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NONLINEAR MIXED EQUATIONS IN MULTIPLY CONNECTED DOMAINS

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Abstract: The present paper deals with the oblique derivative problem for some nonlinear mixed equations with degenerate rank 0 in multiply connected domains. We first give the representation of solutions of the boundary value problem for the equations, and then prove the existence of solutions for the problem by using the complex analytic method.

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1. Formulation of Oblique Derivative Problem

In [1]-[6], the authors introduced and discussed the Tricomi problem for some second order equations of mixed type. In [3], the author posed the exterior Tricomi problem for the mixed equation $K(y)u_{xx}+u_{yy}+r(x,y)u=f(x,y)$ in a doubly connected domain and proved the uniqueness of solutions for the problem. In [6], the authors discussed the solvability of the general Tricomi-Rassias problem and oblique derivative problem for generalized Chaplygin equations. The present paper concerns the oblique derivative problem for some nonlinear mixed equations with degenerate rank 0 in multiply connected domains. Besides by the similar method in [10], we can discuss the Frankl problem for nonlinear mixed equations with degenerate rank 0 in multiply connected domains.

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We consider two functions $K_l(y)\,(l=1,2)$ as follows

$$K_l(y)\begin{cases} >0 \ \text{ for } \ \{y>0\}\cup\{y<-1\}, \\ =0 \ \text{ for } \ \{y=0\}\cup\{y=-1\}, \quad l=1,2, \\ <0 \ \text{ for } \ \{-1<y<0\}, \end{cases}$$

where $K_l(y)\,(l=1,2)$ are continuous in $\overline{D}\cap\{y\geq -1/2\}$ and $\overline{D}\cap\{y\leq -1/2\}$, possess the derivatives $K_l'(y)\,(l=1,2)$ on $y\neq 0,-1/2$ and -1. In this paper we choose $\text{sgn}yK_l(y)=|y|^{m_l}h_l(y), y\geq -1/2, -\text{sgn}(1+y)K_l(y)=|y+1|^{m_l}h_l(y),\ y\leq -1/2$, in which $m_l\,(m_1>[m_2],\ l=1,2)$ are positive numbers, $h_l(y)\,(l=1,2)$ are continuously differentiable positive functions, $K(y)=K_1(y)/K_2(y)$, $H(y)=\sqrt{|K(y)|}$ and $H_l(y)=\sqrt{|K_l(y)|}, l=1,2$. The multiply connected domain $D=G_0\cup G_0'\cup G_1\cup G_1'\cup ...\cup G_N\cup G_N'\cup(A_1B_1\backslash\{a_0,...,a_N\})\cup(A_2B_2\backslash\{a_0',...,a_N'\})$ possesses the exterior boundary $\text{Ext}(D)=\Gamma_0\cup\Gamma_0'\cup\Gamma_1\cup\Gamma_1'\cup\Gamma_{2N}\cup\Gamma_{2N}'$ and interior boundary $Int(D)=\Gamma_2\cup\Gamma_2'\cup\Gamma_3\cup\Gamma_3'\cup...\cup\Gamma_{2N-1}\cup\Gamma_{2N-1}'$. Here A_1B_1, A_2B_2 are two line segments with end points $A_1=(a_0,0), B_1=(a_N,0), A_2=(a_0,-1), B_2=(a_N,-1)$. Moreover:

Γ_0 is the elliptic arc for $y>0$ connecting the points A_1, B_1,

Γ_0' is the elliptic arc for $y<-1$ connecting the points A_2, B_2,

$\Gamma_{2l-1}: x=a_{l-1}-G_1(y)=a_{l-1}-\int_0^y\sqrt{K(t)}dt$ is a characteristic curve for $-1/2<y<0$, $a_{l-1}<x<x_{2l-1}$ emanating from the point $O_{l-1}=(a_{l-1},0)=a_{l-1}$, $l=1,...,N$,

$\Gamma_{2l-1}': x=a_{l-1}+G_2(y)=a_{l-1}+\int_{-1}^y\sqrt{K(t)}dt$ is a characteristic curve for $-1<y<-1/2$, $a_{l-1}<x<x_{2l-1}$ emanating from the point $O_{l-1}'=(a_{l-1},-1)=a_{l-1}'$, $l=1,...,N$,

$\Gamma_{2l}: x=a_l+G_1(y)=a_l+\int_0^y\sqrt{K(t)}dt$ is a characteristic curve for $-1/2<y<0$, $x_{2l}<x<a_l$ emanating from the point $O_l=(a_l,0)=a_l$, $l=1,...,N$,

$\Gamma_{2l}': x=a_l-G_2(y)=a_l-\int_{-1}^y\sqrt{K(t)}dt$ is a characteristic curve for $-1<y<-1/2$, $x_{2l}<x<a_l$ emanating from the point $O_l'=(a_l,-1)=a_l'$, $l=1,...,N$, and $G_0=D_1^+=D\cap\{y>0\}$ is a upper elliptic domain, $G_0'=D_2^+=\{D\cap\{y<-1\}$ is a lower elliptic domain, $G_l=D\cap\{a_{l-1}<x<a_l,-1<y<0\}\,(l=1,...,N)$ are hyperbolic domains with the boundaries $\partial G_0=\Gamma_0\cup(A_1B_1)$, $\partial G_0'=\Gamma_0'\cup(B_2A_2)$, $\partial G_l=\Gamma_{2l-1}\cup\Gamma_{2l-1}'\cup\Gamma_{2l}\cup\Gamma_{2l}'\cup\{a_{l-1}<x<a_l, y=0\}\cup\{a_{l-1}<x<a_l, y=-1\}\,(l=1,...,N)$ respectively. The above characteristic curves intersect at the points: $\Gamma_{2l-1}\cap\Gamma_{2l-1}'=P_{2l-1}=z_{2l-1}=(x_{2l-1},-1/2), \Gamma_{2l}\cap\Gamma_{2l}'=P_{2l}=z_{2l}=(x_{2l},-1/2)\,(l=1,...,N)$. Here note the difference between the domains $G_1, G_2, ...$ and the functions $G_1(y), G_2(y), ...$ In this paper we use the complex number $x+iy$ in elliptic domain D^+, and the hyperbolic number $x+jy$ in hyperbolic domain D^-, where j is the hyperbolic

unit such that $j^2 = 1$ (see [10]).

There is no harm in assuming that the partial boundaries Γ_0, Γ_0' of the domains G_0, G_0' are smooth curves including line segments Re$Z = a_0, a_N$ near $Z = a_0, a_N, a_0 - i, a_N - i$ respectively, because it can be realized through the appropriate conformal mapping as stated in [10]. In general if the inner angle of $D_Z^+ = D_Z \cap \{|y - 1/2| > 1/2\}$ at $Z = a_0, a_N, a_0 - i, a_N - i$ are less than $\pi/2$, then we need multiply the new equation (21) below by $X(Z) = (Z - a_0)(Z - a_N)(Z - a_0 + i)(Z - a_N + i)$, the equation multiplied by $X(Z)$ in the domain D_Z^+ satisfies Condition C, or we assume that the coefficients of (1) satisfy the conditions $\eta/|X(Z)| \in \tilde{L}_\infty(D^+)$, $\eta = a, b, c, d$. If the inner angle of D_Z^+ at $Z = a_0, a_N, a_0 - i, a_N - i$ are greater than $\pi/2$, then the new equation satisfies Condition C still. Besides if we multiply the complex equation (21) and boundary condition (4) below by $X(Z) = \prod_{l=0}^{N}(Z - a_l)(Z - a_l')$, then the index $K = 0$ of the boundary condition (4) in $D^+ \cap \{y > 0\}$ can be transformed into the new index $\tilde{K} = (N - 1)/2$ of the new equation and boundary value conditions in $D_Z \cap \{y > 0\}$, hence we can have $N + 1$ points conditions in the boundary condition (4) below. For the domain $D^+ \cap \{y < -1\}$, the problem can be similarly handled.

Consider the nonlinear mixed equation

$$K_1(y)u_{xx} + |K_2(y)|u_{yy} + a(x,y)u_x + b(x,y)u_y + c_*(x,y,u)u = -d(x,y) \text{ in } D. \quad (1)$$

Suppose that the coefficients of (1) satisfy Condition C, namely

1) For any continuously differentiable function $u(z)$ in $D^* = \overline{D}\backslash\{a_0, a_0', ..., a_N, a_N', P_1P_2, ..., P_{2N-1}P_{2N}\}$, a, b, c, d satisfy:

$$\begin{aligned}
&\tilde{L}_\infty[\eta/|y|^{[m_2]}, D^+] = L_\infty[\eta/|y|^{[m_2]}, D^+] + L_\infty[\eta_x/|y|^{[m_2]}, D^+] \le k_0,\\
&\tilde{C}[\eta/|y|^{[m_2]}, \overline{D^-}] = C[\eta/|y|^{[m_2]}, \overline{D^-}] + C[\eta_x/|y|^{[m_2]}, \overline{D^-}] \le k_0, \quad \eta = a, b, c,\\
&\tilde{L}_\infty[d/|y|^{[m_2]}, D^+] \le k_1, \quad c \le 0 \quad \text{in } D^+, \quad \tilde{C}[d/|y|^{[m_2]}, \overline{D^-}] \le k_1,\\
&a|y|/H_1H_2 = \varepsilon(y), \quad m_1 + m_2 - 2[m_2] \ge 2 \ \text{ in } D^-,
\end{aligned} \quad (2)$$

where $c_* = [c - |u|^\sigma]|y|^{[m_2]}$, σ, k_0, $k_1 (\ge \max[1, 6k_0])$ are positive constants, and $\varepsilon(y) \to 0$ as $y \to 0$.

2) For any functions $u_1(z), u_2(z) \in C(D^*)$, $F(z, u, u_z) = au_x + bu_y + cu + d$ satisfies the condition

$$F(z, u_1, u_{1z}) - F(z, u_2, u_{2z}) = \tilde{a}(u_1 - u_2)_x + \tilde{b}(u_1 - u_2)_y + \tilde{c}(u_1 - u_2) \text{ in } \overline{D},$$

in which $\tilde{a}, \tilde{b}, \tilde{c}$ satisfy conditions similar to those of a, b, c in (2). According to Condition C, the equation (1) is divided by $|y|^{[m_2]}$, we get the equation in the form

$$\hat{K}_1(y)u_{xx} + |\hat{K}_2(y)|u_{yy} + \hat{a}u_x + \hat{b}u_y + \hat{c}u + \hat{d} = 0 \text{ in } D, \quad (3)$$

in which $\hat{K}_1(y) = K_1(y)/|y|^{[m_2]}, \hat{K}_2(y) = K_2(y)/|y|^{[m_2]}$, $\hat{a} = a/|y|^{[m_2]}$, $\hat{b} = b/|y|^{[m_2]}$, $\hat{c} = c_*/|y|^{[m_2]}$, $\hat{d} = d/|y|^{[m_2]}$. Denote $\hat{H}_l(y) = [|y|^{\hat{m}_l} h_l(y)/h_2(y)]^{1/2}$, $\hat{m}_l = m_l - [m_2]$, $l = 1, 2$, $\hat{m} = \hat{m}_1 - \hat{m}_2 = m$, $\hat{H}(y) = \hat{H}_1(y)/\hat{H}_2(y) = H(y)$ and the corresponding function $\hat{G}(y) = G(y)$ with $0 < \hat{m}_2 < 1$. Hence we can only discuss the equation in the form (1) with the condition $0 \le m_2 < 1$ later on. It is clear that equation (1) with the condition $a = b = c_* = d = 0$ is the so-called Chaplygin equation.

The oblique derivative boundary value problem for equation (1) may be formulated as follows:

Problem P. Find a continuous solution $u(z)$ of equation (1) in $\overline{D}$, where u_x, u_y are continuous in $D^* = \overline{D}\backslash\{a_0, a_0', ..., a_N, a_N', P_1P_2, ..., P_{2N-1}P_{2N}\}$, and satisfy the boundary conditions

$$\begin{gathered}\frac{1}{2}\frac{\partial u}{\partial \nu} = \frac{1}{H_1(y)}\mathrm{Re}[\overline{\lambda(z)}u_{\tilde{z}}] = \mathrm{Re}[\overline{\Lambda(z)}u_z] = r(z) \ \text{ on } \Gamma \cup L,\\ \Gamma = \Gamma_0 \cup \Gamma_0',\ L = \tilde{L} \cup \tilde{L}', \tilde{L} = \Gamma_1 \cup \Gamma_3 ... \cup \Gamma_{2N-1}, \tilde{L}' = \Gamma_1' \cup \Gamma_3' ... \cup \Gamma_{2N-1}',\\ \frac{1}{H_1(y)}\mathrm{Im}[\overline{\lambda(z)}u_{\tilde{z}}]|_{z=z_{2l-1}} = b_l, l=1, ..., N, u(t_{lk}) = d_{lk}, l=0,1,...,N, k=1,2.\end{gathered} \tag{4}$$

Herein $t_{l1} = a_l, t_{l2} = a_l'$ $(l = 0, 1, ..., N)$, ν is a given unit vector at every point $z \in \Gamma \cup L$, $u_{\tilde{z}} = [H_1(y)u_x - iH_2(y)u_y]/2$, $\Lambda(z) = \cos(l, x) - i\cos(l, y), \lambda(z) = \mathrm{Re}\lambda(z) + i\mathrm{Im}\lambda(z)$, if $z \in \Gamma$, and $u_{\tilde{z}} = [H_1(y)u_x - jH_2(y)u_y]/2$, $\lambda(z) = \mathrm{Re}\lambda(z) + j\mathrm{Im}\lambda(z)$, if $z \in L = \tilde{L} \cup \tilde{L}'$, b_l $(l \in J_1 = \{1, ..., N\})$, d_{lk} $(l \in J_2 = \{0, 1, ..., N\})$, $k \in J_3 = \{1, 2\})$ are real constants, and $r(z), b_l(l \in J_1), d_{lk}$ $(l \in J_2, k \in J_3)$ satisfy the conditions

$$\begin{gathered}C^1_\alpha[\lambda(z), \Gamma] \le k_0,\quad C^1_\alpha[\lambda(z), L] \le k_0,\quad C^1_\alpha[r(z), \Gamma] \le k_2,\quad C^1_\alpha[r(x), L] \le k_2,\\ \cos(l, n) \ge 0 \ \text{ on } \Gamma \cup L,\quad |b_l| \le k_2 \ (l \in J_1),\quad |d_{lk}| \le k_2 \ (l \in J_2, k \in J_3),\\ \max_{z\in\tilde{L}} \frac{1}{|\mathrm{Re}\lambda(z) - \mathrm{Im}\lambda(z)|} \le k_0,\quad \max_{z\in\tilde{L}'} \frac{1}{|\mathrm{Re}\lambda(z) + \mathrm{Im}\lambda(z)|} \le k_0,\end{gathered} \tag{5}$$

in which n is the outward normal vector at every point on Γ, α $(0 < \alpha < 1)$, k_0, k_2 are non-negative constants.

For convenience we can assume that $\lambda(x)$, $r(x)$ on L are functions of x as $z = x + jy \in L$. It is clear that

$$u(z) = 2\mathrm{Re}\int_{t_{l-1,k}}^{z} [\frac{U(z)}{H_1(y)} + \begin{pmatrix} i \\ -j \end{pmatrix} \frac{V(z)}{H_2(y)}]dz + u(t_{l-1,k}) \text{ in } \begin{pmatrix} \overline{D_k^+} \\ \overline{G_{lk}} \end{pmatrix}, \quad \begin{matrix} l=1, k\in J_3, \\ l\in J_1, k\in J_3, \end{matrix} \tag{6}$$

in which $u(t_{01}) = d_{01} = d_1, u(t_{02}) = d_{02} = d_2$, and here and later on we denote $G_{l1} = G_l \cap \{y \ge -1/2\}$, $G_{l2} = G_l \cap \{y \le -1/2\}, l \in J_1$, and $t_{l-1,k}$ $(l \in J_1,\ k \in$

J_3) are as stated before. It is not difficult to see that Problem P includes the Tricomi problem (Problem T) with the discontinuous Dirichlet boundary condition on $\Gamma \cup L$ as a special case. The number

$$K = \frac{1}{2}(K_0 + K_1 + \cdots + K_N) \tag{7}$$

is called the index of Problem P in $D^+ \cap \{y > 0\}$, where

$$K_j = \left[\frac{\phi_j}{\pi}\right] + J_j, \quad J_j = 0 \text{ or } 1, \quad e^{i\phi_j} = \frac{\lambda(t_j - 0)}{\lambda(t_j + 0)}, \quad \gamma_j = \frac{\phi_j}{\pi} - K_j, \quad j = 0, 1, ..., N, \tag{8}$$

in which $t_l = t_{l1} = a_l, l = 0, 1, ..., N, \lambda(t) = e^{i\pi/2}$ on $L_0 = (a_0, a_N)$ on $x-$axis, $\lambda(t_0 + 0) = \lambda(t_1 - 0) = \cdots = \lambda(t_N - 0) = \exp(i\pi/2)$, where the index $K = 0$ or $-1/2$ on the boundary ∂G_0 of G_0 can be chosen, and can require that $-1/2 \leq \gamma_l < 1/2\,(l = 0, 1, ..., N)$. If $\cos(\nu, n) \equiv 0$ on Γ_0, then $\lambda(t_0 - 0) = -\exp(i\pi/2) = -\lambda(t_N + 0)$, we can select $K_0 = -1, K_1 = \cdots = K_N = 0$, $K = -1/2$, in this case the last two point conditions in (4) can be cancelled. In fact, if $\cos(\nu, n) \equiv 0$ on Γ, from the boundary condition (4), we can determine the value $u(a_N)$ by the value $u(a_0)$, namely

$$u(a_N) = 2\mathrm{Re}\int_{a_0}^{a_N} u_z dz + u(a_0) = 2\int_0^S \mathrm{Re}[z'(s)u_z]ds + d_0 = 2\int_0^S r(z)ds + d_0,$$

in which $\overline{\Lambda(z)} = z'(s)$ on Γ, $z(s)$ is a parameter expression of arc length s of Γ with the condition $z(a_0) = 0$, and S is the length of the boundary Γ_0. In this paper, we discuss Problem P and choose the case of $K = (N-1)/2$ on ∂G_0. In the following we mainly discuss the case of $\tilde{D} = D \cap \{y \geq -1/2\}$, and the case of $\hat{D} = D \cap \{y \leq -1/2\}$ can be similarly discussed. For convenience later on sometimes $G_0 \cup G_0', G_{l1} \cup G_{l2}, G_{l1}, \Gamma_l \cup \Gamma_l'$, are written as $G_0, G_{l1}, G_l, \Gamma_l$ and so on.

Noting that $\lambda(z), r(z) \in C^1_\alpha(\Gamma \cup L)\,(0 < \alpha < 1)$, where $\Gamma = \Gamma_0 \cup \Gamma_0', L = \tilde{L} \cup \tilde{L}'$, we can find two twice continuously differentiable functions $u_0^\pm(z)$ in $\overline{D^\pm}$, for instance, which are the solutions of oblique derivative problems with the boundary condition on $\Gamma \cup L$ in (4) for harmonic equations in $D^\pm$. Thus the functions $v(z) = v^\pm(z) = u(z) - u_0^\pm(z)$ in D is the solution of the boundary value problem (Problem $\tilde{\mathrm{P}}$) for the equation in the form

$$K_1(y)v_{xx} + |K_2(y)|v_{yy} + \tilde{a}v_x + \tilde{b}v_y + \tilde{c}v + \tilde{d} = 0 \text{ in } D \tag{9}$$

satisfying the corresponding boundary conditions

$$\begin{gathered}\mathrm{Re}[\overline{\lambda(z)}W(z)] = R(z) = 0 \text{ on } \Gamma \cup L, \quad v(t_{lk}) = d_{lk} = 0, \quad l \in J_2, \quad k \in J_3, \\ \mathrm{Im}[\overline{\lambda(z_l)}W(z_l)]_{z=z_{2l-1}} = H_1(\mathrm{Im} z_{2l-1})b_l = b_l' = 0, \quad l \in J_1,\end{gathered} \tag{10}$$

where the coefficients of (9) satisfy the conditions similar to Condition C, $W(z) = U + iV = v_{\tilde{z}}^+$ in D^+ and $W(z) = U + jV = v_{\tilde{z}}^-$ in $\overline{D^-}$. Hence later on we only discuss the homogeneous boundary condition (10) and the case of index $K = (N-1)/2$ on ∂G_0 and $\partial G_0'$. From $v(z) = v^{\pm}(z) = u(z) - u_0^{\pm}(z)$ in $\overline{D^{\pm}}$, we have $u(z) = v^{-}(z) + u_0^{-}(z)$ in $\overline{D^{-}}$, $u(z) = v^{+}(z) + u_0^{+}(z)$ in $\overline{D^{+}}$, $v^{+}(z) = v^{-}(z) - u_0^{+}(z) + u_0^{-}(z)$, $v_y^{+} = v_y^{-} - u_{0y}^{+} + u_{0y}^{-} = 2\hat{R}_0(x)$, and $v_y^{-} = 2\tilde{R}_0(x)$ on $\tilde{L}_0 \cup \tilde{L}_0'$, $\tilde{L}_0 = (a_0, a_1) \cup ... \cup (a_{N-1}, a_N)$, $\tilde{L}_0' = (a_0', a_1') \cup ... \cup (a_{N-1}', a_N')$ on $y = 0$ and $y = -1$ respectively, where $\hat{R}_0(x)$, $\tilde{R}_0(x)$ are undetermined real functions.

2. Representation of Solutions of Oblique Derivative Problem

Due to the intersection points of characteristic boundary $P_{2l-1} = \Gamma_{2l-1} \cap \Gamma_{2l-1}'$, $P_{2l} = \Gamma_{2l} \cap \Gamma_{2l}'$ $(l = 1, ..., N)$ are not equal, it needs to give a transformation, such that the bounded domain G_l will be reduced to another bounded domain $\hat{G}_l$, where G_l is bounded by the boundary $a_{l-1}a_l \cup \Gamma_{2l-1} \cup \Gamma_{2l} \cup P_{2l-1}P_{2l}$, and $\hat{G}_l$ is bounded by the boundary $a_{l-1}a_l \cup \hat{\Gamma}_{2l-1} \cup \hat{\Gamma}_{2l}$ $(1 \le l \le N)$, in which

$$\begin{aligned} \Gamma_{2l-1} &= \{x = a_{l-1} - G(y),\ a_{l-1} \le x \le x_{2l-1}\}, \\ \Gamma_{2l} &= \{y + \gamma_l(s) = 0,\ 0 \le s \le s_l'\}, \quad 1 \le l \le N, \end{aligned} \tag{11}$$

where $y = -\gamma(x)$ is $x = a_l + G(y)$ on $x_{2l} \le x \le a_l$ and $\{y = -\gamma(x)\{\cap\{y = 0\}$ on $x_{2l-1} \le x \le x_{2l}$. It is clear that $a_{l-1} < x_{2l-1} < (a_l - a_{l-1})/2 < x_{2l} < a_l$, and

$$\begin{aligned} \hat{\Gamma}_{2l-1} &= \{x = a_{l-1} - G(y),\ a_{l-1} \le x \le (a_l - a_{l-1})/2\}, \\ \hat{\Gamma}_{2l} &= \{x = a_l + G(y),\ (a_l - a_{l-1})/2 \le x \le a_l\}, \quad 1 \le l \le N, \end{aligned} \tag{12}$$

where $G(y)$ is as stated in Section 1. Obviously the curve Γ_{2l} can be expressed by $x = \tau(\mu) = (\mu + \nu)/2$, herein $\nu = x - G(y) = x - Y, \mu = x + G(y) = x + Y$, i.e. $\nu = 2\tau(\mu) - \mu$, $a_{l-1} \le \nu \le x_{2l-1} + \gamma(x_{2l-1})$. We make a transformation

$$\begin{aligned} \hat{\nu} &= (a_{l-1} - a_l)[\nu - 2\tau(\mu) + \mu]/[a_{l-1} - 2\tau(\mu) + \mu] + a_l, \\ \hat{\mu} &= \mu, \quad a_{l-1} \le \nu \le 2\tau(\mu) - \mu, \quad a_{l-1} \le \mu \le a_l, \quad l = 1, ..., N, \end{aligned} \tag{13}$$

where μ, ν are real variables, its inverse transformation is

$$\begin{aligned} \nu &= [a_{l-1} - 2\tau(\mu) + \mu](\hat{\nu} - a_l)/(a_{l-1} - a_l) + 2\tau(\mu) - \mu, \\ \mu &= \hat{\mu},\ a_{l-1} \le \hat{\mu} \le a_l,\ a_{l-1} \le \hat{\nu} \le a_l,\ l = 1, ..., N. \end{aligned} \tag{14}$$

It is not difficult to see that the transformation in (13) maps the domain G onto $\hat{G}$. The transformation (13) and its inverse transformation (14) can be rewritten as

$$\hat{Z} = \hat{x} + j\hat{Y} = f(Z),\ Z = x + jY = f^{-1}(\hat{Z}), \tag{15}$$

where

$$\hat{x}=\frac{1}{2}(\hat{\mu}+\hat{\nu}),\quad \hat{Y}=G(y)=\frac{1}{2}(\hat{\mu}-\hat{\nu}),\quad x=\frac{\mu+\nu}{2},\quad Y=\frac{\mu-\nu}{2}. \tag{16}$$

In this case, the corresponding equation (21) below in G can be rewritten in the form

$$\xi_\mu=\tilde{A}_1\xi+\tilde{B}_2\eta+\tilde{C}_1u+\tilde{D}_1,\ \eta_\nu=\tilde{A}_2\xi+\tilde{B}_2\eta+\tilde{C}_1u+\tilde{D}_2,\ z\in G_l,\ l=1,...,N. \tag{17}$$

Through the transformation (13), we obtain $\xi_{\hat{\mu}}=\xi_\mu, \eta_{\hat{\nu}}=[a_{l-1}-2\tau(\mu)+\mu]\eta_\nu/(a_{l-1}-a_l)$ in $\hat{G}_l\,(l=1,...,N)$, where $\xi=U+V, \eta=U-V$, and then

$$\begin{aligned}\eta_{\hat{\nu}}&=[a_{l-1}-2\tau(\mu)+\mu][\tilde{A}_2\xi+\tilde{B}_2\eta+\tilde{C}_1u+\tilde{D}_2]/(a_{l-1}-a_l),\\ \xi_{\hat{\mu}}&=\tilde{A}_1\xi+\tilde{B}_2\eta+\tilde{C}_1u+\tilde{D}_1\ \text{ in } \hat{G}_l,\ l=1,...,N.\end{aligned} \tag{18}$$

Moreover the boundary condition on $\tilde{L}\cup\tilde{L}_0$ in (4) can be reduced to the form

$$\begin{aligned}\mathrm{Re}[\overline{\lambda(z)}W(z)]&=H_1(y)r(z)=R(z)\ \text{ on } \tilde{L}\cup\tilde{L}_0,\\ \mathrm{Im}[\overline{\lambda(z_{2l-1})}W(z_{2l-1})]&=b_1',\quad l\in J_1,\end{aligned} \tag{19}$$

which $\lambda(z)=1+j$, $R(z)=0$ on $\tilde{L}_0$, $b_l'=H_1(\mathrm{Im}z_{2l-1})b_l\,(l\in J_1)$, $R(z)=H_1(y)r(z)$ on $\tilde{L}$ are as stated in Section 1. Through the transformation (15), the above boundary condition (19) is transformed into

$$\begin{aligned}\mathrm{Re}[\overline{\lambda(z(f^{-1}(\hat{Z})))}W(z(f^{-1}(\hat{Z})))]&=R(z(f^{-1}(\hat{Z}))),\quad \hat{Z}=\hat{x}+j\hat{Y}\in f(\hat{L}\cup\tilde{L}_0),\\ \mathrm{Im}[\overline{\lambda(z(f^{-1}(\hat{Z}_l)))}W(z(f^{-1}(\hat{Z}_l)))]&=b_l',\ l\in J_1,\end{aligned} \tag{20}$$

in which $\hat{Z}_l=f(Z_l)$, $Z_l=x_{2l-1}+jG[-\gamma_1(x_{2l-1})]$, $l\in J_1$. Therefore the boundary value problem (17), (19) is reduced to the boundary value problem (18), (20), i.e. the corresponding Problem A$^-$ in $\hat{G}_l\,(l=1,...,N)$. On the basis of Theorem 4 below, we see that the boundary value problem (18), (20) has a unique solution $W(z(\hat{Z}))$ in $\hat{G}$, and the function

$$W=W(z)\ \text{ in } G_l,\ l\in J_1=\{1,...,N\}$$

is just a solution of Problem A$^-$ for (17) in $G_l\,(l=1,...,N)$ with the boundary condition (19). Noting the relation in (23) below, we can find a solution of Problem P for equation (1). As for the other characteristic curves, it can be similarly handled, but it needs to choose the known value $u(z)=u_0(z)$ on the boundary $\{x_{2l-1}\le x\le x_{2l},\,y=0\}\,(l=1,...,N)$ as the boundary value on $\{x_{21-1}\le x\le x_{2l},\,y=-1/2\}\,(l=1,...,N)$ of $D^-\cap\{y<-1/2\}$, because we first have found the solution $u_0(z)$ of Problem P in $D^-\cap\{y\ge-1/2\}$.

In this paper, we first give the representation of solutions for the oblique derivative problem (Problem P) for equation (1) in D. In the following, we only consider the case of $D\cap\{y\geq -1/2\}$. Noting that

$$W(z)=U+iV=\frac{1}{2}[H_1(y)u_x-iH_2(y)u_y]=\frac{H_1(y)}{2}[u_x-iu_Y]\ \text{ in } D^+,$$
$$W(z)=U+jV=\frac{1}{2}[H_1(y)u_x-jH_2(y)u_y]=\frac{H_1(y)}{2}[u_x-ju_Y]\ \text{ in } D^-,$$

we have

$$\begin{aligned}W_{\tilde z}&=\frac{1}{2}[H_1(y)W_x+iH_2(y)W_y]=\frac{1}{4}\{H_1^2u_{xx}+H_1^2u_{yy}-iH_2[H_{1y}u_x-H_{2y}u_y]\}\\&=\frac{1}{4}\{[\frac{iH_2H_{1y}}{H_1}-\frac{a}{H_1}]H_1u_x+[H_{2y}-\frac{b}{H_2}]H_2u_y-c_*u-d\}\\&=A_1W+A_2\overline{W}+A_3u+A_4=g(Z)\ \text{ in } D^+,\end{aligned}$$

$$\begin{aligned}W_{\bar{\tilde z}}&=\frac{1}{2}[H_1(y)W_x+jH_2(y)W_y]=\frac{1}{2}[H_1[(U+jV)_x+jH_2(U+jV)_y]\\&=H_1[e_1(U+V)_\mu+e_2(U-V)_\nu]=\frac{e_1}{4}\{[\frac{a}{H_1}+\frac{H_2H_{1y}}{H_1}-\frac{b}{H_2}+H_{2y}](U+V)\\&\quad+[\frac{a}{H_1}+\frac{H_2H_{1y}}{H_1}+\frac{b}{H_2}-H_{2y}](U-V)+c_*u+d\}\\&\quad+\frac{e_2}{4}[\frac{a}{H_1}-\frac{H_2H_{1y}}{H_1}-\frac{b}{H_2}+H_{2y}](U+V)\\&\quad+[\frac{a}{H_1}-\frac{H_2H_{1y}}{H_1}+\frac{b}{H_2}-H_{2y}](U-V)+c_*u+d]\ \text{ in } D^-,\end{aligned}\tag{21}$$

where $Z=Z(z)$, $\tau=\mu+i\nu=x+G(y)+j[x-G(y)]=\tau(z)$ are the mappings from D onto the domains D_Z and D_τ respectively. Obviously the complex equation

$$W_{\bar{\tilde z}}=0\ \text{ in } \overline{D}\tag{22}$$

is a special case of (21).

The representation of solutions of Problem P for equation (1) can be written the following theorem.

Theorem 1. *Under Condition C, any solution $u(z)$ of Problem P for equation (1) in D can be expressed as follows*

$$u(z)=u(x)-2\int_{c_l}^{y}\frac{V(z)}{H_2(y)}dy=2\mathrm{Re}\int_{t_{l-1,k}}^{z}[\frac{\mathrm{Re}W}{H_1(y)}+\binom{i}{-j}\frac{\mathrm{Im}W}{H_2(y)}]dz$$
$$+d_{l-1,k}\ \ \text{in}\ \begin{pmatrix}\overline{D_k^+}\\ G_{lk}\end{pmatrix},\ \begin{matrix}l=1,\ k\in J_3,\\ l\in J_1, k\in J_3,\end{matrix}$$

$$
\begin{aligned}
&W(z) = \Phi[Z(z)] + \Psi[Z(z)] = \hat{\Phi}[Z(z)] + \hat{\Psi}[Z(z)],\ \Psi(Z) = T(Z) + \overline{T(\overline{Z})},\\
&\hat{\Psi}(Z) = T(Z) - \overline{T(\overline{Z})},\quad T(Z) = -\frac{1}{\pi}\iint_{D_t^+}\frac{f(t)}{t-Z}d\sigma_t \ \text{ in } \overline{D_Z^+},\\
&W(z) = \phi(z) + \psi(z) = \xi(z)e_1 + \eta(z)e_2 \ \text{ in } \overline{D^-},\\
&\ \xi(z) = \zeta(z) + \int_{c_l}^{y} g_1(z)dy = \int_{S_1} g_1(z)dy + \int_0^y g_1(z)dy\\
&\qquad = \int_{y_1}^{|y|} \hat{g}_1(z)dy,\ z\in s_1,\ \eta(z) = \theta(z) + \int_0^y g_2(z)dy,\ z\in s_2,\\
&g_l(z) = \tilde{A}_l(U+V) + \tilde{B}_l(U-V) + 2\tilde{C}_lU + 2\tilde{D}_lV + \tilde{E}_lu + \tilde{F}_l,\quad l=1,2.
\end{aligned}
\tag{23}
$$

Herein $c_1 = 0, c_2 = -1, U = H_1u_x/2, V = -H_2u_y/2, \Phi(Z)$ *is an analytic function in* $D_Z^+ = Z(D^+)$, *where* $Z(z) = x + iY = x + iY(y)$ *is a mapping from* $z(\in D^+)$ *to* Z, $\xi(z) = \int_{a_{l-1}}^{\mu}[g_1(z)/2H(y)]d\mu\,(l = 1, ..., N)$ *are the integrals along characteristic curves in* s_1 *from a point* $z_1 = x_1 + jy_1\,(\in \Gamma_{2l-1})$ *to the point* $z = x + jy \in \overline{D^-}$, $\phi(z) = \zeta(z)e_1 + \theta(z)e_2$ *is a solution of* (22) *in* D^-, *and* s_1, s_2 *are two families of characteristics in* D^-*:*

$$
s_1:\ \frac{dx}{dy} = \sqrt{K(y)} = H(y),\ s_2:\ \frac{dx}{dy} = -\sqrt{K(y)} = -H(y) \tag{24}
$$

passing through $z = x + jy \in \overline{D^-}$, S_1 *is the characteristic curve from the point on* $\Gamma_{2l-1}\,(1 \le l \le N)$ *to the point on* $\tilde{L}_0$, $\zeta(z) = \int_{S_1} g_1(z)dy$, $\theta(x) = -\zeta(x)$ *on* $\tilde{L}_0$, $\theta(z) = -\xi(x + G(y))$, *and*

$$
\begin{aligned}
&W(z) = U(z) + jV(z) = \frac{1}{2}H_1u_x - \frac{j}{2}H_2u_y,\\
&\xi(z) = \mathrm{Re}\psi(z) + \mathrm{Im}\psi(z),\quad \eta(z) = \mathrm{Re}\psi(z) - \mathrm{Im}\psi(z),\\
&\tilde{A}_1 = \tilde{B}_2 = \frac{1}{4}[\frac{h_{1y}}{h_1} + \frac{h_{2y}}{h_2}],\quad \tilde{A}_2 = \tilde{B}_1 = \frac{1}{4}[\frac{h_{1y}}{h_1} - \frac{h_{2y}}{h_2}],\\
&\tilde{C}_1 = \frac{a}{2H_1H_2} + \frac{m_1}{4y},\quad \tilde{C}_2 = -\frac{a}{2H_1H_2} + \frac{m_1}{4y},\\
&\tilde{D}_1 = -\frac{b}{2H_2^2} + \frac{m_2}{4y},\quad \tilde{D}_2 = \frac{b}{2H_2^2} - \frac{m_2}{4y},\\
&\tilde{E}_1 = -\tilde{E}_2 = \frac{c_*}{2H_2},\quad \tilde{F}_1 = -\tilde{F}_2 = \frac{d}{2H_2},
\end{aligned}
\tag{25}
$$

and

$$
d\mu = d[x + G(y)] = 2H(y)dy \text{ on } s_1, d\nu = d[x - G(y)] = -2H(y)dy \text{ on } s_2.
$$

Proof. From (21) it is not difficult to see that equation (1) in $\overline{D^-}$ can be reduced to the system of integral equations: (23). Moreover we can extend the equation (21) onto the the symmetrical domain $\hat{D}_Z$ of D_Z^- with respect to the real axis $\mathrm{Im}Z = 0$, namely introduce the function $\hat{W}(Z)$ as follows:

$$\hat{W}(Z) = \begin{cases} W[z(Z)], \\ -\overline{W[z(\overline{Z})]}, \end{cases} \quad \hat{u}(z) = \begin{cases} u(Z) \text{ in } D_Z^-, \\ -u(\overline{Z}) \text{ in } \hat{D}_Z, \end{cases} \tag{26}$$

and then the equation (21) is extended as

$$\hat{W}_{\overline{\overline{z}}} = \hat{A}_1\hat{W} + \hat{A}_2\overline{\hat{W}} + \hat{A}_3\hat{u} + \hat{A}_4 = \hat{g}(Z) \text{ in } \overline{D_Z^-} \cup \overline{\hat{D}_Z}, \tag{27}$$

in which

$$\hat{A}_l(Z)=\begin{cases}A_l(Z), \\ \overline{\tilde{A}_l(\overline{Z})},\end{cases} l=1,2,3, \quad \hat{A}_4(Z)=\begin{cases}A_4(Z), \\ -\overline{A_4(\overline{Z})},\end{cases} \quad \hat{g}_l(Z)=\begin{cases}g_l(z) \text{ in } \overline{D_Z^-}, \\ -\overline{g_l(\overline{Z})} \text{ in } \hat{D}_Z,\end{cases} l=1,2,$$

where $\tilde{A}_1(\overline{Z}) = A_2(\overline{Z}), \tilde{A}_2(\overline{Z}) = A_1(\overline{Z}), \tilde{A}_3(\overline{Z}) = A_3(\overline{Z})$, and we mention that $\hat{u}(z)$ on $\tilde{L}_0$ cannot be continuous, but it does not effect the discussion. It is easy to see that the system of integral equations (23) can be written in the form

$$\begin{gathered}\xi(z)=\zeta(z)+\int_0^{\tilde{y}} g_1(z)dy=\int_{y_1}^{\hat{y}} \hat{g}_1(z)dy, \\ \eta(z)=\theta(z)+\int_0^{y} g_2(z)dy=\int_{y_1}^{\hat{y}} \hat{g}_2(z)dy, \hat{z}=x+j\hat{y}=x+j|y| \in \overline{D_Z^-}\cup\overline{\hat{D}_Z},\end{gathered} \tag{28}$$

where $x_1 + jy_1$ is the intersection point of Γ_{2l-1} and the characteristic curve s_1 passing through $z = x + jy \in \overline{D^-}$, the function $\theta(z)$ is determined by $\zeta(z)$, i.e. the functions $\theta(z)$ will be defined by $\theta(z) = -\zeta(z) = -\zeta(x + G(y))$ in $G_l\,(1 \le l \le N)$, for the extended integral, which can be appropriately defined in D_Z^-. For convenience later on the numbers $\hat{y} - y_1, \hat{t} - y_1$ will be written by $\tilde{y}, \tilde{t}$ respectively. □

3. Solvability of Oblique Derivative Problem

In this section, we mainly prove the existence of solutions of Problem P for equation (1).

Theorem 2. *Suppose that equation* (1) *satisfies Condition C. Then the homogeneous problem (Problem P_0) of Problem P for* (1) *with $d = 0$ only has the trivial solution in D.*

Proof. Let $u(z)$ be any two solutions of Problem P_0 for (1) with $d = 0$. By Theorem 1, it is easy to see that $u(z)$ and $W(z) = u_{\tilde{z}}$ satisfy the homogeneous equation and boundary conditions

$$K_1(y)u_{xx}+|K_2(y)|u_{yy}+au_x+bu_y+c_*u=0, \text{ i.e. } w_{\overline{\overline{z}}}=A_1w+A_2\overline{w}+A_3u \text{ in } D, \tag{29}$$

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial u}{\partial \nu} = \dfrac{1}{H_1(y)}\text{Re}[\overline{\lambda(z)}u_{\tilde{z}}] = \text{Re}[\overline{\lambda(z)}W(z)] = 0 \text{ on } \Gamma \cup L, \\ \text{Im}[\overline{\lambda(z)}u_{\tilde{z}}]|_{z=z_{2l-1}} = 0,\ l \in J_1,\ u(t_{lk}) = 0,\ l \in J_2,\ k \in J_3, \end{cases} \tag{30}$$

where the function $w(z) = U(z) + jV(z) = [H_1 u_x - jH_2 u_y]/2$ in the hyperbolic domain D^- can be expressed in the form

$$\begin{aligned} W(z) &= \phi(x) + \psi(z) = \xi(z)e_1 + \eta(z)e_2, \\ \xi(z) &= \zeta(z) + \int_0^y [\tilde{A}_1(U+V) + \tilde{B}_1(U-V) + 2\tilde{C}_1 U + 2\tilde{D}_1 V + \tilde{E}_1 u]dy,\ z \in s_1, \\ \eta(z) &= \theta(z) + \int_0^y [\tilde{A}_2(U+V) + \tilde{B}_2(U-V) + 2\tilde{C}_2 U + 2\tilde{D}_2 V + \tilde{E}_2 u]dy,\ z \in s_2, \end{aligned} \tag{31}$$

in which $\phi(z)$ is a solution of (22). By using the method as stated in Section 2, Chapter V, [10], we can verify $u(z) = 0$ in D^-, and then $u_y = 0$ on $\tilde{L}_0$.

We first verify that the above solution $u(z) \equiv 0$ in D^+. If the maximum $M = \max_{\overline{D^+}} u(z) > 0$, it is clear that the maximum point $z^* \notin D^+$. If the maximum M attains at a point $z^* \in \Gamma$ and $\cos(\nu, n) > 0$ at z^*, we get $\partial u/\partial \nu > 0$ at z^*, which contradicts the first formula of (30). If $\cos(\nu, n) = 0$ at z^*, denote by Γ' the longest curve of Γ including the point z^*, so that $\cos(\nu, n) = 0$ and $u(z) = M$ on Γ_0, then there exists a point $z' \in \Gamma\backslash\Gamma_0$, such that at z', $\cos(\nu, n) > 0, \partial u/\partial n > 0, \cos(\nu, s) > 0\,(< 0)$, $\partial u/\partial s \geq 0\,(\leq 0)$, hence

$$\frac{\partial u}{\partial \nu} = \cos(\nu, n)\frac{\partial u}{\partial n} + \cos(\nu, s)\frac{\partial u}{\partial s} > 0 \text{ at } z' \tag{32}$$

holds, where s is the tangent vector of at $z' \in \Gamma_0$, and this is impossible. This shows $z^* \notin \Gamma_0$. Moreover on the basis of the Hopf Lemma (see Lemma 2.3, Chapter II, [10]), we can verify the positive maximum point $z^* \notin \tilde{L}_0$, hence $u(x) = 0$ on $\tilde{L}_0$, and then $\max_{\overline{D^+}} u(z) = 0$. Similarly we can verify $\min_{\overline{D^+}} u(z) = 0$, thus $u(z) = 0$ in $\overline{D^+}$, this shows $u(z) = 0$ in $\overline{D}$. This completes the proof. □

In order to prove the existence of solutions of Problem P for equation (1) with some conditions, we try to discuss the problem by using a new method. As stated in Section 1, it suffices to discuss Problem $\tilde{\text{P}}$ for (1), it is clear that Problem $\tilde{\text{P}}$ is equivalent to Problem A for the complex equation

$$W_{\tilde{\bar{z}}} = A_1(z,u,W)W + A_2(z,u,W)\overline{W} + A_3(z,u,W)u + A_4(z,u,W) \text{ in } D, \tag{33}$$

with the relation

$$u(z) = \begin{cases} 2\text{Re}\displaystyle\int_{a_0}^{z} [\frac{\text{Re}W(z)}{H_1(y)} + i\frac{\text{Im}W(z)}{H_2(y)}]dz + d_1 \text{ in } \overline{D^+}, \\ u(x) - \displaystyle\int_0^y \frac{\text{Im}W(z)}{H_2(y)}dy \text{ in } \overline{D^-}, \end{cases} \tag{34}$$

and the boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}W(z)] = R(z) \text{ on } \Gamma\cup L,\\ \operatorname{Im}[\overline{\lambda(z)}u_{\bar z}]|_{z=z_{2l-1}} = b_l', l\in J_1,\ u(t_l)=d_l, l\in J_2,\end{gathered}\tag{35}$$

where the coefficients in (33) are as stated in (21), $\lambda(z), R(z), z_{2l-1}, b_l, t_l = t_{l1}, d_l = d_{l1}$ are as stated before, but

$$R(z)=0 \text{ on } \Gamma\cup L,\ b_l'=0, l\in J_1,\ d_l=0,\ l\in J_2. \tag{36}$$

We can divided Problem A into two parts, i.e. Problem A$^+$: (33), (34) in D^+ with the boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}W(z)] = R(z) \text{ on } \Gamma_0\cup\Gamma_0',\\ \operatorname{Re}[iW(z)]=H_2(y)\hat R_0(x) \text{ on } \tilde L_0,\ u(t_l)=d_l,\ l\in J_2,\end{gathered}\tag{37}$$

and Problem A$^-$: (33), (34) in D^- with the boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}W(z)]=R(z) \text{ on } L, \operatorname{Im}[\overline{\lambda(z_{2l-1})}W(z_{2l-1})]=b_l', l\in J_1,\\ \operatorname{Re}[(-j)W(z)] = H_2(y)\tilde R_0(x) \text{ on } \tilde L_0,\end{gathered}\tag{38}$$

where we mention that the corresponding functions $f(x) = -g(x)$ on $\tilde L_0$ in (38), and

$$R(z)=0 \text{ on } \Gamma\cup L,\ b_l'=0, l\in J_1,\ d_l=0, l\in J_2. \tag{39}$$

On the basis of the similar way as in [7]-[10], but it needs to some modifications, we can verify the existence of solutions of Problem A$^+$. In [8], the author mainly discussed the equations of mixed type without degenerate line. In the following we shall sketchy prove that there exists a solution of the above Problem A$^-$ in D^-, and the detailed proof is similar to that as in [9]-[10].

Theorem 3. *If equation* (1) *satisfies Condition C, then there exists a solution* $[w(z), u(z)]$ *of Problem* A$^-$ *for* (33), (34), (38), (39).

Proof. By using the result in [8], we may only discuss the problem in the closed domain

$$D_0 = D^-\cap\{d_0\le x\le d_1, -\delta\le y\le 0\},$$

and s_1, s_2 are the characteristics of families in Theorem 1 emanating from any two points $(d_0, 0), (d_1, 0)(a_{l-1} < d_0 = a_{l-1}+\delta_0 < d_1 = a_l-\delta_0)$, which intersect at a point $z\in\overline{D^-}$, where δ, δ_0 are sufficiently small positive constants.

We discuss the case of $K(y) = -|y|^m h(y)$, $m = m_1-m_2$, $h(y) = h_1(y)/h_2(y)$ are as stated in (2), (3). In order to find a solution of the system of integral equations (23), we rewrite the last condition in (2), namely

$$\frac{ay}{H_1(y)H_2(y)}=o(1), \text{ i.e. } \frac{|a|}{H_1(y)H_2(y)}=\frac{\varepsilon(y)}{|y|},\quad m_1+m_2-2[m_2]\ge 2, \tag{40}$$

where $\varepsilon(y)\to 0$ as $y\to 0$. It is clear that for two characteristics s_1, s_2 passing through a point $z = x+jy \in \overline{D^-}$ and x_1, x_2 are the intersection points with the axis $y = 0$ respectively, for any two points $\tilde{z}_1 = \tilde{x}_1 + j\tilde{y} \in s_1, \tilde{z}_2 = \tilde{x}_2 + j\tilde{y} \in s_2$, we have

$$\begin{aligned}&|\tilde{x}_1-\tilde{x}_2| \le |x_1-x_2| = 2|\int_0^y \sqrt{K(t)}dt| \le \frac{2k_0}{m+2}|y|^{1+m/2} \le \frac{k_1}{12}|y|^{1+m/2},\\ &|y|^{1+m/2} \le \tfrac{k_0(m+2)}{2}|x_1-x_2|,\end{aligned} \tag{41}$$

From Condition C, we can assume that the coefficients in (23) are continuously differentiable with respect to $x \in L_0$ and satisfy the conditions

$$\begin{cases} ||y|^{\hat{m}_2}\tilde{A}_l|, ||y|^{\hat{m}_2}\tilde{A}_{lx}|, ||y|^{\hat{m}_2}\tilde{B}_l|, ||y|^{\hat{m}_2}\tilde{B}_{lx}|, ||y|^{\hat{m}_2}\tilde{E}_l|,\\ ||y|^{\hat{m}_2}\tilde{E}_{lx}| \le k_0 \le k_1/12,\ ||y|^{\hat{m}_2}\tilde{F}_l|,\ ||y|^{\hat{m}_2}\tilde{F}_{lx}| \le k_1/2,\\ 2\sqrt{h},\ 1/\sqrt{h},\ |h_y/h| \le k_0 \le k_1/12 \ \text{in}\ \overline{D^-},\ l = 1,2. \end{cases} \tag{42}$$

And later on we shall use the positive constants

$$\begin{aligned} M &= 4\max[M_1, M_2, M_3],\\ M_1 &= \max[8(k_1 d)^2/(1-m'), M_3/k_1],\\ M_2 &= (2+m)k_0 d\delta^{-2-m}[4k_1\delta+4\varepsilon_0+m]/\delta,\\ M_3 &= 2k_1^2|y_1'|^{-m'}[|y_1'|d + 1/2H(y_1')],\\ \gamma &= \max[4k_1 d\delta^{\beta_1}+(4\varepsilon(y)+m)/2\beta'] < 1, |y| \le \delta, \end{aligned} \tag{43}$$

where

$$\begin{aligned} &m' = m_2-[m_2]+\beta_1 < 1,\ \beta = 1-\beta_1,\ \beta' = (1+m/2)(1-3\beta_1),\\ &\varepsilon_0 = \max_{\overline{D^-}}\varepsilon(z),\ 1/2H(y_1') \le k_0[(m+2)\delta_0/k_0]^{-m/(2+m)}, \end{aligned}$$

and δ, β_1 are appropriate small positive constants, such that $(2+m)\beta_1 < 1$, and d is the diameter of D^-, the positive number δ is small enough. We choose $v_0 = 0, \xi_0 = 0, \eta_0 = 0$ and substitute them into the corresponding positions of v, ξ, η in the right-hand sides of integral equations in (23), and by the successive approximation, we find the sequences of functions $\{v_k\}, \{\xi_k\}, \{\eta_k\}$, which satisfy the relations

$$\begin{aligned} v_{k+1}(z) &= v_{k+1}(x)-2\int_0^y V_k(z)dy = v_{k+1}(x)+\int_0^y (\eta_k - \xi_k)dy,\\ \xi_{k+1}(z) &= \zeta_{k+1}(z) + \int_0^y g_{1k}(z)dy = \int_{y_1}^{\hat{y}} \hat{g}_{lk}dy,\\ \eta_{k+1}(z) &= \theta_{k+1}(z) + \int_0^y g_{2k}(z)dy = \int_{y_1}^{|y|} \hat{g}_{2k}(z)dy,\\ g_{lk}(z) &= \tilde{A}_l\xi_k+\tilde{B}_l\eta_k+\tilde{C}_l(\xi_k+\eta_k)+\tilde{D}_l(\xi_k-\eta_k)+\tilde{E}_l v_k+\tilde{F}_l,\\ &l = 1,2,\ k = 0,1,2,\dots. \end{aligned} \tag{44}$$

Here $v(x) = u(x)-u_0(x)$ on $\tilde{L}_0$ as stated before, $z_1 = x_1+jy_1$ is a point on L_{2l-1}, which is the intersection of L_{2l-1} and the characteristic curve s_1 passing through the point $z = x + jy \in \overline{D^-}$. Setting that $\tilde{g}_{lk+1}(z) = g_{lk+1}(z) - g_{lk}(z)(l = 1,2)$, and

$$\begin{aligned}\tilde{y} = \hat{y}-y_1,\ \tilde{t}=\hat{t}-y_1,\ \tilde{v}_{k+1}(z)=v_{k+1}(z)-v_k(z),\\ \tilde{\xi}_{k+1}(z) = \xi_{k+1}(z) -\xi_k(z),\ \tilde{\eta}_{k+1}(z)=\eta_{k+1}(z)-\eta_k(z),\\ \tilde{\zeta}_{k+1}(z) = \zeta_{k+1}(z) - \zeta_k(z),\ \tilde{\theta}_{k+1}(z)=\theta_{k+1}(z)-\theta_k(z),\end{aligned} \tag{45}$$

denoting by $(k-m')!$ the product $(1-m')...(k-m')$, and according to the method as in Section 5, Chapter VI, [10], we can prove that $\{\tilde{v}_k\}$, $\{\tilde{\xi}_k\}$, $\{\tilde{\eta}_k\}$, $\{\tilde{\zeta}_k\}$, $\{\tilde{\theta}_k\}$ in D_0 satisfy the estimates

$$\begin{cases}|\tilde{v}_k(z)-\tilde{v}_k(x)|, |\tilde{\xi}_k(z)-\tilde{\zeta}_k(z)|, |\tilde{\eta}_k(z)-\tilde{\theta}_k(z)| \le M'\gamma^{k-1}|y|^{1-m'},\\ |\tilde{\xi}_k(z)|, |\tilde{\eta}_k(z)| \le MM_2^{k-1}|\tilde{y}|^{k-m'}/(k-m')!, \text{ or } M'\gamma^{k-1},\\ |\tilde{\xi}_k(z_1)-\tilde{\xi}_k(z_2)-\tilde{\zeta}_k(z_1)-\tilde{\zeta}_k(z_2)|, |\tilde{\eta}_k(z_1)-\tilde{\eta}_k(z_2)-\tilde{\theta}_k(z_1)-\tilde{\theta}_k(z_2)|\\ \le M'\gamma^{k-1}[|x_1-x_2|^{1-m'}+|x_1-x_2|^{\beta}|t|^{\beta'}],\ |y|\le\delta,\ |\tilde{\xi}_k(z_1)-\tilde{\xi}_k(z_2)|,\\ |\tilde{\eta}_k(z_1)-\tilde{\eta}_k(z_2)| \le MM_2^{k-1}|\tilde{t}|^{k-m'}|x_1-x_2|^{1-\beta}/(k-m')!,\\ \text{or } M'\gamma^{k-1}|x_1-x_2|^{\beta}|t|^{\beta'}, |\tilde{\xi}_k(z)+\tilde{\eta}_k(z)-\tilde{\zeta}_k(z)-\tilde{\theta}_k(z)|,\ |\tilde{\xi}_k(z)+\tilde{\eta}_k(z)|\\ \le MM_2^{k-1}|\tilde{y}|^{k-m'}|x_1-x_2|^{1-\beta}/(k-m')!, \text{ or } M'\gamma^{k-1}|x_1-x_2|^{\beta}|y|^{\beta'},\end{cases} \tag{46}$$

where $z = x + jy$, $z = x + jt$ is the intersection point of s_1, s_2 in (24) passing through z_1, z_2, γ is as stated in (43), d is the diameter of D^-, and M, M' is a sufficiently large positive constants.

From the estimate (46), we see that for any two points $z_1 = x_1+jG(y_1)$, $z_2 = x_2 + jG(y_2) \in D_0$, where $x_1 < x_2$, $y_1 = y < y_2(< 0)$. Setting that $z_3 = x_3 + jG(y_1)$ is the intersection point of the characteristic line $s_1 : x - G(y) = x_2 - G(y_2)$ and the straight line $G(y) = G(y_1)$, then we have

$$\begin{aligned}|\xi_k(z_1)-\xi_k(z_2)| &\le |\xi_k(z_1)-\xi_k(z_3)| + |\xi_k(z_2)-\xi_k(z_3)|\\ &\le M(M_2|\tilde{y}|)^{k-1}[|x_1-x_3|^{\beta}|y|^{\beta'} + |y|^{1-\beta}|y_1-y_2|^{\beta}]/(k-1)!\\ &\le M(M_2|\tilde{y}|)^{k-1}(|y|^{\beta'}+|y|^{1-\beta})|z_1-z_2|^{\beta}/(k-1)!\\ &\le M''(M_2|\tilde{y}|)^{k-1}|z_1-z_2|^{\beta}/(k-1)!,\end{aligned}$$

where $M'' = \max_{\overline{D_-}}[|y|^{\beta'}+|y|^{1-\beta}]$. For other case, we can similarly get the estimates:

$$|v_k(z_1)-v_k(z_2)|, |\eta_k(z_1)-\eta_k(z_2)| \le M''(M_2|\tilde{y}|)^{k-1}|z_1-z_2|^{\beta}/(k-1)!.$$

This shows the uniform boundedness and equicontinuous of $\{v_k(z)\}$, $\{\xi_k(z)\}$, $\{\eta_k(z)\}$. Hence we can choose the subsequences of $\{v_n(z)\}$, $\{\xi_n(z)\}$, $\{\eta_n(z)\}$, which in D_0 uniformly converge to $v_*(z)$, $\xi_*(z)$, $\eta_*(z)$ satisfying the system of integral equations

$$\begin{aligned}
&v_*(z)=v_*(x)-2\int_0^y V_* dy=u_*(x)+\int_0^y(\eta_*-\xi_*)dy,\\
&\xi_*(z)=\zeta_*(z)+\int_0^y[\tilde{A}_1\xi_*+\tilde{B}_1\eta_*+\tilde{C}_1(\xi_*+\eta_*)+\tilde{D}_1v_*+\tilde{E}_1]dy, z\in s_1,\\
&\eta_*(z)=\theta_*(z)+\int_0^y[\tilde{A}_2\xi_*+\tilde{B}_2\eta_*+\tilde{C}_2(\xi_*+\eta_*)+\tilde{D}_2v_*+\tilde{E}_2]dy, z\in s_2.
\end{aligned}$$

By using the result in [8] and noting the arbitrariness of δ_0, we can derive that the function $[W_*(z), v_*(z)] = [(\xi_*+\eta_*+j\xi_*-j\eta_*)/2, v_*(z)]$ is a solution of Problem A$^-$ for equation (33) in D^-. Moreover the function $u(z) = v(z)+u_0(z)$ is a solution of Problem P for (1) in D^-. The proof is finished.

From the above discussion, we obtain the following theorem.

Theorem 4. *Let equation* (1) *satisfy Condition C. Then Problem P for (1) has a solution.*

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