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VARIATIONAL ITERATION METHOD FOR SOLVING SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract: In this work, He's variational iteration method is used for the solution of singularly perturbed two-point boundary value problems. Some problems are solved to demonstrate the applicability of the method. It is observed that a good choice of the freely selected initial approximation in the VIM leads to exact solutions for some examples or gives a very well approximate solutions by using only one iteration. VIM has been applied to the initial value problems

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until now. In this work, VIM is used for the solution of singularly perturbed two-point boundary value problems.

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1. Introduction

We consider the singularly perturbed two-point boundary value problems

$$\epsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad x \in [a,b], \tag{1}$$

subject to the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \tag{2}$$

where α, β are real constants and ϵ is a small positive parameter $(0 < \epsilon \ll 1)$. We assume that $p(x), q(x)$ and $f(x)$ are sufficiently continuously differentiable functions on $[a,b]$. This type of two-point boundary value problems (1)-(2) are generally encountered in fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process, and geophysics. In these problems a small parameter is multiplied to a highest derivative. The solution varies rapidly in some parts and varies slowly in some other parts. There are thin transition boundary or interior layers where the solutions can change rapidly, while away from the layers the solution behaves regularly and vary slowly. There is a wide variety of techniques for solving singular perturbation problems [1], [11], [14], [16], [8], [17], [15], [10]. Furthermore different numerical methods have been proposed by various authors for singularly perturbed two-point boundary value problems, such as non-uniform mesh tension spline methods [12], non-uniform mesh compression spline numerical method [13], and least squares methods based on Bézier control points [2]. The aim of our study is to introduce He's variational iteration method [7], [3], [5], [6], [4] as an alternative to existing methods in solving singularly perturbed two-point boundary value problems and the method is implemented to four numerical examples. According to the authors knowledge this paper represents the first application of the variational iteration method to singularly perturbed two-point boundary value problems.

The rest of the paper is organized as follows: In Section 2, we give a description of the method. In Section 3, we have solved four numerical examples to demonstrate the applicability of the present method. The discussion of our results is given in Section 4.

2. Variational Iteration Method

The variational iteration method (VIM) is proposed by the Chinese mathematician Ji-Huan He and it has recently been intensively studied by many scientists to apply different type of problems. By He's method we introduce the following correction functional corresponding to (1)

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\{\epsilon y''(t) + p(t)y'(t) + q(t)y(t) - f(t)\}dt, \tag{3}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory. By making the correction functional stationary with restricted variations $\delta y_n(x) = 0, \delta y_n'(x) = 0$, we obtain

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(x,t)\{\epsilon y''(t) + p(t)y'(t) + q(t)y(t) - f(t)\}dt$$

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \epsilon \int_0^x \lambda(x,t)\frac{d^2}{dt^2}\delta y_n(t)dt + \int_0^x \lambda(x,t)\frac{d}{dt}p(t)\delta y_n(t)dt \\ &+ \int_0^x \lambda(x,t)\delta q(t) y_n(t)dt. \end{aligned} \tag{4}$$

Integrating (4) by parts yields

$$\begin{aligned} \delta y_{n+1}(x) &= \left(1 - \epsilon\frac{\partial\lambda(x,t)}{\partial t} + p(t)\lambda(x,t)\right)\delta y_n(t)_{\downarrow t=x} + \epsilon\lambda(x,t)\frac{d}{dt}\delta y_n(t)_{\downarrow t=x} \\ &+ \int_0^x \left(\epsilon\frac{\partial^2\lambda(x,t)}{\partial t^2} - p(t)\frac{\partial\lambda(x,t)}{\partial t} + q(t)\lambda(x,t)\right)\delta y_n(t)dt. \end{aligned} \tag{5}$$

Therefore, by imposing the above restricted variation terms to (5) we obtain the following Euler-Lagrange equation

$$\begin{gathered} \epsilon\frac{\partial^2\lambda(x,t)}{\partial t^2} - p(t)\frac{\partial\lambda(x,t)}{\partial t} + q(t)\lambda(x,t) = 0, \\ \left(1 - \epsilon\frac{\partial\lambda(x,t)}{\partial t} + p(t)\lambda(x,t)\right)_{\downarrow t=x} = 0, \\ \lambda(x,t)_{\downarrow t=x} = 0. \end{gathered} \tag{6}$$

3. Applications and Results

To incorporate our discussion above, four special cases of the singularly perturbed two-point boundary value problem (1)-(2) will be studied. We do comparison between one-iterative variational iteration solutions and the exact solution for each application. All the results are calculated by using the symbolic calculus software *Mathematica*.

Example 1. We consider the following simple example

$$\epsilon y''(x) = 2, \quad y(0) = 1, \quad y(1) = 2.$$

Solving (6) using the coefficients $p(x) = 0$, $q(x) = 0$, then λ can easily identified as

$$\lambda(x,t) = \frac{t-x}{\epsilon}.$$

Therefore, we have the following iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\{\epsilon y_n''(t) - 2\}dt.$$

Now, we begin with the following initial approximation

$$y_0(x) = a_1 + a_2 x,$$

where a_1 and a_2 are constants to be determined. By the above iteration formula, we have

$$y_1(x) = a_1 + a_2 x + \frac{x^2}{\epsilon}.$$

Applying the boundary conditions yield $a_1 = 1$ and $a_2 = 1 - \frac{1}{\epsilon}$. Thus

$$y_1(x) = 1 + \left(1 - \frac{1}{\epsilon}\right) x + \frac{x^2}{\epsilon},$$

which is the exact solution.

Example 2. As a second example we study the singularly equation [9]

$$\epsilon y''(x) - y(x) = 0 \ \ ; \ \ y(0) = 1 \ \ , \ \ y(1) = 0.$$

Solving (6) using the coefficients $p(x) = 0$, $q(x) = -1$, then λ can easily identified as

$$\lambda(x,t) = \frac{e^{\frac{t-x}{\sqrt{\epsilon}}} - e^{\frac{x-t}{\sqrt{\epsilon}}}}{2\sqrt{\epsilon}}.$$

Therefore, we have the following iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\{\epsilon y_n''(t) - y_n(t)\}dt.$$

Now, we begin with the following initial approximation

$$y_0(x) = b_1 e^{\frac{x}{\sqrt{\epsilon}}} + b_2 e^{\frac{-x}{\sqrt{\epsilon}}},$$

where b_1 and b_2 are constants to be determined. By the above iteration formula, we have

$$y_1(x) = b_1 e^{\frac{x}{\sqrt{\epsilon}}} + b_2 e^{\frac{-x}{\sqrt{\epsilon}}}.$$

Applying the boundary conditions yield

$$b_1 = -b_2 = \frac{e^{\frac{1}{\sqrt{\epsilon}}}}{e^{\frac{2}{\sqrt{\epsilon}}} - 1}.$$

Thus

$$y_1(x) = \frac{e^{\frac{1-x}{\sqrt{\epsilon}}}\left(e^{\frac{2x}{\sqrt{\epsilon}}} - 1\right)}{e^{\frac{2}{\sqrt{\epsilon}}} - 1},$$

which is the exact solution.

Example 3. We consider the following singularly perturbed two-point boundary value problem

$$\epsilon y''(x) + y(x) = 0, \quad y(0) = 0, \quad y(1) = 1.$$

Solving (6) using the coefficients $p(x) = 0$, $q(x) = 1$, then λ can easily identified as

$$\lambda(x,t) = \frac{1}{\sqrt{\epsilon}} \sin \frac{t-x}{\sqrt{\epsilon}}.$$

Therefore, we have the following iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\{\epsilon y_n''(t) + y_n(t)\}dt.$$

Now, we begin with the following initial approximation

$$y_0(x) = c_1 \sin \frac{x}{\sqrt{\epsilon}} + c_2 \cos \frac{x}{\sqrt{\epsilon}},$$

where c_1 and c_2 are constants to be determined. By the above iteration formula, we have

$$y_1(x) = c_1 \sin \frac{x}{\sqrt{\epsilon}} + c_2 \cos \frac{x}{\sqrt{\epsilon}}.$$

Applying the boundary conditions yield

$$\begin{aligned} c_1 &= \csc \frac{1}{\sqrt{\epsilon}}, \\ c_2 &= 0. \end{aligned}$$

Thus

$$y_1(x) = \csc\frac{1}{\sqrt{\epsilon}}\sin\frac{x}{\sqrt{\epsilon}},$$

which is the exact solution.

Example 4. Now, we consider the following example [16]

$$\epsilon y''(x) - y'(x) = 0; \quad y(0) = 1, \quad y(1) = 1.$$

The exact solution is given by

$$y(x) = \frac{e^{\frac{x-1}{\epsilon}} - 1}{e^{\frac{-1}{\epsilon}} - 1}.$$

Solving (6) using the coefficients $p(x) = -1$, $q(x) = 0$, then λ can easily identified as

$$\lambda(x,t) = 1 - e^{\frac{x-t}{\epsilon}}.$$

Therefore, we have the following iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t)\{\epsilon y_n''(t) - y_n'(t)\}dt.$$

Now, we begin with the following initial approximation

$$y_0(x) = d_1 + d_2 e^{-\frac{x}{\epsilon}},$$

where d_1 and d_2 are constants to be determined. By the above iteration formula, we have

$$y_1(x) = (d_1 - 2d_2) + 4d_2 e^{-\frac{x}{\epsilon}} - d_2 e^{\frac{x}{\epsilon}}.$$

Applying the boundary conditions yield

$$d_1 = \frac{-2 + 4e^{-\frac{1}{\epsilon}} - e^{\frac{1}{\epsilon}}}{-3 + 4e^{-\frac{1}{\epsilon}} - e^{\frac{1}{\epsilon}}},$$

$$d_2 = \frac{-1}{-3 + 4e^{-\frac{1}{\epsilon}} - e^{\frac{1}{\epsilon}}}.$$

Thus

$$y_1(x) = \frac{4\left(e^{-\frac{1}{\epsilon}} - e^{-\frac{x}{\epsilon}}\right) - e^{\frac{1}{\epsilon}} + e^{\frac{x}{\epsilon}}}{-3 + 4e^{-\frac{1}{\epsilon}} - e^{\frac{1}{\epsilon}}},$$

The error differences between the exact solution $y(x)$ and the obtained approximate solution $y_1(x)$ for different interval values of ϵ are shown in Figures 1 and 2. It can be seen that the solution obtained by the present method is nearly identical with the exact solution. Also, it is to be noted that only the first order term of the variational iteration solution was used in evaluating the approximate solutions. It is evident that the efficiency of this approach can be

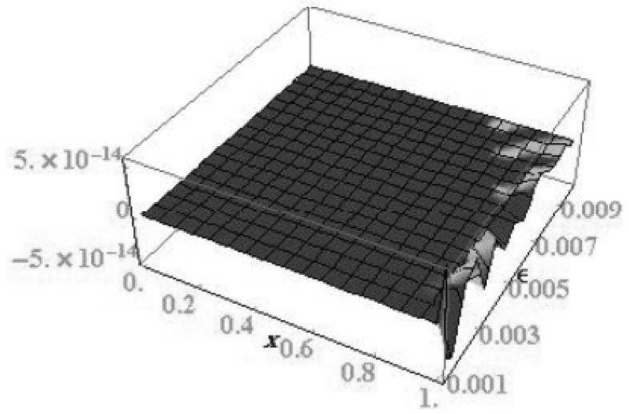

Figure 1: The error for Example 4 using only 1-iterative solutions for $0.001 \leq \epsilon \leq 0.01$

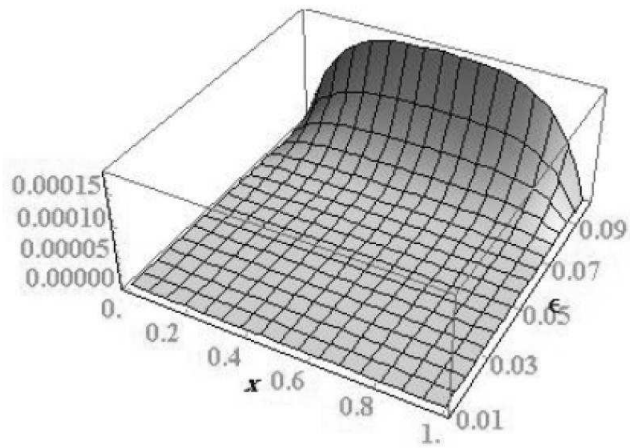

Figure 2: The error for Example 4 using only 1-iterative solutions for $0.01 \leq \epsilon \leq 0.1$

dramatically enhanced by either computing further terms of $y_n(x)$ with freely selection of $y_0(x)$, or by a good choice of the initial approximation function where in the presented examples $y_0(x)$ was the general form of the obtained Lagrange multiplier function.

4. Conclusion

This present analysis exhibits the applicability of the variational iteration method to solve linear singularly perturbed two-point boundary value problems. It may be concluded that the method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of singularly perturbed two-point boundary value problems.

Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear singularly perturbed two-point boundary value problems.

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