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ASYMPTOTIC BEHAVIOR OF NON-OSCILLATORY SOLUTIONS OF CERTAIN SECOND ORDER NON-LINEAR DIFFERENCE EQUATIONS

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Abstract: In this paper, we investigate the asymptotic behavior of all non-oscillatory solutions of the second order non-linear difference equation

$$\Delta[p(n)\phi(\Delta x(n))] + q(n+1)f(x(n+1)) = 0, \quad n \geq n_0.$$

Also we provide conditions under which $\Delta x(n)$ is oscillatory whenever $x(n)$ is a solution of the above equation. Suitable examples are provided to illustrate the results.

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1. Introduction

In recent years, the investigation of the theory of difference equations has assumed a greater importance. Many results in the theory of difference equations have been obtained more or less natural discrete analogous of corresponding results of differential equations. Nevertheless, the theory of difference equa-

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tions is richer than the corresponding theory of differential equations. In recent years oscillation, asymptotic behavior and stability of discrete models have become a very popular subject, see for example monographs [1, 2] and the papers [4] - [14].

In this paper, we consider the following nonlinear second order difference equation of the form

$$\Delta[p(n)\phi(\Delta x(n))] + q(n+1)f(x(n+1)) = 0, \quad n \geq n_0. \tag{1}$$

We assume that:

(H_1) $p(n)$ is a sequence of positive real numbers and $q(n)$ is a sequence of real numbers where $q(n)$ is not identically zero for $n \geq n_0 > 0$;

(H_2) $f : \mathbb{R} \to \mathbb{R}, xf(x) > 0$ for $x \neq 0$ and $f(u) - f(v) = g(u, v)(u - v)$ for all $u, v \neq 0$ where g is a non-negative function;

(H_3) $\text{sgn}\phi(u) = \text{sgn}|u|$;

(H_4) $\phi(u)\text{sgn}\ u$ has the inverse function $\psi(u)$.

By a solution of (1), we mean a sequence $\{x(n)\}$ which is defined for $n \geq n_0$ and satisfies (1) for $n \geq n_0$. A solution $\{x(n)\}$ of equation (1) is said to be oscillatory if the terms $\{x(n)\}$ of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is said to be non-oscillatory.

When $p(n) = 1$ and $\phi(u) = u$, equation (1) becomes

$$\Delta^2 x(n) + q(n+1)f(x(n+1)) = 0. \tag{E_1}$$

If $f(x) = x^\gamma$ and $f(x) = x$, we obtain respectively

$$\Delta^2 x(n) + q(n+1)x^\gamma(n+1) = 0 \tag{E_2}$$

and

$$\Delta^2 x(n) + q(n+1)x(n+1) = 0. \tag{E_3}$$

Taking $\phi(u) = u^\alpha$ and $f(x) = x^\alpha$, equation (1) can be expressed as

$$\Delta\left[p(n)(\Delta x(n))^\alpha\right] + q(n+1)x^\alpha(n+1) = 0. \tag{E_4}$$

Letting $\phi(u) = u$ and $f(x) = x^\alpha$, we obtain

$$\Delta\left[p(n)\Delta x(n)\right] + q(n+1)x^\alpha(n+1) = 0. \tag{E_5}$$

If $\phi(u) = u^\sigma$, we get

$$\Delta\left[p(n)(\Delta x(n))^\sigma\right] + q(n+1)f(x(n+1)) = 0. \tag{E_6}$$

Oscillatory behavior of second order difference equation (E_3) has been investigated in [11] and [14]. In [6], the authors discussed the oscillatory behavior of (E_4).

Asymptotic behavior of non-oscillatory solution of second order difference equations has been dealt by many authors. In particular the papers [5], [7], [10] and [12] deal with the asymptotic behavior of difference equations of the form (E_2) and (E_5).

In Section 2, we provide lemmas which are needed in the proofs of the Theorems in Section 3 and Section 4. In Section 3, we establish sufficient conditions for the asymptotic behavior of non-oscillatory solutions of equation (1). In Section 4, conditions are established to ensure the oscillation of $\Delta x(n)$ whenever $x(n)$ is a solution of (1).

2. Lemmas

In this section we provide lemmas which are needed in the proofs of the theorems in Section 3.

The following result is extracted from [3].

Lemma 1. *If ϕ is sub-multiplicative on $[0,\infty)$, then ϕ satisfies $\phi\left(\frac{y}{x}\right) \geq \frac{\phi(y)}{\phi(x)}$ for all $x, y > 0$ and the inverse function ψ of ϕ is super multiplicative on $[0,\infty)$, that is, $\psi(x,y) \geq \psi(x)\psi(y)$ for all $x, y \geq 0$. Moreover, ψ satisfies $\psi\left(\frac{y}{x}\right) \leq \frac{\psi(y)}{\psi(x)}$ for all $x, y > 0$.*

Lemma 2. *Suppose that $x(n)$ is a non-oscillatory solution of (1) and $q(n) \geq 0$. Then $\Delta x(n)$ is non-oscillatory.*

Proof. Suppose to the contrary that $\Delta x(n)$ is oscillatory. Without loss of generality, we assume that $x(n) > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. Now $\Delta[p(n)\phi(\Delta x(n))] = -q(n+1)f(x(n+1)) < 0$ for $n \geq n_1$. Hence $p(n)\phi(\Delta x(n))$ is non-increasing for $n \geq n_1$. Hence there is a $n_2 \geq n_1$ such that $p(n)\phi(\Delta x(n)) = 0$ for $n \geq n_2$. This implies that $q(n) = 0$ for $n \geq n_2$.

This contradicts the assumption that $q(n) \geq 0$. Hence $\Delta x(n)$ is non-oscillatory. □

3. Asymptotic Behavior of Non-Oscillatory Solutions

In this section we discuss the asymptotic behavior of non-oscillatory solutions of equation (1).

Theorem 1. *Assume that (H_1)-(H_3) hold and*

$$\sum_{j=n_0}^{\infty} q(j+1) = \infty. \tag{2}$$

If $x(n)$ is an eventually positive solution of equation (1), then $\lim_{n\to\infty} x(n) = 0$.

Proof. Let $x(n) > 0$ for $n \geq n_0$. From Lemma 2 and equation (1), we obtain $\Delta x(n)$ is non-oscillatory. Now

$$\Delta\left[\frac{p(n)\phi(\Delta x(n))}{f(x(n))}\right] = \frac{f(x(n))\Delta[p(n)\phi(\Delta x(n))] - p(n)\phi(\Delta(x(n)))g(x(n+1),x(n))\Delta x(n)}{f(x(n))f(x(n+1))}$$

Hence

$$\Delta\left[\frac{p(n)\phi(\Delta x(n))}{f(x(n))}\right] = -q(n+1) - \frac{p(n)\phi(\Delta(x(n)))g(x(n+1),x(n))\Delta x(n)}{f(x(n))f(x(n+1))} \text{ for all } n \geq n_0. \tag{3}$$

Now we consider the following two cases.

Case (i). Suppose that $\Delta x(n)$ is eventually positive. We assume without loss of generality that $\Delta x(n) > 0$ for $n \geq n_1$. By (H_1), (H_2) and (3) we have

$$\Delta\left[\frac{p(n)\phi(\Delta x(n))}{f(x(n))}\right] \leq -q(n+1) \text{ for all } n \geq n_1. \tag{4}$$

Summing the inequality (4) from n_1 to $n-1$, we obtain

$$\frac{p(n)\phi(\Delta x(n))}{f(x(n))} - \frac{p(n_1)\phi(\Delta x(n_1))}{f(x(n_1))} \leq -\sum_{i=n_1}^{n-1} q(i+1),$$

$$\frac{p(n)\phi(\Delta x(n))}{f(x(n))} \leq \frac{p(n_1)\phi(\Delta x(n_1))}{f(x(n_1))} - \sum_{i=n_1}^{n-1} q(i+1).$$

Letting $n \to \infty$ and using (2), we obtain a contradiction. Hence $\Delta x(n)$ is eventually negative.

Case (ii). Suppose that $\Delta x(n) < 0$ is eventually negative. We assume without loss of generality that $\Delta x(n) < 0$ for $n \geq n_1$ and there exists a real number $\alpha \geq 0$ such that

$$\lim_{n\to\infty} x(n) = \alpha. \tag{5}$$

We claim that $\alpha = 0$. Otherwise $x(n) \geq \alpha > 0$ for $n \geq n_1$. By equations (1) and (5), we get

$$\Delta[p(n)\phi(\Delta x(n))] = -q(n+1)f(x(n+1)) \leq -q(n+1)f(\alpha)$$

for $n \geq n_1$. Hence

$$\Delta[p(n)\phi(\Delta x(n))] \leq -q(n+1)f(\alpha). \tag{6}$$

By summing the inequality (6) from n_1 to $n-1$, we obtain

$$p(n)\phi(\Delta x(n)) - p(n_1)\phi(\Delta x(n_1)) \leq -f(\alpha)\sum_{i=n_1}^{n-1} q(i+1),$$

$$p(n)\phi(\Delta x(n)) \leq p(n_1)\phi(\Delta x(n_1)) - f(\alpha)\sum_{i=n_1}^{n-1} q(i+1) \text{ for all } n \geq n_1. \tag{7}$$

By (2), the right hand side of (7) tends to $-\infty$ as $n \to \infty$ where the left hand side of (7) remains positive which is a contradiction. This contradiction completes the proof. □

Theorem 2. *Assume that (H_1)-(H_4) and*

$$\sum_{j=n_1}^{\infty} \psi\left(\frac{k}{p(j)}\right) \operatorname{sgn}k = \infty \text{ for every } \ k \neq 0 \tag{8}$$

hold. If $x(n)$ is an eventually negative solution of equation (1), then $\lim_{n\to\infty} x(n) = -\infty$.

Proof. Let $x(n) < 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. By Lemma 2, $\Delta x(n)$ is non-oscillatory. We consider the following two cases.

Case (i). Suppose that $\Delta x(n)$ is eventually positive. We assume without loss of generality that $\Delta x(n) > 0$ for $n \geq n_1$. It follows from equation (1) that $p(n)\phi(\Delta x(n))$ is non-decreasing for $n \geq n_1$. Then

$$p(n)\phi(\Delta x(n)) \geq p(n_1)\phi(\Delta x(n_1)) \text{ for } n \geq n_1.$$

Hence

$$\Delta x(n) \geq \psi\left[\frac{p(n_1)\phi\left(\Delta x(n_1)\right)}{p(n)}\right]. \tag{9}$$

Summing (9) from n_1 to $n-1$, we obtain

$$x(n) \geq x(n_1) + \sum_{i=n_1}^{n-1} \psi\left[\frac{p(n_1)\phi(\Delta x(n_1))}{p(i)}\right]. \tag{10}$$

It follows from (8) and (10) that $\lim_{n\to\infty} x(n) = \infty$ which contradicts the fact that $x(n) < 0$ for $n \geq n_1$. Hence $\Delta x(n)$ is eventually negative.

Case (ii). Suppose that $\Delta x(n)$ is eventually negative. Without loss of generality, we assume that $\Delta x(n) < 0$ for $n \geq n_1$. It follows from equation (1) that $p(n)\phi(\Delta x(n))$ is non-decreasing for $n \geq n_1$. Then

$$p(n)\phi(\Delta x(n)) \geq p(n_1)\phi(\Delta x(n_1)) \text{ for } n \geq n_1.$$

Hence

$$\Delta x(n) \leq \psi\left(\frac{-p(n_1)\phi(\Delta x(n_1))}{p(n)}\right). \tag{11}$$

Taking $u = p(n_1)\phi(\Delta x(n_1))$ and summing the above inequality from n_1 to $n-1$, we obtain $x(n) - x(n_1) \leq \sum_{i=n_1}^{n-1} \psi\left(-\frac{u}{p(i)}\right)$,

$$x(n) \leq x(n_1) + \sum_{i=n_1}^{n-1} \psi\left(-\frac{u}{p(i)}\right). \tag{12}$$

It follows from (12) that $\lim_{n\to\infty} x(n) = -\infty$. This completes the proof of our theorem. □

Theorem 3. *Assume that* (H_1)-(H_4) *hold. Suppose that*

$$\sum_{j=n}^{\infty} \psi\left(\frac{k}{p(j)}\right) < \infty \text{ for every } k > 0 \tag{13}$$

hold. If $x(n)$ is an eventually positive solution of equation (1), then $x(n)$ is bounded.

Proof. Let $x(n) > 0$ for $n \geq n_1$ where $n_1 \geq n_0$. By Lemma 2, $\Delta x(n)$ is non-oscillatory. Hence $\Delta x(n)$ is either eventually negative or eventually positive. If $\Delta x(n)$ is eventually negative, then $x(n)$ is bounded. If $\Delta x(n)$ is eventually positive, we assume that without loss of generality that $\Delta x(n) > 0$ for $n \geq n_1$. It follows from equation (1) that $p(n)\phi(\Delta x(n))$ is non-increasing for $n \geq n_1$. Then

$$\begin{aligned} p(n)\phi(\Delta x(n)) \leq & p(n_1)\phi(\Delta x(n_1)), \quad n \geq n_1, \\ \Delta x(n) \leq & \psi\left(\frac{p(n_1)\phi(\Delta x(n_1))}{p(n)}\right), \\ \Delta x(n) \leq & \psi\left(\frac{u}{p(n)}\right). \end{aligned}$$

Summing the above inequality from n_1 to $n-1$, we obtain,

$$x(n) \leq x(n_1) + \sum_{i=n_1}^{n-1} \left(\frac{u}{p(i)}\right).$$

It follows from (13) that $x(n)$ is bounded. This completes the poof of the theorem. □

Theorem 4. *Let* (H_1)*-*(H_4) *hold and* ϕ *be sub-multiplicative. Suppose that*

$$\sum^{\infty} q(i+1) < \infty \tag{14}$$

and

$$\sum^{\infty} \psi \left[\frac{1}{p(j)} \sum_{i=j}^{\infty} q(i+1) \right] = \infty. \tag{15}$$

If

$$\int_{\varepsilon}^{\infty} \frac{du}{\psi(f(u))} < \infty \text{ for each } \varepsilon > 0 \tag{16}$$

and $x(n)$ *is an eventually positive solution of (1), then* $\lim_{n\to\infty} x(n) = 0$.

Proof. Let $x(n) > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. By Lemma 2, $\Delta x(n)$ is non-oscillatory. We consider the following cases.

Case (i). Suppose that $\Delta x(n) > 0$ for $n \geq n_1$. It follows from (4) that

$$\Delta \left(\frac{p(n)\phi(\Delta x(n))}{f(x(n))} \right) \leq 0.$$

Hence $\frac{p(n)\phi(\Delta x(n))}{f(x(n))}$ is positive and decreasing for $n \geq n_1$. Thus

$$0 \leq \frac{p(n)\phi(\Delta x(n))}{f(x(n))} + \sum_{i=n_1}^{n-1} q(i+1) \leq \frac{p(n_1)\phi(\Delta x(n_1))}{f(x(n))}$$

for $n \geq n_1$. Letting $n \to \infty$, we obtain

$$\sum_{i=n_1}^{\infty} q(i+1) \leq \frac{p(n_1)\phi(\Delta x(n_1))}{f(x(n_1))}.$$

From Lemma 1, we see that

$$\frac{\Delta x(n)}{\psi(f(x(n))} = \frac{\psi(\phi(\Delta x(n)))}{\psi(f(x(n)))} \geq \psi\left(\frac{\phi(\Delta x(n))}{f(x(n))} \right) \geq \psi\left(\frac{1}{p(n)} \sum_{i=n}^{\infty} q(i+1) \right).$$

Thus

$$\sum_{j=n_1}^{n-1} \psi\left(\frac{1}{p(j)} \sum_{i=n}^{\infty} q(i+1) \right) \leq \sum_{j=n_1}^{n-1} \frac{\Delta x(n)}{\psi(f(x(n)))} \leq \int_{x(n_1)}^{x(n)} \frac{dt}{\psi(f(t))}. \tag{17}$$

Letting $n \to \infty$, right hand side of (17) is finite, where the left hand side of

(17) tends to ∞. Hence we obtain a contradiction.

Case (ii) Suppose that $\Delta x(n) < 0$ for $n \geq n_1$. Then (5) holds, i.e. $\lim_{n\to\infty} x(n) = \alpha$. We claim that $\alpha = 0$. Otherwise there exists a number $0 \leq \alpha \leq \beta$ such that $x(n) \geq \beta$ for $n \geq n_1$. It follows from (1) and (H_2) that

$$\Delta[p(n)\phi(\Delta x(n))] = -q(n+1)f(x(n+1)) \leq -q(n+1)f(\beta) \text{ for } n \geq n_1.$$

Summing from n_1 to $n-1$, we obtain

$$p(n)\phi(\Delta x(n)) \leq p(n_1)\phi(\Delta x(n_1)) - f(\beta)\sum_{i=n_1}^{n-1} q(i+1).$$

Letting $n \to \infty$, we obtain

$$p(n)\phi(\Delta x(n)) \geq f(\beta)\sum_{i=n_1}^{\infty} q(i+1).$$

It follows from Lemma 1 that

$$\begin{aligned}\Delta x(n) = \psi[-\phi(\Delta x(n))] \leq & \psi\left(-f(\beta)\frac{1}{p(n)}\sum_{i=n}^{\infty} q(i+1)\right)\\ \leq & \psi\left[-f(\beta)\right]\psi\left(\frac{1}{p(n)}\sum_{i=n}^{\infty} q(i+1)\right).\end{aligned}$$

Summing the above inequality from n_1 to $n-1$, we obtain

$$x(n) \leq x(n_1) + \psi\left(-f(\beta)\right)\sum_{j=n_1}^{n-1}\psi\left(\frac{1}{p(j)}\sum_{i=j}^{\infty} q(i+1)\right).$$

By (15), $\lim_{n\to\infty} x(n) = -\infty$ which is a contradiction. Hence $\lim_{n\to\infty} x(n) = 0$. □

Theorem 5. *Let (H_1)-(H_2) hold and ϕ be sub-multiplicative. Suppose that*

$$\sum^{\infty}\psi\left(\frac{1}{p(i)}\right) < \infty \tag{18}$$

and

$$P(n) = \sum_{i=n}^{\infty}\psi\left(\frac{1}{p(i)}\right). \tag{19}$$

If

$$\sum^{\infty}\psi\left(\frac{1}{p(i)}\sum^{n} q(i+1)f(\lambda p(i))\right) = -\infty \tag{20}$$

and $x(n)$ is an eventually negative solution of equation (1) then $\lim_{n\to\infty} x(n) = -\infty$.

Proof. Let $x(n) < 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. By Lemma 2, $\Delta x(n)$ is non-oscillatory. We consider the following two cases.

Case (i). Suppose that $\Delta x(n)$ is eventually positive. We assume without loss of generality that $\Delta x(n) > 0$ for $n \geq n_1$. As in the proof of Theorem 3, (10) holds for $n \geq n_1$. Hence

$$x(n) \geq x(n_1) + \sum_{i=n_1}^{n-1} \psi\left(\frac{p(n_1)\phi(\Delta x(n_1))}{p(i)}\right).$$

By Lemma 2 and (19), we obtain $-x(n) \geq \alpha - x(n) \geq -\lambda P(n)$ for $n \geq n_1$ where $P(n) = \sum_{i=n}^{\infty} \psi\left(\frac{1}{p(i)}\right)$ and $\lambda = -p(n_1)\phi(\Delta x(n_1)) < 0$ and $\lim_{n\to\infty} x(n) = \alpha \leq 0$.

Then $x(n) \leq \lambda P(n) < 0$ for $n \geq n_1$. From equation (1)

$$\Delta[p(n)\phi(\Delta x(n))] = q(n+1)f(x(n+1)) \geq -q(n+1)f(\lambda P(n)) \text{ for } n \geq n_1.$$

Summing the above inequality from n_1 to $n-1$, we obtain,

$$p(n)\phi(\Delta x(n)) - p(n_1)\phi(\Delta x(n_1)) \geq -\sum_{i=n_1}^{n-1} q(i+1)f(\lambda P(i+1)),$$

$$\begin{aligned} p(n)\phi(\Delta x(n)) &\geq p(n)\phi(\Delta x(n)) - p(n_1)\phi(\Delta x(n_1)) \\ &\geq -\sum_{i=n_1}^{n-1} q(i+1)f(\lambda P(i+1)). \end{aligned}$$

By Lemma 2,

$$\begin{aligned} \Delta x(n) = \psi[\phi(\Delta x(n))] &\geq \psi\left(-\frac{1}{p(n)}\sum_{i=n_1}^{n-1} q(i+1)f(\lambda P(i+1))\right) \\ &\geq \psi(-1)\psi\left(\frac{1}{p(n)}\sum_{i=n_1}^{n-1} q(i+1)f(\lambda P(i+1))\right). \end{aligned}$$

Summing from n_1 to $n-1$, we obtain

$$x(n) \geq x(n_1) + \psi(-1)\sum_{j=n_1}^{n-1} \psi\left(\frac{1}{p(n)}\sum_{i=n_1}^{j-1} q(i+1)f(\lambda P(i+1))\right).$$

Thus we obtain, $\lim_{n\to\infty} x(n) = \infty$ which contradicts the fact that $x(n) < 0$ for $n \geq n_1$.

Case (ii). Suppose that $\Delta x(n)$ is eventually negative. We assume without loss of generality that $\Delta x(n) < 0$ for $n \geq n_1$. It follows (1) that $p(n)\phi(\Delta(n))$ is non-decreasing for $n \geq n_1$. We assume without loss of generality that $x(n) < -P(n) < 0$ for $n \geq n_1$. Similar to the proof of Case (ii) in the previous theorem, we have

$$\Delta x(n) = \psi\left[-\phi(\Delta x(n))\right] \leq \psi\left[\frac{1}{p(n)}\sum_{i=n_1}^{n-1} q(i+1) f(-P(i+1))\right].$$

Summing again from n_1 to $n-1$, we obtain

$$x(n) \leq x(n_1) + \sum_{j=n_1}^{n-1} \psi\left[\frac{1}{p(j)}\sum_{i=n_1}^{j-1} q(i+1) f(-P(i+1))\right].$$

Thus, we obtain $\lim_{n\to\infty} x(n) = -\infty$. This completes the proof. □

4. Oscillatory Behavior

In this section, we assume that:

(H_5) $u\phi(u) \geq 0$ for $u \geq 0$.

Theorem 6. *Assume that*

$$\sum_{i=n_0}^{\infty} q(i+1) = \infty \tag{21}$$

and

$$\sum_{i=n_0}^{\infty} \phi^{-1}\left[\frac{k}{p(i)}\right] = -\infty \text{ for every } k < 0. \tag{22}$$

If $x(n)$ is a solution of (1), then $\Delta x(n)$ is oscillatory.

Proof. If $x(n)$ is an oscillatory solution of (1) for $n \geq n_0$, then $\Delta x(n)$ is also oscillatory for $n \geq n_0$. Without loss of generality, we may assume that $x(n)$ is an eventually positive solution of (1). Thus $x(n) \geq 0$ for $n \geq n_0$. Hence $f(x(n+1)) > 0$. We divide the proof in to the following two cases.

(i) $\Delta x(n) > 0$ for $n \geq n_0$.

(ii) $\Delta x(n) < 0$ for $n \geq n_0$.

Case (i). Define $w(n) = \frac{p(n)\phi(\Delta x(n))}{f(x(n))}$. Then $\Delta w(n) \leq -q(n+1)$ for all $n \geq n_0$. Summing the above inequality from n_0 to $n-1$, we obtain

$$w(n) - w(n_0) \leq -\sum_{i=n_0}^{n-1} q(i+1).$$

Allowing $n \to \infty$, we find that $w(n) < 0$. Hence $\Delta x(n) < 0$ for n large enough. This is a contradiction.

Case (ii). Let $\Delta x(n) < 0$ for n large enough. It follows from (21) that there exists a number $n_1 \geq n_0$ such that $\sum_{i=n_1}^{n-1} q(i+1) \geq 0$ for $n \geq n_1$. Now from (1), we obtain,

$$\Delta[p(n)\phi(\Delta x(n))] = -q(n+1)f(x(n+1)) \leq 0,$$
$$\Delta[p(n)\phi(\Delta x(n))] \leq 0.$$

Summing the above inequality from n_1 to $n-1$, we obtain,

$$p(n)\phi(\Delta x(n)) \leq p(n_1)\phi(\Delta x(n_1)) < 0,$$
$$\phi(\Delta x(n)) \leq \frac{k}{p(n)} \quad \text{for some constant } k < 0,$$
$$\Delta x(n) \leq \phi^{-1}\left(\frac{k}{p(n)}\right).$$

Summing the above inequality from N to $n-1$, we get

$$x(n) - x(N) \leq \sum_{i=N}^{n-1} \phi^{-1}\left(\frac{k}{p(i)}\right).$$

The above inequality together with (22) imply that $\lim_{n\to\infty} x(n) = -\infty$. This contradicts $x(n)$ being eventually positive for $n \geq n_0$. Hence $\Delta x(n)$ is oscillatory. □

Taking $\phi(u) = f(u) = |u|^{\alpha-2} u$, (1) is reduced to the form

$$\Delta[p(n)\,|\Delta x(n)|^{\alpha-2}\,\Delta x(n)] + q(n+1)\,|x(n+1)|^{\alpha-2}\,x(n+1) = 0. \tag{23}$$

Clearly, the inverse of ϕ is $\phi^{-1}(s) = |s|^{\frac{1}{\alpha-1}}\operatorname{sgn}(s)$. Thus we have the following theorem.

Theorem 7. *Assume that* $\sum_{i=n_0}^{\infty} q(i+1) = \infty$ *and* $\sum_{i=n_0}^{\infty} \left|\frac{k}{p(i)}\right|^{\frac{1}{\alpha-1}} = \infty$ *for every* $k < 0$ *hold. If* $x(n)$ *is a solution of (23), then* $\Delta x(n)$ *is oscillatory.*

5. Examples

In this section, we provide examples to illustrate the results established in Sections 3 and 4.

Example 1. We consider the following difference equation

$$\Delta\left[n3^n\Delta x(n)\right] + \frac{3^n(n+2)}{2}x(n+1) = 0, \quad n \geq 1, \tag{24}$$

for which the conditions of Theorem 1 are satisfied. Hence $\lim_{n\to\infty} x(n) = 0$ if $x(n)$ is an eventually positive solution of equation (24). For the difference equation (24), $x(n) = \left(\frac{2}{3}\right)^{n+1}$ is an eventually positive solution such that $\lim_{n\to\infty} x(n) = 0$.

Example 2. We consider the following difference equation

$$\Delta\left[\frac{1}{n}(\Delta x(n))^{\sigma}\right] + \frac{1}{n(n+1)^2}x(n+1) = 0, \quad n \geq 1, \tag{25}$$

where $\sigma \geq 1$ is any quotient of odd integers. Here $\phi(u) = u^{\sigma}$ and $f(x) = x$. Also $p(n) = \frac{1}{n}$ and $q(n+1) = \frac{1}{n(n+1)^2}$ such that $\sum_{i=1}^{\infty} q(i+1) = 2 - \frac{\pi^2}{6} < \infty$ which violates the condition (2) in Theorem 1. Hence, eventhough $x(n) = n$ is an eventually positive solution of (25), we have $\lim_{n\to\infty} x(n) \neq 0$.

Example 3. For the difference equations

$$\Delta\left[\frac{1}{3^n}\Delta x(n)\right] + \frac{1}{2 \times 3^{n+1}}x(n+1) = 0, \quad n \geq 1, \tag{26}$$

the conditions of Theorem 2 are satisfied. Hence $\lim_{n\to\infty} x(n) = -\infty$ if $x(n)$ is an eventually negative solution of (26). The equation (26) has an eventually negative solution $x(n) = -\left(\frac{3}{2}\right)^n$ such that $\lim_{n\to\infty} x(n) = -\infty$.

Example 4. In this example, we consider the difference equation

$$\Delta\left[\frac{1}{2^n}\Delta x(n)\right] + \frac{1}{2^{n+1}(n+1)^3}x^3(n+1) = 0, \quad n \geq 1, \tag{27}$$

in which $\phi(u) = u$ and $f(x) = x^3$. Also $p(n) = \frac{1}{2^n}$ and $q(n+1) = \frac{1}{2^{n+1}(n+1)^3}$. Hence by Theorem 2, we see that, if $x(n)$ is an eventually negative solution of the equation (27), then $\lim_{n\to\infty} x(n) = -\infty$. In fact $x(n) = -n$ is an eventually negative solution of (27).

Example 5. In the following difference equation (28), $p(n) = n(n+1)$ and $q(n) = \frac{2}{n+3}$ such that $\sum_{i=n_0}^{\infty} q(i+1) = \infty$.

$$\Delta[n(n+1)\Delta x(n)] + \frac{2}{n+3}x(n+1) = 0, \quad n \geq 1. \tag{28}$$

We find that the conditions of Theorem 3 are satisfied and hence if $x(n)$ is an eventually positive solution of (28), then $x(n)$ is bounded. In fact $x(n) = \frac{1}{n+1}$ is a solution which is bounded between 0 and 1.

Example 6. In the following difference equation

$$\Delta[n^2\Delta x(n)] + \frac{1}{(n+2)^2}x(n+1) = 0, \quad n \geq 1, \tag{29}$$

we have, $p(n) = n^2$ and $q(n+1) = \frac{1}{(n+2)^2}$. Hence

$$\sum_{j=n}^{\infty} \psi\left(\frac{k}{p(j)}\right) = \sum_{j=n}^{\infty}\left(\frac{k}{p(j)}\right) = k\sum_{j=1}^{\infty}\left(\frac{1}{n^2}\right) = k\frac{\pi^2}{6} < \infty.$$

By Theorem 3, we conclude that, if $x(n)$ is an eventually positive solution of (29), then $x(n)$ is bounded. For the equation (29), $x(n) = \frac{n+1}{n}$ is one such eventually positive solution which is bounded.

Example 7. Consider the difference equation

$$\Delta[(n+1)(\Delta x(n))^2] + q(n+1)x^3(n+1) = 0, \quad n \geq 1, \tag{30}$$

for which a solution is given by

$$x(n) = \begin{cases} -2 & : n \text{ is odd}, \\ -1 & : n \text{ is even}, \end{cases}$$

and

$$q(n) = \begin{cases} \frac{1}{8} & : n \text{ is odd}, \\ 1 & : n \text{ is even}. \end{cases}$$

We see that conditions of Theorem 6 are satisfied and hence $\{\Delta x(n)\}$ is oscillatory.

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