International Journal of Pure and Applied Mathematics
Volume 58 No. 1 2010, 99-112

THE SOLUTIONS TO A BI-FRACTIONAL BLACK-SCHOLES-MERTON DIFFERENTIAL EQUATION

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Abstract: A model for option pricing of a two parameter (γ, α)-fractional Black-Scholes-Merton differential equation is established based on the stock price modeled by $(dS_t)^\alpha = \mu(S_t)^\alpha(dt)^\alpha + \sigma(S_t)^\alpha\ dW_\alpha(t)$, where $\alpha > 0$, μ, σ are constants and $dW_\alpha(t) = \varepsilon(dt)^{\alpha/2}$, fractional Wiener process, ε obeys standard normal distribution. We solve the bi-fractional Black-Scholes-Merton differential equation obtained under the key boundary condition $C(S,t) = \max(S-K, 0)$ for call option and $P(S,t) = \max(K-S, 0)$ for put option at time T, the maturity date of the option, and obtain the explicit option pricing formulas for European call option and put option for $\gamma > 0, 1 \le \alpha \le 2$.

AMS Subject Classification: 35K57, 35K99
Key Words: option pricing, Black-Scholes-Merton differential equation, fractional derivatives, Taylor series of fractional order

Received: December 22, 2009

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1. Introduction

The framework for option pricing was developed by Black and Scholes [1] and Merton [7] when the underlying asset follows a diffusion process. It is based on Samuelson's stock price model,

$$dS = \mu S dt + \sigma S dB, \tag{1}$$

where μ, σ are constants, and B is a Brownian motion with the unit variance, and assume "ideal conditions" in the market for the stock and for the option and prove in [1] and [7] that the value $V(S,t)$ of the option in terms of the price of the stock satisfies the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2}$$

where S is the price of the stock underlying, r is riskless interest rate (constant), and σ is volatility of the stock price.

Recently, under the assumptions of the dynamical equation of the stock value $S(t)$ satisfying the fractional differential equation

$$d^\alpha S = \hat{r} S t^{1-\alpha}(dt)^\alpha + \hat{\sigma} S w(t)(dt)^{\alpha/2}, \tag{3}$$

and

$$dS = rSdt + \sigma S w(t)(dt)^{\alpha/2}, \tag{4}$$

where $0 < \alpha \leq 1$, $\hat{r} := \Gamma^{-1}(1-\alpha)r =: r/(1-\alpha)!$ and $\hat{\sigma} := (\alpha!)\sigma$, and $w(t)$ is a normalized Gaussian white noise, i.e. with zero mean and the unit variance, and r the interest rate. By using the equation (3) and equation (4) combined with the Itô Lemma and the fractional Taylor's series of the price of the stock option $V(S,t)$, Guy Jumarie derives two fractional Black-Scholes equations, see [4],

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \left(\frac{rV}{(1-\alpha)!} - rS^\alpha \frac{\partial^\alpha V}{\partial S^\alpha}\right) t^{1-\alpha} - \frac{(\alpha!)^3[(1-\alpha)!]^2}{(2\alpha)!}\sigma^2 S^{2\alpha}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}}, \tag{5}$$

and

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \left(rV - rS\frac{\partial V}{\partial S}\right)\frac{t^{1-\alpha}}{(1-\alpha)!} - \frac{\alpha!}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, \tag{6}$$

respectively, and their solutions, see [5], where $\frac{\partial^\alpha V}{\partial t^\alpha}$ and $\frac{\partial^\beta V}{\partial S^\beta}$ are Riemann-Liouville fractional derivatives, see [12].

In this paper, we will consider the option pricing problem while the dynamics of stock price S follows a fractional Itô process, i.e.

$$(dS)^\alpha = \mu S^\alpha (dt)^\alpha + \sigma S^\alpha dW_\alpha(t), \tag{7}$$

where $\alpha > 0$, μ, σ are constants, $dW_\alpha(t) = \varepsilon(dt)^{\alpha/2}$, is called fractional Wiener process, ε obeys standard normal distribution.

This paper is organized as follows. We derive a single parameter fractional Black-Scholes-Merton partial differential equation in Section 2 and a bi-parameters fractional Black-Scholes-Merton differential equation in Section 3, respectively. In Section 4, we obtain the explicit option pricing formulas for European call option and put option, respectively.

2. The Fractional Black-Scholes-Merton Differential Equation

Let $V(S,t)$ denote the price of an option on a stock: S is the price of the stock at time t. Let V(S,t): $0 < S < +\infty, 0 \le t < \infty$, $(S,t) \to V(S,t)$ has fractional derivative of order $k\alpha$, in position S and in t for any positive integer k and any $\alpha(> 0)$, then we have, see [4],

$$\begin{aligned} d^\alpha V(S,t) =& \frac{1}{\Gamma(1+\alpha)}\frac{\partial^\alpha V}{\partial t^\alpha}(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)}\frac{\partial^\alpha V}{\partial S^\alpha}(dS)^\alpha \\ &+ \frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}}(dS)^{2\alpha} + \frac{1}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha} V}{\partial t^{2\alpha}}(dt)^{2\alpha} \\ &+ \frac{1}{\Gamma^2(1+\alpha)}\frac{\partial^{\alpha+\alpha} V}{\partial S^\alpha \partial t^\alpha}(dS)^\alpha (dt)^\alpha + \text{higher order terms.} \end{aligned} \tag{8}$$

Since

$$(dS)^\alpha = \mu(S,t)(dt)^\alpha + \sigma(S,t)dW_\alpha(t), \tag{9}$$

$$(dS)^{2\alpha} = \sigma^2(S,t)(dt)^\alpha + o((dt)^\alpha), \tag{10}$$

and

$$(dS)^\alpha (dt)^\alpha = o((dt)^\alpha), \tag{11}$$

thus, we have the fractional Itô Lemma:

$$\begin{aligned} d^\alpha V(S,t) =& \left(\frac{1}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\mu(S,t)}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial S^\alpha} + \frac{\sigma^2(S,t)}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}}\right)(dt)^\alpha \\ &+ \frac{\sigma(S,t)}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial S^\alpha} dW_\alpha(t), \quad \alpha > 0, \end{aligned} \tag{12}$$

where $\Gamma_\delta = \Gamma(1+\delta)$.

This shows that if S_t obeys fractional Itô process then $d^\alpha V(S,t)$ also obeys fractional Itô process with α-order fluctuation rate

$$\frac{1}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\mu(S,t)}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial S^\alpha} + \frac{\sigma^2(S,t)}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}},$$

and α-order variance rate

$$\left(\frac{\sigma(S,t)}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial S^\alpha}\right)^2.$$

Thus, assume that the price $S(t)$ of stock obeys the following fractional Itô process

$$(dS)^\alpha = \mu S^\alpha (dt)^\alpha + \sigma S^\alpha dW_\alpha(t), \quad \alpha > 0, \tag{13}$$

by the above fractional Itô Lemma, for the price of option on stock S_t we have

$$d^\alpha V(S,t) = \left(\frac{1}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\mu S^\alpha}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial S^\alpha} + \frac{\sigma^2 S^{2\alpha}}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}}\right)(dt)^\alpha + \frac{\sigma S^\alpha}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial S^\alpha} dW_\alpha(t), \quad \alpha > 0. \tag{14}$$

In order to derive the fractional Black-Scholes-Merton differential equation, we set up a riskless portfolio Π of $V(S,t)$ and S such that

$$d^\alpha \Pi = d^\alpha V - \delta (dS)^\alpha. \tag{15}$$

According to the discussion above, choosing $\delta = \frac{1}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial S^\alpha}$, we have

$$d^\alpha \Pi = \left[\frac{1}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\sigma^2 S^{2\alpha}}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}}\right](dt)^\alpha. \tag{16}$$

This shows that the investment portfolio Π must be riskless during time dt. From the assumption in absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate r, it follows that

$$d^\alpha \Pi = r\Pi (dt)^\alpha. \tag{17}$$

For $0 \le m < \alpha \le m+1$, m is integer, assume $d^j(\Pi - V)|_{S=0} = 0, (j = 0, 1, 2, \cdots, m-1)$, from (15) we have, see [4],

$$\Gamma_{\alpha-m}(\Pi - V) = -\delta \int_o^S (dS)^\alpha = -\delta\alpha \int_0^S (S-\tau)^{\alpha-1} d\tau = -\delta S^\alpha. \tag{18}$$

Thus, from (16), (18) and (17), we obtain the fractional Black-Scholes-Merton differential equation

$$\frac{1}{\Gamma_\alpha}\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\sigma^2 S^{2\alpha}}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}} + \frac{r S^\alpha}{\Gamma_\alpha \Gamma_{\alpha-m}}\frac{\partial^\alpha V}{\partial S^\alpha} - rV = 0, \tag{19}$$

where $(m < \alpha \le m+1)$,

$$\frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{f(\xi) - f(0)}{(x-\xi)^\alpha} d\xi, \tag{20}$$

for $o < \alpha < 1$, and

$$\frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_0^x \frac{f(\xi) - f(0)}{(x-\xi)^{\alpha-n+1}} d\xi, \tag{21}$$

for $\alpha \geq 1, \quad n = [\alpha] + 1$, is called Modified Riemann-Liuoville fractional deravatives of order α, left handed. It is the rate of the output at time t for input $f(x)$ at start-terminal $x = 0$.

In particular, when $\alpha = 1$ this reduces to the Black-Scholes differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{22}$$

In order to solve this Black-Scholes-Merton equation under the key terminal boundary condition $V(S,T) = V_0(S) = \max(S-K, 0)$ for call option, and $V(S,T) = V_0(S) = \max(K-S, 0)$ for put option, respectively, we need to derive a bi-fractional Black-Scholes-Merton differential equation.

3. The Bi-Fractional Black-Scholes-Merton Differential Equation

In this section, we will derive the bi-fractional Black-Scholes-Merton differential equation by using heuristic argument of Le Mehute [6] and Giona and Roman [3].

Let $V(S,t)$ denote the price of an option on a stock. We consider the price change of option price in the financial market as a fractal transmission system. The relationship between the total flux of option price current rate $Y(S,t)$ of order α per unit time from time $t = 0$ to time t and the option price $V(S,t)$, considering as the output and the input at start terminal $t = T$ of the fractal transmission system of option price (see [6], [3]), should satisfy the following equation

$$\int_t^T Y(S,t')dt' = S^{d_f - 1}\int_t^T H(t'-t)[V(S,t) - V(S,T)]dt', \tag{23}$$

where $H(t)$ is the transmission function and d_f the Hausdorff dimension of the fractal transmission system considered. This is a conservation equation containing an explicit reference to the history of the diffusion process of the option price on the fractal structure. We assume that the diffusion sets are underlying fractal (underlying fractals denote self-similar sets in [11] or net fractals [9]) and the transmission function on the underlying fractal should behave as

$$H(t) = \frac{A_\gamma}{\Gamma(1-\gamma)t^\gamma}, \qquad 0 < \gamma < 1, \tag{24}$$

where γ is a transmission exponent and A_γ is a constant that can be determined, see [10]. From equation (23), we have that

$$-Y(S,t) = S^{d_f-1}\frac{d}{dt}\int_t^T H(t'-t)[V(S,t)-V(S,T)]dt'. \tag{25}$$

On the other hand, the option price current rate $Y(S,t)$ is given by the Black-Scholes-Merton differential equation (19)

$$Y(S,t) = -\frac{\partial^\alpha V}{\partial t^\alpha} = \Gamma_\alpha\left\{\frac{\sigma^2 S^2}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha}V}{\partial S^{2\alpha}} + \frac{rS^\alpha}{\Gamma_\alpha\Gamma_{\alpha-m}}\frac{\partial^\alpha V}{\partial S^\alpha} - rV\right\}. \tag{26}$$

Thus, from (24), (25) and (26), we have the fractional in time-space Black-Scholes-Merton differential equation

$$\frac{A_\gamma S^{d_f-1}}{\Gamma_\alpha}\frac{\partial^\gamma V}{\partial t^\gamma} + \frac{\sigma^2 S^{2\alpha}}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha}V}{\partial S^{2\alpha}} + \frac{rS^\alpha}{\Gamma_\alpha\Gamma_{\alpha-m}}\frac{\partial^\alpha V}{\partial S^\alpha} - rV = 0 \tag{27}$$

for $m < \alpha \leq m+1$, where

$$\frac{\partial^\gamma V}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_t^T \frac{V(S,t')-V(S,T)}{(t'-t)^\gamma}dt' \tag{28}$$

is called the Modified Riemann-Liouville fractional derivative of order $\gamma, 0 < \gamma < 1$, right-handed. It is the rate of the output at time t for input $f(S,t)$ at start-terminal $t = T$. For $\gamma \geq 1$, we define the Modified Reimann-Liouville fractional derivatives of order γ $\frac{\partial^\gamma V}{\partial t^\gamma}$ as follows:

$$\frac{\partial^\gamma V}{\partial t^\gamma} = \frac{(-1)^{n-1}}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n\int_t^T \frac{V(S,t')-V(S,T)}{(t'-t)^{\gamma-n+1}}dt', \tag{29}$$

$n = [\gamma]+1$.

The above time-fractional Black-Scholes-Merton differential equation has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The particular derivative that is obtained when the equation is solved depends on the boundary conditions that are used. These specify the value of the derivative at the boundaries of possible values of S and time t.

The key terminal boundary condition is

$$V(S,t) = \max(S-K,0) \quad \text{when} \quad t = T,$$

for European call option, and

$$V(S,t) = \max(K-S,0) \quad \text{when} \quad t = T,$$

for European put option, T is the maturity date of the option, and K is the exercise price of the option.

4. The Solution of the Bi-Fractional Black-Scholes-Merton Differential Equation

In this section, we discuss the solution of the bi-fractional Black-Scholes-Merton differential equation. For simplicity, we assume $A_\gamma = d_f = 1$, it satisfies the

$$\frac{1}{\Gamma_\alpha}\frac{\partial^\gamma V}{\partial t^\gamma} + \frac{\sigma^2 S^{2\alpha}}{\Gamma_{2\alpha}}\frac{\partial^{2\alpha} V}{\partial S^{2\alpha}} + \frac{rS^\alpha}{\Gamma_\alpha \Gamma_{\alpha-m}}\frac{\partial^\alpha V}{\partial S^\alpha} - rV = 0, \tag{30}$$

where for $\gamma > 0, m < \alpha \le m+1$,

$$\frac{\partial^\gamma V}{\partial t^\gamma} = \begin{cases} \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_t^T \frac{[V(S,t')-V(S,T)]}{(t'-t)^\gamma}dt', & 0<\gamma<1, \\ \frac{(-1)^{n-1}}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n \int_t^T \frac{V(S,t')-V(S,T)}{(t'-t)^{\gamma-n+1}}dt', & n=[\gamma]+1, \gamma\ge 1, \end{cases} \tag{31}$$

and

$$\frac{\partial^\beta V}{\partial S^\beta} = \begin{cases} \frac{1}{\Gamma(1-\beta)}\frac{d}{dS}\int_0^S \frac{V(S',t)}{(S-S')^\beta}dS', & 0<\beta<1, \\ \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dS}\right)^n \int_0^S \frac{V(S',t)}{(S-S')^{\beta-n+1}}dS', & n=[\beta]+1, \beta\ge 1. \end{cases} \tag{32}$$

In order to eliminate S^α and $S^{2\alpha}$ in equation (30), we let

$$S = Ke^x, \quad t = T - \tau/\frac{1}{2}\sigma^2, \quad V(S,t) = Ku(x,\tau).$$

Noting that $\frac{\partial^\gamma V}{\partial t^\gamma} = -K(\frac{1}{2}\sigma^2)^\gamma \frac{\partial^\gamma u}{\partial \tau^\gamma}$ and $S^\beta \frac{\partial^\beta V}{\partial S^\beta} = K\frac{\partial^\beta u}{\partial x^\beta}$, see [4], formula (3.24), from equation (30), for $-\infty < x < +\infty$, we have

$$a_1\frac{\partial^\gamma u}{\partial \tau^\gamma} = a_2\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + a_3\frac{\partial^\alpha u}{\partial x^\alpha} - ru \quad (m<\alpha\le m+1), \tag{33}$$

where

$$a_1 = \frac{(\sigma^2/2)^\gamma}{\Gamma(1+\alpha)}, \quad a_2 = \frac{\sigma^2}{\Gamma(1+2\alpha)}, \quad a_3 = \frac{r(\Gamma(1+\alpha))^{-1}}{\Gamma(1+\alpha-m)}, \tag{34}$$

and

$$\frac{\partial^\gamma u}{\partial \tau^\gamma} = \begin{cases} \frac{1}{\Gamma(1-\gamma)}\frac{d}{d\tau}\int_0^\tau \frac{[u(x,\tau')-u(x,0)]}{(\tau-\tau')^\gamma}d\tau', & 0<\gamma<1, \\ \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{d\tau}\right)^n \int_0^\tau \frac{u(x,\tau')-u(x,0)}{(\tau-\tau')^{\gamma-n+1}}d\tau', & n=[\gamma]+1, \gamma\ge 1, \end{cases} \tag{35}$$

$$\frac{\partial^\beta u}{\partial x^\beta} = \begin{cases} \frac{1}{(1-\beta)}\frac{d}{dx}\int_{-\infty}^x \frac{u(x',\tau)}{(x-x')^\beta}dx', & 0<\beta<1, \\ \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n \int_{-\infty}^x \frac{u(x',\tau)}{(x-x')^{\beta-n+1}}dx', & n=[\beta]+1, \beta\ge 1. \end{cases} \tag{36}$$

The corresponding key boundary condition is $u(x,0) = u_0(x) = K\max(e^x - 1, 0)$ for call option, and $u(x,0) = u_0(x) = K\max(1-e^x, 0)$ for put option, respectively.

We derive now the explicit solution of equation (33) by the Laplace transform and the Fourier transform. By the Laplace transform and noting that the formula of Laplace transform for the Reimann-Liouville fractional derivative of order $\gamma > 0$, see [8],

$$L\left[\frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{d\tau}\right)^n\int_0^\tau\frac{u(x,\tau')}{(\tau-\tau')^{\gamma-n+1}}d\tau'\right] = p^\gamma u(x,p) - \sum_{k=0}^{n-1}p^k b_k(x), \quad (37)$$

and

$$L\left[{}_0D_\tau^\gamma u_0(x)\right] = b_n p^{\gamma-1}u_0(x), \quad (38)$$

where $n = [\gamma]+1$, $b_n = (n-\gamma-1)(n-\gamma-2)\dots(1-\gamma)\frac{\Gamma(1-\gamma)}{\Gamma(n-\gamma)}$,

$$b_k(x) = \left[\frac{\partial^{\gamma-k-1}u(x,\tau)}{\partial\tau^{\gamma-k-1}}\right]_{\tau=0}. \quad (39)$$

Thus, from (37), (38) and equation (33), we have

$$a\frac{\partial^{2\alpha}u(x,p)}{\partial x^{2\alpha}} + b\frac{\partial^\alpha u(x,p)}{\partial x^\alpha} + cu(x,p) = h(x) \quad (-\infty < x < +\infty), \quad (40)$$

where $u(x,p) = L[u(x,\tau)] = \int_0^\infty e^{-p\tau}u(x,\tau)d\tau, a = a_2, b = a_3, c = -(r + a_1p^\gamma)$, and

$$h(x) = -a_1\left(\sum_{k=0}^{n-1}p^k b_k(x) + b_n p^{\gamma-1}u_0(x)\right). \quad (41)$$

By the exponential Fourier transforms $f_e(\omega) = F_e[f(x)] = \int_{-\infty}^{+\infty}e^{i\omega x}f(x)dx$, and the Fourier transform of fractional derivatives, and noting that

$$F_e[\frac{\partial^\beta u}{\partial x^\beta}] = (-i\omega)^\beta u_e(\omega,p), \quad \beta > 0, \text{ see [8]},$$

it follows from (40) that

$$u_e(\omega,p) = h_e(\omega)g_e(\omega), \quad (42)$$

where $h_e(\omega) = F_e[h(x)]$, and

$$g_e(\omega) = \frac{1}{aq^{2\alpha}+bq^\alpha+c}, \quad q = -i\omega, \quad (43)$$

to be called the fractional Green function of equation (40).

Since the Fourier transform of the convolution

$$h(x)*g(x) = \int_{-\infty}^{+\infty}h(x-y)g(y)dy = \int_{-\infty}^{+\infty}h(y)g(x-y)dy \quad (44)$$

of the two functions $h(x)$ and $g(x)$, which are defined in $(-\infty,+\infty)$, is equal to the product of their Fourier transforms:

$$F[h(x)*g(x)] = F[h(x)]F[g(x)] = h_e(\omega)g_e(\omega) \quad (45)$$

under the assumption that both $h_e(\omega)$ and $g_e(\omega)$ exist, by (45) it follows from

(42) that

$$u(x,p)=\int_{-\infty}^{+\infty}h(y)g(x-y)dy, \tag{46}$$

where $g(x)=F^{-1}[g_e(\omega)]$.

By the residual theorem of one complex variable function, it is proved that

$$\begin{aligned} g(x) &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{xq}}{aq^{2\alpha}+bq^{\alpha}+c}dq \\ &= \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty}\frac{e^{xq}}{aq^{2\alpha}+bq^{\alpha}+c}dq - Res\left(e^{xq}g_e(\omega),q_1\right), \end{aligned} \tag{47}$$

where $s>|b/a|^{1/\alpha}$, and $q_1=[\frac{-b+\sqrt{b^2-4ac}}{2a}]^{1/\alpha}$ is an unique pole of $g_e(\omega)$ in the domain $0<Re(q)<s$ for $1\le\alpha\le 2$, since $g_e(\omega)$ is analytic for $Re(q)>s$. Noting that $\frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty}\frac{e^{xq}}{aq^{2\alpha}+bq^{\alpha}+c}dq=L^{-1}[g_e(\omega)]$, and that $g_e(\omega)$ can be written in the form

$$g_e(\omega)=\frac{1}{c}\frac{cq^{-\alpha}}{aq^{\alpha}+b}\frac{1}{1+\frac{cq^{-\alpha}}{aq^{\alpha}+b}}=\frac{1}{a}\sum_{k=0}^{\infty}(-1)^k(\frac{c}{a})^k\frac{q^{-\alpha k-\alpha}}{(q^{\alpha}+\frac{b}{a})^{k+1}}, \tag{48}$$

the term-by-term inversion, based on the general expansion theorem for the Laplace transform given in [2], Section 22, using the formula of the Laplace transform of the function $x^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm zx^{\alpha})$, $\quad (E^{(k)}_{\alpha,\beta}(y)\equiv\frac{d^k}{dy^k}E_{\alpha,\beta}(y))$ (see [8], 1.80):

$$\int_0^{\infty}e^{(-pt)}t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm at^{\alpha})dt=\frac{k!p^{(\alpha-\beta)}}{(p^{\alpha}\mp a)^{k+1}}, \tag{49}$$

$Re(p)>|a|^{1/\alpha}$, produces

$$g_1(x)\equiv L^{-1}[g_e(q)]=\frac{1}{a}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{c}{a})^k g_{1,k}(x), \tag{50}$$

$Re(p)>|b/a|^{1/\alpha}$, where

$$g_{1,k}(x)=x^{2\alpha(k+1)-1}E^{(k)}_{\alpha,2\alpha+k\alpha}(-\frac{b}{a}x^{\alpha}), \tag{51}$$

$$E^{(k)}_{\lambda,\mu}(y)\equiv\frac{d^k}{dy^k}E_{\lambda,\mu}(y)=\sum_{j=0}^{\infty}\frac{(j+k)!y^j}{j!\Gamma(\lambda j+\lambda k+\mu)}.$$

Using the Taylor expansion of the function $(1+z)^{\alpha}=e^{\alpha\log(1+z)}$ $(\log 1=0)$ at $z=0$:

$$(1+z)^{\alpha}=1+\alpha z+C_2^{\alpha}z^2+C_3^{\alpha}z^3+\cdots+C_n^{\alpha}z^n+\ldots\quad(|z|<1),$$

where $C_n^\alpha = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)/n!$, we have

$$Res\left(\frac{e^{xq}}{aq^{2\alpha}+bq^\alpha+c}, q_1\right) = \frac{e^{xq_1}}{a\alpha q_1^{\alpha-1}(q_1^\alpha - q_2^\alpha)}. \tag{52}$$

Thus, we have

$$\begin{aligned} u(x,p) &= \int_{-\infty}^{+\infty} h(y)g(x-y)dy \\ &= -a_1 \int_{-\infty}^{+\infty} \left[\sum_{j=0}^{n-1} p^j b_j(y) + b_n p^{\gamma-1} u_0(y)\right] g(x-y)dy, \end{aligned} \tag{53}$$

where $g(x) = g_1(x) - g_2(x)$, $\quad g_{2,k}(x) = x^k$, and

$$g_2(x) = \frac{e^{xq_1}}{a\alpha q_1^{\alpha-1}(q_1^\alpha - q_2^\alpha)} = \sum_{k=0}^{\infty} \frac{1}{\alpha k!}\left(\frac{q_1^{k+1-\alpha}}{\sqrt{b^2-4ac}}\right) g_{2,k}(x).$$

It can be written in the form as follows

$$\begin{aligned} u(x,p) = &-\sum_{k=0}^{\infty} c_k (r + a_1 p^\gamma)^k \left[\sum_{j=0}^{n-1} p^j g_{j;1,k}(x) + b_n p^{\gamma-1} g_{0;1,k}(x)\right] \\ &+ a_1 \sum_{k=0}^{\infty} \frac{1}{\alpha k!}\left(\frac{q_1^{k+1-\alpha}}{\sqrt{b^2-4ac}}\right)\left[\sum_{j=0}^{n-1} p^j g_{j;2,k}(x) + b_n p^{\gamma-1} g_{0;2,k}(x)\right], \end{aligned} \tag{54}$$

where, $c_k = \frac{a_1}{a^{k+1}k!}$,

$$g_{j;i,k}(x) = \int_{-\infty}^{+\infty} b_j(y) g_{i,k}(x-y)dy, \tag{55}$$

$(j = 0, 1, \dots, n-1; i = 1, 2)$ and

$$\begin{aligned} g_{0;i,k}(x) &= \int_{-\infty}^{+\infty} u_0(y) g_{i,k}(x-y)dy, \quad (i = 1, 2) \\ &= \begin{cases} \int_0^\infty (e^y - 1) g_{i,k}(x-y)dy, & \text{for call option,} \\ \int_{-\infty}^0 (1 - e^y) g_{i,k}(x-y)dy, & \text{for put option.} \end{cases} \end{aligned} \tag{56}$$

The term-by-term inversion of Laplace transform, based on the general expansion theorem for the Laplace transform given in Doetsch [2], Section 22, produces

$$u(x,\tau) = -\sum_{k=0}^{\infty} c_k \left[\sum_{j=0}^{n-1} g_{j;1,k}(x) h_{j;1,k}(\tau) + b_n g_{0;1,k}(x) h_{0;1,k}(\tau)\right]$$

$$+\sum_{k=0}^{\infty}\frac{a_1}{\alpha k!}\left[\sum_{j=0}^{n-1}g_{j;2,k}(x)h_{j;2,k}(\tau)+b_n g_{0;2,k}(x)h_{0;2,k}(\tau)\right], \quad (57)$$

where,

$$h_{j;1,k}(\tau)=\sum_{\ell=0}^{k}C_\ell^k r^{k-\ell}a_1^\ell\frac{\tau^{-1-j-\ell\gamma}}{\Gamma(-j-\ell\gamma)},$$

$$h_{0;1,k}(\tau)=\sum_{\ell=0}^{k}C_\ell^k r^{k-\ell}a_1^\ell\frac{\tau^{-\gamma-\ell\gamma}}{\Gamma(1-\gamma-\ell\gamma)}, \quad (58)$$

$$h_{j;2,k}(\tau)=h_k\left\{\frac{t^{-j-1-\frac{\gamma(\alpha_k-1)}{2}}}{\Gamma(-j-\frac{\gamma(\alpha_k-1)}{2})}+\sum_{m=1}^{\infty}\frac{e_m t^{\frac{\gamma(m-\alpha_k+1)}{2}-j-1}}{\Gamma(\frac{\gamma(m-\alpha_k+1)}{2}-j)}\right\}, \quad (59)$$

and

$$h_{0;2,k}(\tau)=h_k\left\{\frac{t^{-\frac{\gamma(\alpha_k+1)}{2}}}{\Gamma(1-\frac{\gamma(\alpha_k+1)}{2})}+\sum_{m=1}^{\infty}\frac{e_m t^{\gamma(m-\alpha_k-1)/2)}}{\Gamma(\frac{\gamma(m-\alpha_k-1)}{2}+1)}\right\}, \quad (60)$$

where $h_k=\frac{a_1^{(\alpha_k-1)/2}a^{-(\alpha_k+1)/2}}{2}$, $\quad \alpha_k=\frac{(k+1-\alpha)}{\alpha}$, and $\{e_m\}_{m=1}^{\infty}$ is defined by

$$\frac{[(-b+\sqrt{b^2-4ac})/2)]^{\alpha_k}}{\sqrt{b^2-4ac}}\equiv d_k p^{\frac{\gamma(\alpha_k-1)}{2}}\left(1+\sum_{m=1}^{\infty}\frac{e_m}{p^{m\gamma/2}}\right), \quad (61)$$

herein, $d_k=\frac{1}{2}a_1^{(\alpha_k-1)/2}a^{-(\alpha_k+1)/2}$.

Hence, for European call option, we obtain that

$$C(S,t)=-K\sum_{k=0}^{\infty}c_k\left[\sum_{j=0}^{n-1}g_{j;1,k}(x)\,h_{j;1,k}(\tau)+b_n g_{0;1,k}^C(x)\,h_{0;1,k}(\tau)\right]$$
$$+K\sum_{k=0}^{\infty}\frac{a_1}{\alpha k!}\left[\sum_{j=0}^{n-1}g_{j;2,k}(x)\,h_{j;2,k}(\tau)+b_n g_{0;2,k}^C(x)\,h_{0;2,k}(\tau)\right]; \quad (62)$$

for European put option, we have

$$P(S,t)=-K\sum_{k=0}^{\infty}c_k\left[\sum_{j=0}^{n-1}g_{j;1,k}(x)\,h_{j;1,k}(\tau)+b_n g_{0;1,k}^P(x)\,h_{0;1,k}(\tau)\right]$$
$$+K\sum_{k=0}^{\infty}\frac{a_1}{\alpha k!}\left[\sum_{j=0}^{n-1}g_{j;2,k}(x)\,h_{j;2,k}(\tau)+b_n g_{0;2,k}^P(x)\,h_{0;2,k}(\tau)\right], \quad (63)$$

where $x = \ln(S/K)$, $\tau = \frac{1}{2}\sigma^2(T-t)$, and

$$g^C_{0;j,k}(x) = \int_0^{\infty} (e^y - 1) g_{j,k}(x-y) dy \quad (j = 1,2), \tag{64}$$

$$g^P_{0;j,k}(x) = \int_{-\infty}^{0} (1 - e^y) g_{j,k}(x-y) dy \quad (j = 1,2). \tag{65}$$

In particular, if $0 < \gamma < 1$, or $b_k(x) = \left[\frac{\partial^{\gamma-k-1} u(x,\tau)}{\partial \tau^{\gamma-k-1}}\right]_{\tau=0} = 0$, $(k = 0, 1, 2, \ldots, [\gamma])$, then we have

$$\begin{aligned} C(S,t) &= -K \sum_{k=0}^{\infty} c_k g^C_{0;1,k}(x)\, h_{0;1,k}(\tau) \\ &+ K \sum_{k=0}^{\infty} \frac{a_1}{\alpha k!} g^C_{0;2,k}(x)\, h_{0;2,k}(\tau), \end{aligned} \tag{66}$$

and

$$\begin{aligned} P(S,t) &= -K \sum_{k=0}^{\infty} c_k g^P_{0;1,k}(x)\, h_{0;1,k}(\tau) \\ &+ K \sum_{k=0}^{\infty} \frac{a_1}{\alpha k!} g^P_{0;2,k}(x)\, h_{0;2,k}(\tau). \end{aligned} \tag{67}$$

5. Results and Discussion

In this paper, we establish a model for option pricing of a two parameters (γ, α)-fractional Black-Scholes-Merton differential equation driven by the dynamics of stock price $S(t)$ satisfying $(dS_t)^\alpha = \mu(S_t)^\alpha (dt)^\alpha + \sigma(S_t)^\alpha dW_\alpha(t)$, where $\alpha > 0$, μ, σ are constants and $dW_\alpha(t) = \varepsilon(dt)^{\alpha/2}$ to be called fractional Wiener process, ε obeys standard normal distribution. We obtain the explicit option pricing formulas for European call option and put option for $\gamma > 0, 1 \le \alpha \le 2$.

It is worth to point out that the fractional Black-Scholes-Merton differential equation equations (19), (27) hold also for the stock price S_t satisfying $(dS_t)^{2H} = \mu(S_t)^{2H}(dt)^{2H} + \sigma(S_t)^{2H} dB_H(t)$, $B_H(t)$ is the fractional Brownian motion, H the Hurst exponent. Consequently, the option pricing formulas equations (62) and (63) with $\alpha = 2H$ are valid for H, $\frac{1}{2} \le H \le 1$.

The option pricing formula for $\alpha, 0 < \alpha < 1$ and for $H, 0 < H < \frac{1}{2}$ are still open problems.

Acknowledgments

This work is supported by the NSFC (No. 10771071, No. 10571121, No. 10571028, No. 10871047) and by Shanghai Leading Academic Discipline Project (No. B407).

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