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GENERAL SOLUTION FOR MULTIPLE FOLDINGS OF HEXAFLEXAGONS

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Abstract: This article explains hexaflexagons: how to make them, how to operate them, and their mathematical theory. Hexaflexagons are known to be surfaces with no inside or outside, similar to Möbius strips. Referring to the articles of Gardner and Madachy the author discovered a general solution for multiple foldings of hexaflexagons, which is described.

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1. Surfaces with no Inside or Outside

In the December 1990 edition of the 'Basic mathematics' magazine, I introduced a handmade puzzle known as a hexaflexagon under the title 'Folding Paper Hexaflexagons' [4]. It has been 10 years since then. The theoretical work related this puzzle has advanced significantly, and the puzzle is now understood. A new folding technique has in fact been developed. I would like to introduce this puzzle to those readers who do not know it, and explain its close relation to mathematics.

The puzzle was devised in 1939 by the English mathematician Arthur H. Stone, and is known as a hexaflexagon. Perhaps because the name hexaflexagon

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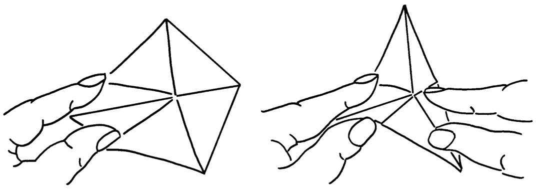

Figure 1: Revealing a new face

sounds unfamiliar it is often called an 'origami hexagon' or 'pleated origami' in Japanese, which all refer to the same thing.

The 'hexa' in hexaflexagon means six, and 'flexagon' indicates something that is flexible, easy to bend, and can take many shapes. There are flexagons in shapes other than hexagons, such as tetraflexagons, which are square, but the most interesting from both a theoretical and practical perspective, is the hexaflexagon.

I first heard how interesting this puzzle is in 1985, from a report by Shin'ichi Ikeno appearing in 'Mathematical Science' [2]. In fact the puzzle is not new to Japan, and resembles an old toy known as a *byoubugai*. The puzzle is made of paper and has a hexagonal shape. The hexagon is constructed from six triangles, and by squeezing two adjacent triangles between the thumb and index finger as shown in Figure 1, a new face can be revealed from the center.

Figure 2 illustrates a face which from a topological perspective has no inside or outside. Known as a Möbius strip, it is a normal loop glued together with a 180 degree twist, and was devised by the German astronomer A.F. Möbius (1790-1869). The twist may be to the left or right, and yields a connected surface for which an inside and outside cannot be distinguished. The Möbius strip involves a 180 twist but the hexaflexagon is made with a 540 degree twist; 540 degrees is 3 times 180 degrees. In general, gluing together a strip with an odd multiple of 180 degree twists yields a surface with no inside or outside, while an even multiple yields a face with an inside and an outside.

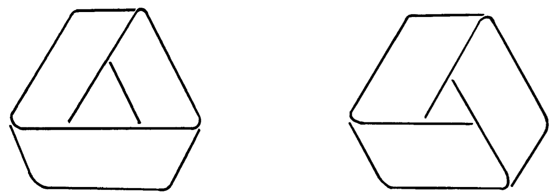

Figure 2: Surfaces with no inside or outside

2. 3 Face Folding

Now, fundamentals are important. The 3 face fold used for the hexaflexagon is a basic among basics, so I'd like for the reader to master it completely.

Ten equilateral triangles with sides of 6 cm are lined up sideways as shown in Figure 3(1). It should be possible to draw a diagram of this complexity with a ruler and compasses. The right hand edge of the 10 triangles is for gluing, so in fact 9 triangles are involved in the puzzle. The triangles each have inner and outer faces, so there are a total of $9 \times 2 = 18$ triangles. The hexaflexagon on the other hand, is composed of 6 triangles; $18 \div 6 = 3$, so mathematically, it is natural that it constitutes a 3 face folding.

While it may tally mathematically however, the appropriate arrangement of the triangles is key, and is explained below. Let's focus on the correct folding technique first. Make a valley fold along line $a - b$ (Figure 3(2)), a valley fold along line $c - d$ (Figure 3(3)), then without restricting the ¡glue¿ part, make a valley fold along line $e - f$ and glue (see Figure 3 (4)). This involves 3 valley folds which is a twist of $180 \times 3 = 540$ degrees.

Squeezing two adjacent triangles of the glued hexaflexagon in the way shown in Figure 1 causes a new face to appear naturally from the center. If it doesn't appear, try sliding back one of the triangles (at 60 degrees from the central angle) without pulling too hard. If it still doesn't appear then the hexaflexagon was constructed incorrectly and should be remade according to Figure 3.

Let's confirm that the hexaflexagon performs correctly. Fill in the numbers on the hexagonal face as shown in Figure 4, with '1' on the first face, '2' on the next face to appear, and '3' on the next. The cyclic order $1 \to 2 \to 3 \to 1 \to 2 \to 3$ of faces appearing is characteristic.

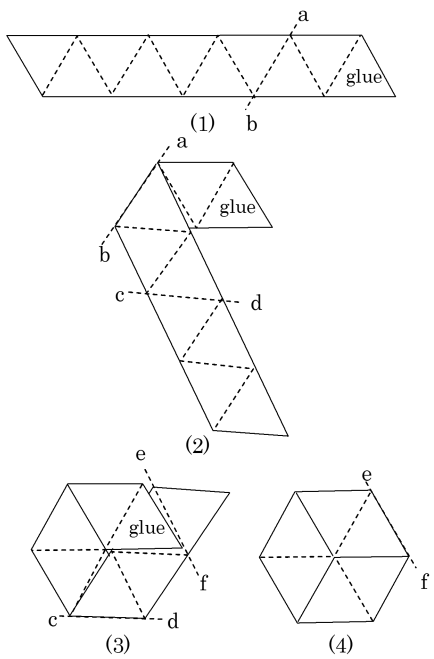

Figure 3: Folding order (3 face folding)

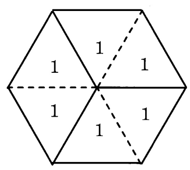

Figure 4: Numbering the faces

It is interesting to know the actual positions of the faces numbered 1 to 3. Figure 5 shows a hexaflexagon that has been peeled open and spread out again. It shows the flaps of paper with the numbers (1), (2) and (3) written on the underside. The same numbers are not written on continuous areas, but pairs of two are lined up in equally spaced positions on both sides. Considering the folding relationship shown in Figure 1, for every fold, the triangle in Figure 5 is offset by 2 steps. The hexaflexagon is thus a single long thin segmented face seen in a staggered manner.

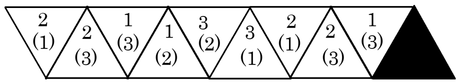

Figure 5: The relationship among the three faces

3. Martin Gardner's Paper Templates

The report from 1990 introduced above only discussed 3 face folding. My own interest moved on to the question of whether there are folding methods for larger numbers of faces.

Martin Gardner's 'The Scientific American Book of Mathematical Puzzles and Diversions' contains an article introducing the hexaflexagon, and on page 25 there are paper templates for between 4 and 7 face folds [1]. The book only contains paper templates for folding diagrams and doesn't include an explanation of the folding technique. Since no solution is printed it is necessary to find one through one's own efforts. After repeatedly failing many times, and thinking to myself *'not like thisc not like thatc',* I eventually succeeded in making these models.

When making hexaflexagons with many faces ($n \geq 4$), it becomes clear that not only the theory, but also the actual paper used for construction, and techniques for making diagrams and so on also become problematic. When I first heard of the puzzle in around 1985, I used drawing paper, a ruler and compasses to make the diagrams. This is reasonable when handling only a three face fold, but as the number of faces increases, the accuracy of the diagrams becomes more of a requirement. The lead in a pencil is 0.3 mm, and the graduations on a ruler are in units of 1 mm, so no matter how carefully the diagram is drawn the error in a hand drawn diagram must be at least around 0.1 mm. Even supposing that the error in a single triangle is 0.1 mm, when 10 triangles are included the error accumulates and reaches 1 mm. When making a 12 face fold, the number of triangles is 37 so the error is 3.7 mm and cannot be ignored. Also, drawing paper was used at first, but while drawing paper appears to be strong, it is surprisingly useless. It often tears during bending and folding.

Based on these experiences I abandoned the ruler and compasses, and instead made the diagrams using the language known as Visual Basic. When a computer is used, the hand drawing error of 0.1 mm and accumulated error of

3.7 mm do not arise and the result is considerably more accurate. Drawing paper is weak when it comes to bending and folding, so normal photocopy paper was used instead. I suppose that the quality of the fibrous material must be different. Lastly, although numbers were first written on the faces in order to distinguish them, I gradually realized that classifying them by color was more appealing and therefore filled them in using colored pencils. Copy paper is thin however, and the color shows through to the other side, so colored origami paper was attached using glue.

4. Reduction to a Fundamental Pattern

Now, allow me to explain how I achieved the 4 to 8 face foldings. Figure 6 shows the arrangement of a paper template for a particular representative example. The black triangle is used as an overlap for gluing, and has no relation to the actual appearance.

The 6 face folding is comparatively easy, so let's begin there. The template for the 6 face folding is simply two templates for the 3 face folding (Figure 3(1)) glued together side by side. The number of triangles is 18, but there is one extra used as an overlap (colored black) so the total is in fact 19. If the model is folded from the right hand edge in an orderly manner using a right twist rule, it is the same as the 3 face folding. The 6 face folding may thus be achieved by applying the 3 face folding.

Long straight paper strips such as the 3 face folding and the 6 face folding are referred to as 'straight models' by Joseph Madachy [3]. These straight models are formed according to the following equation

$$n = 3 \times 2^p \qquad (p \geq 0, 1, 2, \cdots).$$

Substituting the shown values for p yields $n = 3, 6, 12, 24, \cdots$, meaning that the 3 face, 6 face, 12 face, and 24 face foldings are possible with this method. Indeed, $n = \infty$, that is to say a model with an infinite number of faces, is also possible in theory.

The basis of the remaining models is a reduction to the fundamental pattern of the straight models (Figure 7). Regarding the folding technique, let's look at the 4 face and 7 face foldings.

For the 4 face folding, by taking the 3 parts below the dotted lines in bottom-up order, and folding using a right twist rule 3 times, the 3 face folding may be applied. The layered parts are indicated in gray, and since these parts

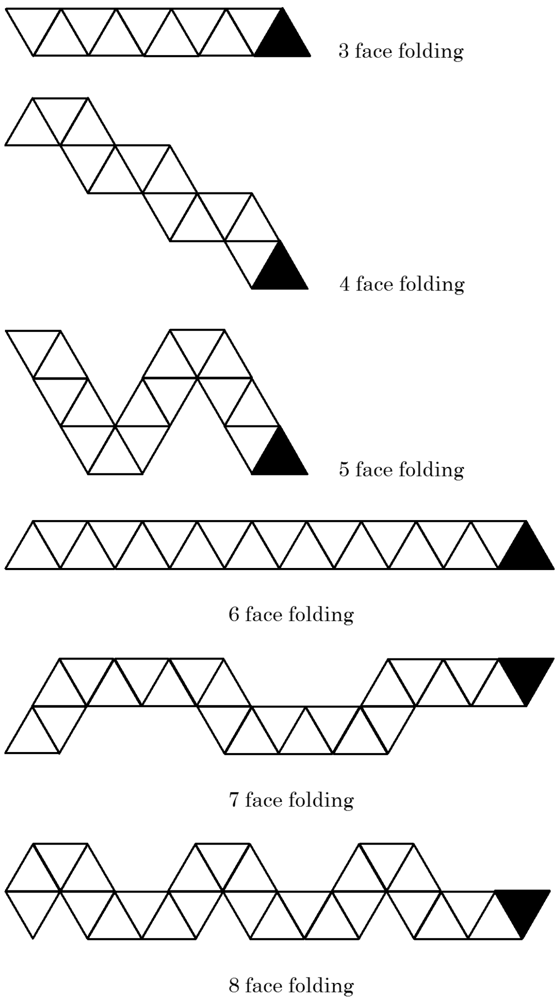

Figure 6: 3 to 8 face foldings (paper templates)

form a new face, they are marked '4'. The 4 face folding may be completed by applying the 3 face folding to the layered state.

For the 7 face folding, by taking the 3 parts inside the dotted lines in right-left order, and folding using a right twist rule 3 times, the template for the 6 face folding may be applied. The number '7' was written on the layered parts. This 6 face folding template may be completed by transforming it and applying

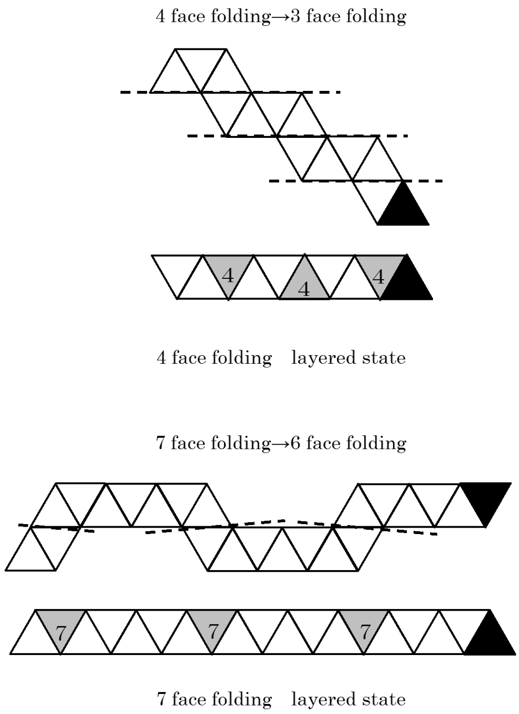

Figure 7: Reduction to the fundamental pattern

the template for the 3 face folding. In short, this is a 7 face folding → 6 face folding → 3 face folding procedure.

5. Transition Diagram

If the model is folded up as above, it is certain that only the target number of faces will be revealed. What however, is the order in which the faces appear? The answer may be found by referring to the transition diagram in Figure 8. I drew up this diagram by referring to the work of Joseph Madachy [3].

In the case of the 3 face folding ($n = 3$) the transition diagram is expressed as a triangle. The numbers 1, 2, and 3 written at the tips of the triangle are the numbers of the faces. There is a plus (+) symbol inside the triangle, and this

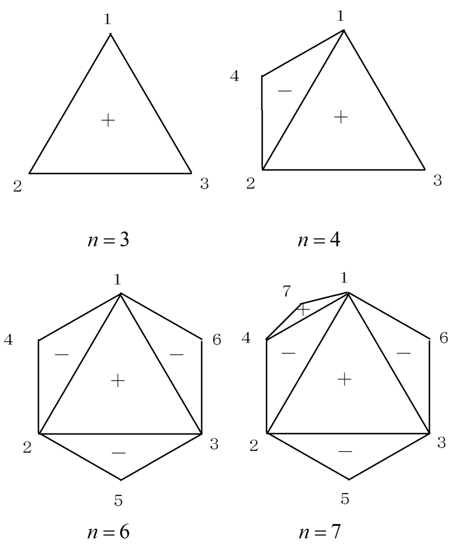

Figure 8: Transition diagrams

signifies that the face numbers cycle in an anticlockwise manner $1 \to 2 \to 3$.

In the case of the 4 face folding ($n = 4$), a new triangle has been added to the transition diagram of the 3 face folding ($n = 3$) in the area between tips 1 and 2. This is the triangle related to the new face with number 4. The triangle is marked inside with a minus ($-$) symbol, signifying that the face numbers cycle in a clockwise manner $1 \to 2 \to 4$. There are thus two cycles existing in the 4 face folding: the plus (+) cycle $1 \to 2 \to 3$, and the minus ($-$) cycle $1 \to 2 \to 4$. For example, to go from 3 to 4 it is not possible to advance directly through $3 \to 1 \to 4$. Instead, by advancing in the $1 \to 2 \to 3$ plus (+) cycle through $3 \to 1 \to 2$, and then advancing in the $1 \to 2 \to 4$ minus ($-$) cycle through $2 \to 4$, the target can be reached. In this case, 2 acts as a relay point.

In the case of the 6 face folding ($n = 6$), three triangles are added to the transition diagram of the 3 face folding. Around the plus (+) cycle $1 \to 2 \to 3$, there are three minus ($-$) cycles $1 \to 2 \to 4$, $2 \to 3 \to 5$, and $1 \to 6 \to 3$.

In the case of the 7 face folding ($n = 7$), a triangle with a plus cycle $1 \to 7 \to 4$ is added to the outer edge of the transition diagram for the 6 face folding ($n = 6$).

The transition diagram for an n face folding thus complies with an n sided polygon, and this n sided polygon is partitioned into $n-2$ triangles such that the adjacent triangles have a different symbol (indicating the cycle direction). By constructing the transition diagram, the operations needed to reveal a particular face may be performed smoothly.

6. General Solution for Multiple Foldings

Paper templates for the 3 to 8 face foldings are shown in Figure 6, and an explanation summarizing the folding processes is shown in Figure 7, but how should foldings for 9 or more faces be handled? Allow me to explain how to make templates for foldings of more than 9 faces.

To begin with, the existence of the fundamental pattern of the straight model is just as stated above. Expressed as $n = 3 \times 2^p$ $(p \geq 0, 1, 2, \cdots)$ the values are $n = 3, 6, 12, 24, \cdots$ and so on. The templates for the 12 and 24 face foldings are long thin strips. So what happens with larger values of n? Just as the 7 to 11 face foldings may be reduced to the fundamental pattern with the 6 face folding as a base, so the 13 to 23 face foldings may be reduced with the 12 face folding as a base.

The straight model which is the base for this process is colored gray where layered parts occur, and opening out these areas reciprocally yields the template for the desired n face folding. For more details refer to [5].

I was able to construct paper templates for all the models such that $9 \leq n \leq 24$, and by folding them confirm that they could all be produced in accordance with theory. I proceeded to complete the simple models first, and the 19 face model remained unresolved until the end. When $n = 19$ it is prime, and I was worried that this model might not be possible, but I settled on the positions by trial and error, and producing an expansion diagram revealed a snake-like form (Figure 9).

It was demonstrated above that the cases when $3 \leq n \leq 24$ are possible, but this does not constitute mathematical proof for the case of arbitrary. Diligently investigating the cases when $n \geq 25$ will probably not reveal any problems, but using actual materials to make the models and confirm their construction is painful, and this may be thought of as the limit.

Drawing the templates using a ruler and compasses takes time and leads to errors. I therefore made a versatile model that may be applied to all the templates (Figure 10). This was achieved using about 30 lines of *Visual Basic*

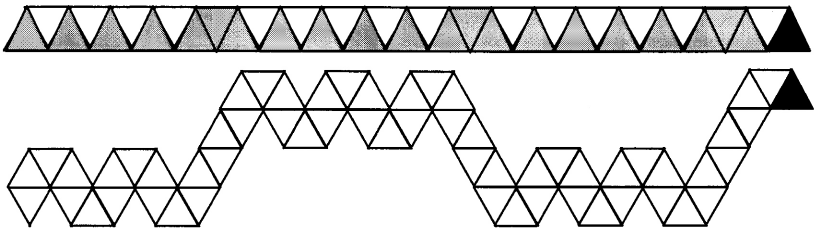

Figure 9: Template for the 19 face folding

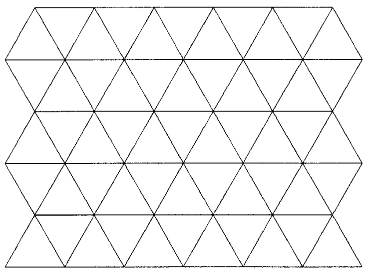

Figure 10: Versatile template made with *Visual Basic*

instructions. The paper template needed for an face folding may be cut out from this template using a pair of scissors.

First of all, it is impressive see the phenomenon of the hexaflexagon with the 3 face folding. One may next wonder if a 4 face folding is possible. Seeing that the 4 face folding is possible, one may wonder whether 5, 6, and arbitrary n face foldings are possible. This thought process is similar to the methods of *extension, generalization, continuity,* and *equivalence* used in mathematics. The fundamental 3 face folding is quite impressive by itself, and I'd be very pleased if those readers who have not experienced this puzzle would try it and see.

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