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THE OPTIMIZATION OF EIGENVALUE PROBLEMS FOR OPERATORS INVOLVING THE p-LAPLACIAN

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Abstract: In this paper we survey results concerning the following optimization problem: given a bounded domain $\Omega \subset \mathbb{R}^n$, numbers $p > 1$, $\alpha \geq 0$ and $A \in [0, |\Omega|]$, find a subset $D \subset \Omega$, of measure A, for which the first eigenvalue of the operator

$$u \mapsto -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \alpha\chi_D|u|^{p-2}u$$

with the Dirichlet boundary condition is as small as possible. We also address the question of symmetry of optimal solutions.

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1. Introduction

In this article we describe the results of papers [1, 9, 10], concerning the optimal pairs of an eigenvalue problem involving the p-Laplacian. The research has been originated by the paper [1] dealing with the optimizaton of eigenvalues for the linear case of $p = 2$. The paper [9] is available online at www.im.uj.edu.pl/actamath, the paper [10] – at www.opuscula.agh.edu.pl.

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Let Ω be a bounded domain (i.e. open and connected set) in the space $\mathbb{R}^n$ ($n \geq 1$) with the closure $\overline{\Omega}$ and boundary $\partial\Omega$. We denote by $|\Omega|$ the Lebesgue measure of Ω. Given numbers $p > 1$, $\alpha \geq 0$ and a measurable subset D of Ω, we shall be concerned with the eigenvalue problem of the form

$$\begin{cases} -\Delta_p(u) + \alpha\,\chi_D\,\varphi_p(u) = \lambda\,\varphi_p(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ_p is the p-Laplacian, χ_D is the characteristic function of D, while φ_p is a function defined by

$$\varphi_p(u) := \begin{cases} |u|^{p-2}u, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

The p-Laplacian is a nonlinear differential operator of the form

$$\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(\varphi_p(\nabla u)),$$

which coincides with the Laplacian Δ for $p = 2$.

In this paper we deal with real function spaces only. In particular, we use standard Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, with $1 < p < \infty$. It is customary to use solutions of (1) in a weak sense. Any nontrivial function $u\colon \Omega \to \mathbb{R}$ is said to be an eigenfunction of problem (1) if and only if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega \varphi_p(\nabla u)\nabla v + \alpha \int_\Omega \chi_D\,\varphi_p(u)\,v = \lambda \int_\Omega \varphi_p(u)\,v, \qquad \forall\, v \in W_0^{1,p}(\Omega).$$

The eigenvalue problems like (1) were investigated by many authors (see, e.g., [3, 7, 8] and the references therein). It is known that there exists a positive number $\lambda(\alpha, D)$, called the *first* or *principal* eigenvalue, such that the eigenvalue problem (1) has a positive solution u with a positive eigenvalue λ if and only if $\lambda = \lambda(\alpha, D)$. Moreover, $\lambda(\alpha, D)$ is simple in the sense that the set of all solutions to problem (1) is a one-dimensional subspace of the Sobolev space $W_0^{1,p}(\Omega)$. Consequently, the positive eigenfunction u is unique up to a scalar multiple. Let us fix $A \in [0, |\Omega|]$ and define

$$\Lambda(\alpha, A) := \inf\{\,\lambda(\alpha, D)\colon D \subset \Omega,\ |D| = A\,\}. \tag{2}$$

Any minimizer in (2) is called an *optimal configuration.* If u is a positive eigenfunction of problem (1) with $\lambda = \Lambda(\alpha, A)$ and with an optimal configuration D, then (u, D) is said to be an *optimal pair* (or *optimal solution*).

2. Optimal Pairs

We recall that Ω is any bounded domain in $\mathbb{R}^n$ (we need no additional assumptions on the regularity of the boundary $\partial\Omega$) and $p \in (1, \infty)$.

Theorem 1. *For any $\alpha \geq 0$ and $A \in [0, |\Omega|]$ there exists an optimal pair. Every optimal pair (u, D) has the following properties:*

(a) $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and ∇u is locally Hölder continuous, i.e., for every compact $K \subset \Omega$ there exists $\beta \in (0, 1)$ such that $\nabla u \in C^{0,\beta}(K)$;

(b) there is a number $t \geq 0$ such that (up to a set of measure zero)

$$D = \{ u \leq t \}. \tag{3}$$

As usual, we write $\{ u < t \}$ instead of $\{ x \in \Omega \colon u(x) < t \}$ and similarly we put $\{ u \leq t \} := \{ x \in \Omega \colon u(x) \leq t \}$.

The proof of the existence of optimal pairs can be found in the paper [9]. The regularity properties of eigenfunctions, stated in assertion (a), are rather well known. In this connection see [9] and the references therein. The equality (3) in the case when $p = 2$ was stated in [1]. The lack of higher regularity of eigenfunctions is a source of difficulty in obtaining more general results. However, the difficulties were overcome in [10] on the base of the following lemma, which can be of independent interest.

Lemma 2. *Assume that $u \in W_{loc}^{1,p}(\Omega)$ is a weak solution of the equation*

$$-\Delta_p(u) = f \qquad \text{in } \Omega$$

with $p > 1$, $f \in L^q(\Omega)$, $q > \frac{n}{p}$, $q \geq 2$. Let

$$Z := \{ x \in \Omega \colon \nabla u(x) = 0 \}.$$

Then $|\nabla u|^{p-1} \in W_{loc}^{1,2}(\Omega)$ and $f(x) = 0$ for almost every $x \in Z$.

This result comes from [6] and its preprint was quoted in [2].

Remark 3. According to (3), our optimization of eigenvalue problem is equivalent to finding the smallest eigenvalue and an associated eigenfunction of the problem

$$\begin{cases} -\Delta_p(u) + \alpha \chi_{\{ u \leq t \}} \varphi_p(u) = \lambda \varphi_p(u) & \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \\ |\{ u \leq t \}| = A, \end{cases}$$

with free variables u and t.

3. Steiner Symmetry

In this section we focus on the linear case with $p = 2$. A well-known result concerning the symmetry of optimal solutions (see [1], Theorem 4) has been extended in [10] to the case of domains with possibly non-smooth boundary.

From now on we shall assume that Ω satisfies the exterior cone condition at each point $x \in \partial\Omega$, which means that there exists a finite right circular cone V_x with vertex x such that $\overline{\Omega} \cap V_x = \{x\}$.

Let us recall that a domain G of $\mathbb{R}^n$ is Steiner symmetric with respect to a hyperplane P iff for any point $x = (x_1, \dots, x_n) \in G$ the segment connecting x and the point x^* reflected with respect to P is contained in G.

Theorem 4. *Let $p = 2$. If the domain Ω is Steiner symmetric with respect to a hyperplane P, then for any optimal pair (u, D) both u and D are symmetric with respect to P, and $D^c = \Omega \setminus D$ is Steiner symmetric with respect to P.*

The paper [10] includes a proof of Theorem 4, in which some ideas of [5] were adopted. The proof uses Theorem 3.6 of [4], which may be stated as follows:

Theorem 5. *Let Ω be bounded, connected and Steiner symmetric relative to the hyperplane $P = \{x = (x_1, x')\colon x_1 = 0\}$. Assume that $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ is a positive weak solution of the boundary value problem*

$$\begin{cases} -\Delta u = f_1(u) + f_2(u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $f_1\colon [0, \infty) \to \mathbb{R}$ is locally Lipschitz continuous, while $f_2\colon [0, \infty) \to \mathbb{R}$ is non-decreasing and is identically zero on an interval $[0, h]$ for some $h > 0$. Then

$$u(-x_1, x') = u(x_1, x') \qquad \text{for } (x_1, x') \in \Omega.$$

Moreover,

$$\frac{\partial u}{\partial x_1}(x_1, x') < 0 \qquad \text{if } (x_1, x') \in \Omega \text{ and } x_1 > 0.$$

It is worth noting that one of the most interesting phenomena studied in the paper [1] is symmetry breaking for certain plane domains. An optimal configuration D may have less symmetry than Ω, in particular in the case of thin annuli and dumbbells with narrow handles.

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