is made by first Hermite-interpolating of a non-algebraic surface (Sin surface) at 5×5 interpolating points (the positions of the interpolating points are seen as small cubes); next, the local Taylor monomial bases of the tensor-product patch centered at every knot are converted to respective Bernstein bases rescaled on the support of the one and only EBFBS basis function attaining its maximum at that knot; finally, the surface is edited by moving its Bezier control polygon (three of the cubes are moved 'out' from the center of the surface, so the surface obtains the shape given on the right-hand side of the figure).

port of F_j, ψ_j ***each of*** F_j, ψ_j ***is*** C^∞***-smooth but not analytic, i.e., the Taylor series expansion of these functions in powers of*** $t - t_{j-1}$ ***in a neighbourhood of*** t_{j-1}***, and in powers of*** $t - t_j$ ***in a neighbourhood of*** t_j***, respectively, is not convergent.***

This property is inherited also by the respective GERBS G_j ***in the three knots*** t_{j-1}, t_j, t_{j+1} ***belonging to supp*** G_j***.***

This property is of key importance for the construction of C^∞-smooth GERBS, for the following reason. A C^∞-smooth GERBS is analytic for all $t \in \mathbb{R}$, $t \neq t_{j-1}$, t_j, t_{j+1}. If G_j was analytic on the boundary of its support (t_{j-1} and/or t_{j+1}), then it would be ***possible to extend*** G_j ***by 0 in the***

interior of its support, which is contradiction. With a little modification, the same argument can be extended also to the 'interior' knot t_j: in this case, the same argument can be applied to G'_j. The importance of this argument increases in the multivariate case, since the boundary of the support of a GERBS will consist entirely of points where the GERBS is C^∞-smooth, but not analytic. This property is key for ***order- and shape-preserving*** approximation. For example, using an ERBS-based Hermite interpolant on a scattered-point set can ensure positivity, monotonicity, etc., preservation by this interpolant, while in general an analogous Hermite polynomial-based interpolant ***must fail*** to preserve order constraints, because ***the polynomial analogues*** of the GERBS basis functions are ***entire analytic*** functions (recall that such functions can be bounded if, and only if, they are constants).

The typical representatives of C^∞-smooth GERBS are ***ERBS*** [6], and they exhibit all of the above-said 'superproperties'.

One final comment here addresses the question about the ***admissible classes of functions which can be used as local functional coefficients in the ERBS-based [6, formula (50)])***. For example, admissible are all analytic functions in (t_{j-1}, t_{j+1}), expanded in Taylor series in powers of $t - t_j$, but they by far do not exhaust all the possible functions that can be used as local functional coefficients in [6, formula (50)]). Other function families consist of Hermite-interpolating trigonometric polynomials, combination of algebraic polynomials, trigonometric polynomials and linear combinations of exponential (expo-affine) functions (which, altogether, constitute (in the univariate case) all possible solutions of linear ODEs with constant coefficients), rational functions with poles on, or out of, the boundary of supp G_j, and other, more general classes. For every $j = 1, \dots, n$ it is possible to identify the exact maximal class of functions which can serve as a local functional coefficient multiplying G_j, as in the following outline. An ***expo-rational density*** ψ_j can be mapped by ***rational transformation of its argument*** to a function from the ***Schwartz class*** $S(\mathbb{R})$ (for example, the expo-rational function corresponding to the default set of the intrinsic parameters (see [6]) can be mapped to the standard Gaussian $\exp[-(\bullet)^2/2]$). The maximal class of C^∞-smooth multipliers on $S(\mathbb{R})$ is well known: this is ***the space of moderately increasing distributions*** (see, e.g. [1]). The respective maximal class of functional coefficients of the ERBS in one of its two intervals of monotonicity will be the image of the space of moderate distributions under the action of the ***inverse rational transformation of the argument***. This procedure has to be repeated for the other interval of monotonicity of the ERBS. The resulting two maximal classes are the maximal ranges of 'the left-hand' and 'the right-hand' part of the functional

coefficient (i.e., on (t_{j-1}, t_j) and (t_j, t_{j+1}), respectively). The local coefficient function for the j-th ERBS will be obtained by selecting one function from each of the 'left' and 'right' class. This includes also cases of ***discontinuity of the local coefficient function*** on (t_{j-1}, t_{j+1}) at t_j, an eventual discontinuity of the local coefficient function at t_j being interpreted as ***coalescing of two adjacent knots in the knot vector, thus covering also the case of non-strictly increasing knot vector***.

6. Concluding Remarks

The classical polynomial B-spline theory does not allow usage of non-regular curves in the interpolation process. The graphs of polynomial B-splines are curves which are either regular everywhere (the tangential vector exists and is not zero) or there are points on the curve (corresponding to values of the parameter at a knot) where the tangential vector does not exist. In this setting, G-regularity is a more general concept than C-regularity.

The C^1-smooth GERBS approach turns that question around. The new B-splines propose a smooth reparametrization of the piecewise affine B-splines, which guarantees the existence of tangential vector everywhere on the curve, but this tangential vector is zero at the knots, i.e., the curve with constant coefficients (2) is not regular at the knots. However, when the constant coefficients (corresponding to Lagrange interpolation at the knots) get upgraded to Taylor polynomial coefficients (corresponding to Hermite interpolation at the knots) and when the interpolated curve is regular at every knot, then so becomes also its GERBS Hermite interpolant (12). Thus, in the new setting of C^1-smooth GERBS allowing absence of regularity at the knots, C-regularity becomes, in a way, a more general concept than G-regularity.

Here we considered only examples of parametric curves and tensor-product surfaces (a first announcement of related results was made in [8]), but the greatest new impact of the GERBS approach is in the most difficult part of Computer Aided Geometric Design (CAGD) and finite element analysis (FEA): it provides a simple and very general technique for generation of smooth convex partitions of unity which works in fairly uniform way for mesh-free data, on triangulations and for more general domain partitions in the multivariate/multidimensional case. (This was the main part of the new results announced in [5].)

At the Seventh International Conference on Mathematical Methods for Curves and Surfaces in Tønsberg, Norway, in June 2008, at the lecture of

Thomas J.R. Hughes, on which the authors were also present, Tom Hughes informed the geometric modelling community of the world of his vision of a united approach to geometric modelling in CAGD and FEA in the modelling and simulation via boundary-value problems for PDEs. The main common tool which Tom Hughes proposed was the current industrial standard in CAGD: Non-Uniform Rational B-splines (NURBS), and the NURBS-based methods proposed by him gave the start of *Isogeometric Analysis*. It was rather symbolic that this happened at the same conference where, on the next day, the first communication [5] on the topic of Generalized Expo-rational B-splines (GERBS) was given, with Tom Hughes being in the audience. What seemed to impress the audience most, was the possibility to easily construct GERBS-based smooth convex partitions of unity on triangulations, where each GERBS had the support of the usual piecewise linear/affine B-spline (i.e., the star-1 neighbourhood of 'its' vertex in the triangulation) while at the same time GERBS was smooth, and multiplication of each GERBS with a coefficient which was not constant, but a Taylor polynomial 'around the vertex of the GERBS' immediately implied Hermite interpolation at this vertex of all derivatives present in the Taylor polynomial. The conversion to 'Bezier form' was also done effortlessly by simply changing the monomial basis in the Taylor polynomial around each vertex with respective *tensor-product* Bernstein basis; moreover, this conversion was done independently for every vertex in the triangulation, i.e., the procedure was readily parallelized.

After the presentation of [5] at this conference there was a lot of interest in the multivariate constructions based on GERBS, but the author of [5] requested (at least) one year more to work on the development of the theory before starting to publish relevant results. The present paper is the first publication in this direction. Summarizing its conclusions, it is our opinion that ***GERBS and their rational forms, Non-Uniform Rational Generalized Expo-Rational B-splines (NURGERBS)*** [1] ***vastly outperform polynomial B-splines and their rational forms NURBS as a potential universal tool of Isogeometric Analysis***.

In line with the afore-mentioned, the present paper is the first in a sequence of publications dedicated to the theory and applications of GERBS.

[1] In the case of ERBS, the respective rational forms have been termed NUERBS [7].

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References

[1] M. Рид, Б. Саймон. *Методы современной метематической физики, т. 1: Функциональный анализ.* Мир, Москва, 1977. (In Russian.)

[2] J. W. Daniel. Constrained approximation and Hermite interpolation with smooth quadratic splines: some negative results. *J. Approximation Theory*, 17(2) (1976), 135–149.

[3] H. E. Daniels. Saddlepoint approximations in statistics, *Annals Math. Statist.*, 25 (1954) 631-650.

[4] L. Dechevsky. Shape-preserving wavelet approximations: state of the art and ongoing research. In: Wavelet applications in Industrial Processing V, F. Truchetet, O. Laligant (eds.). Proceedings of SPIE, 6763 (2007) 67630X 1–14.

[5] L. T. Dechevsky. Generalized expo-rational B-splines. A communication at the Seventh International Conference on Mathematical Methods for Curves and Surfaces, Tønsberg'2008, Norway. url: http://heim.ifi.uio.no/∼cagd/

[6] L. T. Dechevsky, A. Lakså, B. Bang. Expo-rational B-splines. *Int. J. Pure Appl. Math.*, 27(3) (2006) 319–369.

[7] L. T. Dechevsky, A. Lakså, B. Bang. NUERBS form of expo-rational B-splines. *Int. J. Pure Appl. Math.*, 32(1) (2006) 1–22.

[8] L. T. Dechevsky, A. Lakså, B. Bang. Freeform design, generalized expo-rational B-splines. Communication at the 4-th International Conference "Multivariate Approximation: Theory and Applications Cancun, Mexico, 2007.

[9] L. Dechevsky, B. MacGibbon. Asymptotically minimax non-parametric function estimation with positivity constraints I. Les Cahiers du GERAD, G-99-24, Montreal, Canada, 1999.

[10] L. Dechevsky, B. MacGibbon. Asymptotically minimax non-parametric function estimation with positivity constraints I. *Statistics* – Berlin, Taylor & Francis, 43(5) (2009) 423–442.

[11] L. Dechevsky, B. MacGibbon. Asymptotically minimax non-parametric function estimation with positivity constraints II. Les Cahiers du GERAD, G-2001-34, Montreal, Canada, 2001.

[12] L. Dechevsky, B. MacGibbon. Asymptotically minimax non-parametric function estimation with positivity constraints II. *Int. J. Pure Appl. Math.*, 57(2) (2006) 165–200.

[13] L. T. Dechevsky, B. MacGibbon, S. I. Penev. Numerical methods for asymptotically minimax non-parametric function estimation with positivity constraints I. Les Cahiers du GERAD, G-2000-33, Montreal, Canada, 2001.

[14] L. T. Dechevsky, B. MacGibbon, S. I. Penev. Numerical methods for asymptotically minimax non-parametric function estimation with positivity constraints I. Sankhya, Ser. B, 63(2) (2001) 149–180.

[15] L. T. Dechevsky, B. MacGibbon. Numerical methods for asymptotically minimax non-parametric function estimation with positivity constraints II. (In preparation.)

[16] L. T. Dechevsky, S. I. Penev. On shape-preserving probabilistic wavelet approximations. *Stochast. Anal. and Appl.*, 15(2) (1997) 187–215.

[17] L. T. Dechevsky, S. I. Penev. On shape-preserving wavelet estimators of cumulative distribution functions and densities. *Stochast. Anal. and Appl.*, 16(3) (1998) 428-469.

[18] L. T. Dechevsky, J. O. Ramsay, S I. Penev. Penalized wavelet estimation with Besov regularity constraints. *Mathematica Balkanica (N. S.)*, 13(4) (1999) 257–356.

[19] I. P. Gancheva, N. D. Delistoyanova. Euler Beta-function B-splines: Definition, Basic Properties, and Practical Use in Computer Aided Geometric Design. M. Sc. Diploma thesis (supervisor: L. T. Dechevsky). Narvik University College, Norway, 2007.

[20] V. H. Hristov, K. G. Ivanov. Realization of K-functionals on subsets and constrained approximation. *Math. Balkanica (N.S.)* 4(3) (1990) 236–257.

[21] P. P. Korovkin. On the order of the approximation of functions by linear positive operators. *Dokl. Akad. Nauk SSSR (N.S.)* 114 (1957) 1158–1161. (In Russian.)

[22] M.-J. Lai, L. L. Schumaker, On the approximation power of bivariate splines. *Adv. in Comp. Math.*, 9 (1998) 251–279.

[23] M.-J. Lai, L. L. Schumaker. Macro-elements and stable local bases for splines on Clough-Tocher triangulations. *Numer. Math.*, 88 (2001) 105–119.

[24] M.-J. Lai, L. L. Schumaker. Macro-elements and stable local bases for splines on Powell-Sabin triangulations. *Math. Comp.*, 72 (2003) 335–354.

[25] A. Lakså. Basic properties of Expo-Rational B-splines and practical use in Computer Aided Geometric Design. Dissertation of Doctor Philos., Oslo University. Oslo, 2006.

[26] D. Leviatan, I. A. Shevchuk. Counterexamples in convex and higher order constrained approximation. *East J. Approx.* 1(3) (1995) 391–398.

[27] A. Mahmood, D. J. Lynch, L. D. Philipp. A fast banded matrix inversion using connectivity of Schur's complements. In: Proceedings of the IEEE International Conference on Systems Engineereing, Dayton, OH, 1991, pp. 303–306.

[28] D. L. McFadden. Economic Choices. Nobel Prize Lecture, December 8, 2000.

[29] S. Penev, L. Dechevsky. On non-negative wavelet-based density estimators. *Nonparametric Statistics*, 7(4) (1997) 364–394.

[30] I. J. Schoenberg. On interpolation by spline functions and its minimal properties. In: On Approximation Theory (Proceedings of Conference in Oberwolfach, 1963). Birkhäuser, Basel, 1964, 109–129.

[31] I. J. Schoenberg. Notes on spline functions. I. The limits of the interpolating periodic spline functions as their degree tends to infinity. *Indag. Math.*, 34 (1972) 412-422.

[32] I. J. Schoenberg. Cardinal interpolation and spline functions. II. Interpolation of data of power growth. Collection of Articles Dedicated to J. L. Walsh on his 75th Birthday, VIII. *J. Approximation Theory*, 6 (1972) 404-420.

[33] I. J. Schoenberg. Cardinal interpolation and spline functions. IV. The exponential Euler splines. In: *Linear Operators and Approximation*, P. L. Butzer, J. -P. Cahane, B. Szökefalvi-Nagy (eds.). Birkhäuser, Basel, 1972, pp. 382-404.

[34] I. J. Schoenberg. Notes on spline functions. III. On the convergence of the interpolating cardinal splines as their degree tends to infinity. *Israel J. Math.*, 16 (1973) 87-93.