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# REFLECTIVE SPACE-FILLING POLYHEDRA

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**Abstract:** A polyhedron in 3-space is called a *space-filler* or a *space-filling polyhedron* if its infinitely many (directly or reflectively) congruent copies fill the space with no gaps and no (3-dimensional) overlaps. A space-filling polyhedron $P$ is called a *reflective* space-filler if a tiling by congruent copies of $P$ satisfies the following three conditions: (1) the tiling is face-to-face, (2) if two tiles $P_1$ and $P_2$ in the tiling have a common face, $P_1$ is the mirror-image of $P_2$ in the plane containing $P_1 \cap P_2$, and (3) the chromatic number of the tiling is two. H.S. Coxeter [2], [3] proved that there exist only seven types of reflective space-fillers. In this paper, we give an elementary proof of this fact.

**AMS Subject Classification:** 52B10
**Key Words:** space-filler, polyhedron, reflective, tiling

## 1. Preliminaries and Definitions

A polyhedron in the space $\mathbb{R}^3$ is called a *space-filler* or a *space-filling polyhedron* if its infinitely many (directly or reflectively) congruent copies fill the space

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with no gaps and no (3-dimensional) overlaps. Hilbert asked in his eighteenth problem [4]: "What polyhedra exist for which a complete filling of all space is possible by juxtaposition of congruent copies ?" Even if it is restricted to only the tetrahedra, it is still open [1], [5].

**Definition 1.** A convex polyhedron $P$ is called a *reflective space-filler* or a *reflective space-filling polyhedron* if its congruent copies tile space in such a way that:

(1) the tiling is face-to-face,

(2) if the intersection $P_1 \cap P_2$ of two of those copies has a face in common, then $P_1$ is the mirror-image of $P_2$ in the plane with $P_1 \cap P_2$, and

(3) the chromatic number of the tiling is two, that is, all tiles can be assigned two colors so that any two tiles with a face in common have distinct colors.

There are three types of tetrahedral reflective space-fillers [5], two of which were found by D.M.Y. Sommerville [6], [7].

**Definition 2.** We call a tetrahedron $P$ with vertex set $\{A, B, C, D\}$ the $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron if $|AB| = |CD| = 2$ and $|AC| = |AD| = |BC| = |BD| = \sqrt{3}$ (Figure 1), where $|XY|$ means the length of the edge $XY$ joining two points $X, Y$ in the space. Then four faces of $P$ are congruent to each other, the dihedral angles of $AB$ and $CD$ are $\pi/2$, and those of the remaining edges are $\pi/3$.

Let $E$ be the midpoint of the edge $AB$. We call the tetrahedron with the vertex set $\{A, E, C, D\}$ the *half* $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron (Figure 2). Then $|CD| = 2$, $|AE| = 1$, $|AC| = |AD| = \sqrt{3}$ and $|CE| = |DE| = \sqrt{2}$. The dihedral angles of $AE, CE$, and $DE$ are $\pi/2$, those of $AC$ and $AD$ are $\pi/3$, and that of $CD$ is $\pi/4$.

Let $F$ be the midpoint of the edge $CD$. We call the tetrahedron with the vertex set $\{A, E, C, F\}$ the quarter $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron (Figure 3). Then $|AE| = |EF| = |CF| = 1$, $|AF| = |CE| = \sqrt{2}$ and $|AE| = \sqrt{3}$. The dihedral angles of $AE, CE$, and $EF$ are $\pi/2$, those of $AE$ and $CF$ are $\pi/4$ and that of $AC$ is $\pi/3$.

The following theorem was proved by H.S. Coxeter [2], [3].

**Theorem.** (see [2], [3]) *There exist only seven types of reflective space-fillers among polyhedra. They are the $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron, the half-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron, the quarter-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron, three triangular right prisms whose baces are an equilateral triangle, an isosceles right triangle, or a right triangle with angles $\pi/6$ and $\pi/3$, and cuboids.*

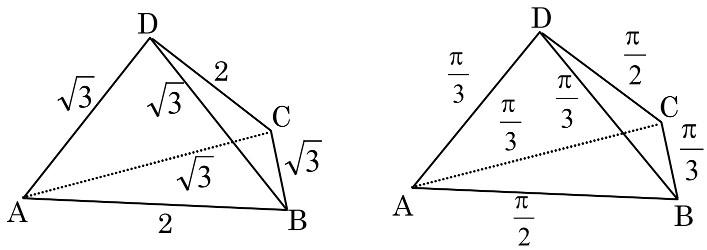



Figure 1: The $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron with lengths and dihedral angles of edges

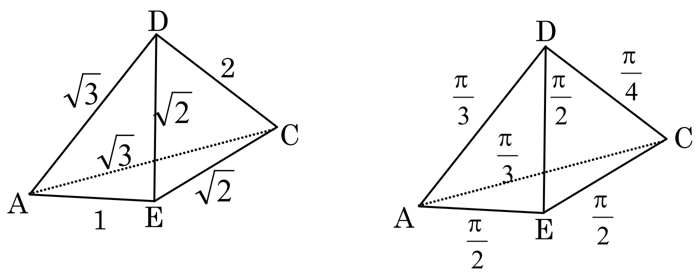



Figure 2: The half-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron with lengths and dihedral angles of edges

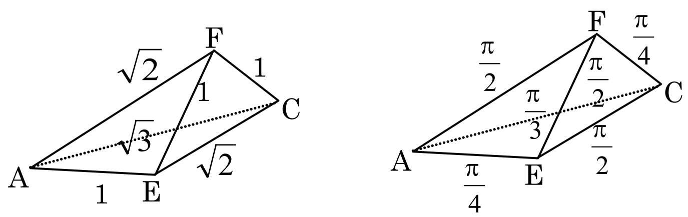



Figure 3: The quarter-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron with lengths and dihedral angles of edges

H.S. Coxeter proved the theorem by using positive semidefinite matrices. The proof includes more general cases for $n$ dimensional spaces. In this paper, we restrict the result to three dimensional space only and give an elementary proof which is self-contained.

## 2. Proof of Theorem

For the proof of Theorem we need several lemmas and propositions.

**Lemma 1.** *Every reflective space-filling polyhedron $P$ is convex and every dihedral angle of $P$ is $\pi/k$ for some integer $k \geq 2$, and the angle is at most $\pi/2$.*

*Proof.* Every reflective space-filling polyhedron $P$ is convex by the conditions (1) and (2) in Definition 1. Since every edge of a congruent copy of $P$ in the reflective tiling, is surrounded by the same number of directly or reflectively congruent copies of $P$ by the condition (3) in Definition 1, its dihedral angle is $\pi/k$ for some integer $k \geq 2$, and it is at least $\pi/2$. □

**Definition 3.** The *edge graph* of a polyhedron $P$ is the topological graph defined by its vertices and edges naturally. The *sphere graph* $G(P)$ of $P$ is the geometric graph which is defined as follows: The vertex set of $G(P)$ is the set of unit normal vectors of the faces of $P$, which is described on the unit sphere in $\mathbb{R}^3$ with center at the origin. For two vertices $a$ and $b$ of $G(P)$, if their corresponding faces have an edge in common, then $a$ and $b$ are joined by the shortest arc on the unit sphere. The edge set of $G(P)$ is the set of all such arcs. Notice that the edge graph of $P$ and the geometrical graph $G(P)$ are topologically dual graphs of each other.

**Lemma 2.** *Let $P$ be a reflective space-filling polyhedron.*

*(1) The length of an edge $ab$ in $G(P)$ is $(1-1/k)\pi$ where $\pi/k$ is the dihedral angle of the two faces corresponding to $a$ and $b$ in $P$.*

*(2) The origin is an interior point of the convex hull of the vertex set of $G(P)$.*

*(3) $G(P)$ is 3-connected.*

*Proof.* Lemma 2 (1) holds by the definition of the edge set of $G(P)$. Since $P$ is a convex polyhedron, Lemma 2 (2) holds. Since by deleting any two faces of $P$, the set of remaining faces of $P$ are still connected, $G(P)$ is 3-connected. □

**Lemma 3.** *The edge graph of any reflective space-filling polyhedron $P$ is 3-regular, that is, all faces of $G(P)$ are triangles.*

*Proof.* Let $P$ be a reflective space-filling polyhedron and let $\mathcal{P}$ be the tiling by congruent copies of $P$, which satisfies conditions (1), (2) and (3) in Definition 1. Let $A$ be a vertex of a congruent copy of $P$ in $\mathcal{P}$. Describe a sphere $S$ with

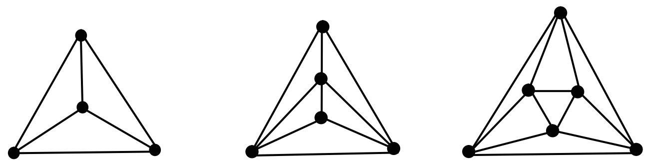


Figure 4: Sphere graphs of reflective space-filling polyhedra

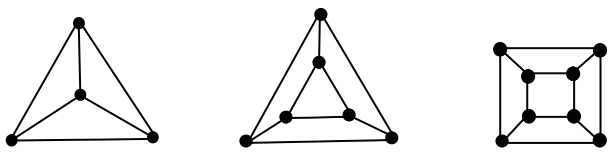


Figure 5: Edge graphs of reflective space-filling polyhedra

the center $A$ and a sufficiently small radius. Denote by $H$ the geometrical graph obtained by the intersection of $S$ and faces of polyhedra in the tiling $\mathcal{P}$. Denote the numbers of vertices, edges, and faces of $H$ by $v, e$, and $f$, respectively. Then $v + f - e = 2$ by Euler's formula. Since $P$ is a reflective space-filler, each face of the graph $H$ is surrounded by the same number (denoted by $l$) of edges and $l \geq 3$.

Suppose $l \geq 4$. Then by counting the number of edges in $H$ twice we get $2e = lf \geq 4f$ and hence $f \leq e/2$. Since each edge of a polyhedron in the tiling $\mathcal{P}$ has dihedral angle at most $\pi/2$ by Lemma 1, it is surrounded by at least four congruent copies of $P$ in $\mathcal{P}$. Hence the degree of any vertex in $H$ is at least four. By counting the number of edges in $H$ twice we get $2e \geq 4v$. Hence $v \leq e/2$. We have a contradiction

$$2 = v + f - e \leq e/2 + e/2 - e = 0.$$

Therefore, $l < 4$ and so $l = 3$. All faces of $H$ are triangles, the edge graph of $P$ is 3-regular, and all faces of $G(P)$ are triangles. □

**Lemma 4.** *The sphere graph $G(P)$ of every reflective space-filling polyhedron $P$ is topologically isomorphic to one of the graphs showed in Figure 4. Hence the edge graph of $P$ is isomorphic to one of three graphs shown in Figure 5.*

*Proof.* Let $v, f$, and $e$ be the numbers of vertices, faces, and edges in $G(P)$, respectively for a reflective space-filling polyhedron $P$. Since all faces of the sphere graph $G(P)$ of a reflective space-filling polyhedron $P$ are triangles by Lemma 3, we have $3f = 2e$. Since the lengths of edges in $G(P)$ are at least $\pi/2$ by Lemma 1, any triangular face of $G(P)$ has area at least one eighth of unit sphere. Hence $f \leq 8$. By $3f = 2e$, $f$ is even and hence $f = 4, 6$, or 8. By the Euler formula $v + f - e = 2$, we have $v = 4, 5$, or 6 with respect to $(f, e) = (4, 6), (6, 9)$, or $(8, 12)$. Since $G(P)$ is 3-connected, $G(P)$ is isomorphic to one of graphs shown in Figure 4. Hence the edge graph of $P$ which is the dual graph of $G(P)$, is isomorphic to one of three graphs showed in Figure 5. □

For a point on the unit sphere $S$ with center at the origin, we use both the spherical coordinates $(\theta, \phi)$ and the orthogonal coordinates $[x, y, z]$, where $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$, $x = \cos\phi \sin\theta, y = \sin\phi \sin\theta$, and $z = \cos\theta$. For two points $u, v$ on $S$ the symbol $\| uv \|$ stands for the shortest arc length between $u$ and $v$ on $S$. For a real number $0 < \alpha < \pi$ we denote the circle on $S$ with center $v$ by

$$\Gamma(v, \alpha) = \{u \in S : \| uv \| = \alpha\}.$$

A subset of $S$ is called *$\alpha$-slice* if it is congruent to the set $\{(\theta, \phi) : 0 \leq \theta \leq \pi, 0 \leq \phi \leq \alpha\}$, whose boundary consists of two semicycles. For example, a $\pi$-slice is a hemisphere.

**Lemma 5.** *Let $P$ be a reflective space-filling polyhedron and let $u, v$ be two distinct adjacent vertices in $G(P)$. Then there exists a $(\pi- \| uv \|)$-slice which includes all vertices in $G(P)$ except $u$ and $v$.*

*Proof.* Let $u, v$ be any two distinct vertices in $G(P)$ for a reflective space-filling polyhedron $P$. Let $w$ ($w \neq u$ and $w \neq v$) be a vertex in $G(P)$. By Lemma 1 we have $\| uw \| \geq \pi/2$ and $\| vw \| \geq \pi/2$. Hence all vertices in $G(P)$ except $u$ and $v$ are included in a $(\pi- \| uv \|)$-slice bounded by two semicircles of $\Gamma(u, \pi/2)$ and $\Gamma(v, \pi/2)$. □

We prove the theorem through the following six propositions. From Proposition 1 to Proposition 4, we assume that the sphere graph $G(P)$ of a reflective space-filling polyhedron $P$ has four vertices $v_i$ ($i = 1, 2, 3, 4$). We can assume, without loss of generality, that the edge $v_1v_2$ is the longest edge in $G(P)$, and that the spherical coordinates of the vertices are $v_1 = (0, 0)$ and $v_2 = (\| v_1v_2 \|, 0)$. Let

$$\| v_1v_2 \| = (1 - 1/k)\pi$$

for some integer $k \geq 2$ by Lemma 2 (1). Then $v_3$ and $v_4$ are included in a $\pi/k$-slice by Lemma 5.

**Proposition 1.** *We have $k \leq 5$.*

*Proof.* Suppose $k \geq 6$. Then $v_2 = (1 - 1/k)\pi, 0)$ for $5\pi/6 \leq (1 - 1/k)\pi \leq \pi$.

Let $v = v_i$ ($i = 3$ or $4$) and let $v = (\theta, \phi)$ for the spherical coordinates. By Lemma 2 (1),

$$v \in \Gamma(v_1, (1 - 1/l)\pi) \text{ and } v \in \Gamma(v_2, (1 - 1/m)\pi)$$

for some integers $l \geq 2$ and $m \geq 2$. Since $v_2 = (1 - 1/k)\pi, 0)$ for $5\pi/6 \leq (1 - 1/k)\pi \leq \pi$, we get $\theta = \pi/2$ and $v \in \Gamma(v_1, \pi/2)$.

Similarly, we get $v \in \Gamma(v_2, \pi/2)$. Therefore $v_3$ and $v_4$ are two points with spherical coordinates $(\pi/2, \pi/2)$ and $(\pi/2, 3\pi/4)$. This contradicts Lemma 2 (2). Therefore $k \leq 5$. ☐

**Proposition 2.** *We have* $3 \leq k \leq 4$.

*Proof.* Suppose $k = 2$. Then $\| v_i v_j \| = \pi/2$ for all $i, j (\neq i) \in \{1, 2, 3, 4\}$. This contradicts Lemma 2 (2). Hence $3 \leq k$.

Suppose $k = 5$. Then $v_2 = (4\pi/5, 0) = (\sin(4\pi/5), 0, \cos(4\pi/5)$. Since the edge $v_1 v_2$ is the longest edge in $G(P)$ and $\| v_1 v_2 \| = 4\pi/5$, for any $i$ and $j(\neq i)$ in $\{1, 2, 3, 4\}$, we have

$$\| v_i v_j \| \in \{(1 - 1/l)\pi : l = 2, 3, 4, 5\}.$$

Since $|v_3 v_2| \geq \pi/2$, we have $v_3 v_1 \leq \pi + \pi/5 - \pi/2 = 7\pi/10$. Since $\| v_3 v_1 \| = (1 - 1/l)\pi$ for some integer $l \geq 2$,

$$\| v_3 v_1 \| = \pi/2 \text{ or } v_3 v_1 = 2\pi/3,$$

that is, $v_3 \in \Gamma(v_1, \pi/2) \cup \Gamma(v_1, 2\pi/3)$. Similarly, we get $v_4 \in \Gamma(v_1, \pi/2) \cup \Gamma(v_1, 2\pi/3)$. Exchange $v_1$ and $v_2$ in these statements. The resulting statements also hold. Hence we have

$$\{v_3, v_4\} \subset \bigcap_{i=1}^{2} \{\Gamma(v_i, \pi/2) \cup \Gamma(v_i, 2\pi/3)\}.$$

The right set in the above has six points and we denote them as follows:

$$p_1 = (\pi/2, \pi/2) = [0, 1, 0], \ p_2 = (\pi/2, 3\pi/2) = [0, -1, 0],$$

$$p_3 = (\pi/2, \alpha) = [\cos\alpha, \sin\alpha, 0],$$

$$p_4 = (\pi/2, 2\pi - \alpha) = [\cos\alpha, -\sin\alpha, 0],$$

$$p_5 = (2\pi/3, \beta) = [\sqrt{3}/2 \cos\beta, \sqrt{3}/2 \sin\beta, -1/2],$$

and

$$p_6 = (2\pi/3, 2\pi - \beta) = [\sqrt{3}/2 \cos\beta, -\sqrt{3}/2 \sin\beta, -1/2],$$

where $\pi/2 < \alpha, \beta < \pi$ (Figure 6).

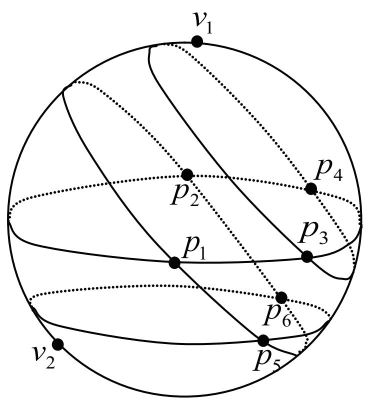



Figure 6: Possible six points for $v_3$ and $v_4$

By Lemma 2 (2) the set $\{v_3, v_4\}$ is $\{p_1, p_4\}$, $\{p_1, p_6\}$, $\{p_2, p_3\}$, $\{p_2, p_5\}$, $\{p_3, p_4\}$, $\{p_3, p_6\}$, or $\{p_4, p_5\}$.

Since $\| v_2p_3 \|= 2\pi/3$ for $p_3 \in \Gamma(v_2, 2/3\pi)$ and $\| v_2p_5 \|= \pi/2$ for $p_5 \in \Gamma(v_2, \pi/2)$, calculating inner products we get

$$\sin 4\pi/5 \cos\alpha = -1/2, \quad \sqrt{3}/2 \sin 4\pi/5 \cos\beta - 1/2 \cos 4\pi/5 = 0.$$

Hence, we have

$$\alpha = \cos^{-1}(-0.8506\cdots) \text{ and } \beta = \cos^{-1}(-0.7946\cdots),$$

which imply

$\| p_1p_4 \|= (2\pi - \alpha) - \pi/2 = 3\pi/2 - \alpha \notin \{(l-1)\pi/l : l = 2, 3, 4, 5\}$,

$\| p_1p_6 \|= \cos^{-1}(-\sqrt{3}/2 \sin\beta) = \cos^{-1}(-0.52\cdots) \notin \{(l-1)\pi/l : l = 2, 3, 4, 5\}$,

$\| p_3p_4 \|= 2\pi - 2\alpha \notin \{(l-1)\pi/l : l = 2, 3, 4, 5\}$, and

$\| p_3p_6 \|= \cos^{-1}(\sqrt{3}/2 \cos\alpha\cos\beta - \sqrt{3}/2 \sin\alpha \sin\beta) \notin \{(l-1)\pi/l : l = 2, 3, 4, 5\}$.

This contradicts Lemma 1. Therefore $k \leq 4$ and so $2 \leq k \leq 4$. □

**Proposition 3.** *If $k = 4$, then $P$ is the half-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron or the quarter-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron.*

*Proof.* Suppose $k = 4$. Then $v_2 = (3\pi/4, 0) = [1/\sqrt{2}, 0, -1/\sqrt{2}]$. Since the edge $v_1v_2$ is the longest edge in $G(P)$ and $\| v_1v_2 \|= 3\pi/4$, for any $i$ and $j(\neq i)$

in $\{1,2,3,4\}$, we have

$$\| v_i v_j \| \in \{(1-1/l)\pi : l = 2,3,4\}.$$

The intersection of the two circles $\Gamma(v_1, \pi/2)$ and $\Gamma(v_2, 3\pi/4)$ is the point $(\pi/2, \pi)$ and the intersection of the two circles $\Gamma(v_1, 3\pi/4)$ and $\Gamma(v_2, \pi/2)$ is the point $(3\pi/4, \pi)$. None of these two points is $v_3$ nor $v_4$ by Lemma 2 (2). Hence

$$\{v_3, v_4\} \subset \bigcap_{i=1}^{2} \{\Gamma(v_i, \pi/2) \cup \Gamma(v_i, 2\pi/3)\}.$$

We denote the six points in the right set above by $p_i$ $(1 \le i \le 6)$ similar to the proof of Proposition 2.

Since $\| v_2 p_3 \| = 2\pi/3$ and $\| v_2 p_5 \| = \pi/2$, by calculating inner products we get $1/\sqrt{2}\cos\alpha = -1/2$, $(\sqrt{3}/2\sqrt{2})\cos\beta + 1/2\sqrt{2} = 0$. Then $\alpha = 3\pi/4$, $\cos\beta = -1/\sqrt{3}$ and $\sin\beta = \sqrt{2/3}$. These imply

$$\| p_1 p_4 \| = 3\pi/2 - \alpha = 3\pi/4, \quad \| p_3 p_4 \| = 2\pi - 2\alpha = \pi/2,$$

$$\| p_3 p_6 \| = \cos^{-1}(\sqrt{3}/2\cos\alpha\cos\beta - \sqrt{3}/2\sin\alpha\sin\beta) = \cos^{-1}(-1/2\sqrt{2} \\ -1/2) \notin \{(1-1/l)\pi : l = 2,3,4,5\}.$$

Therefore $\{v_3, v_4\} = \{p_1, p_4\}$ or $\{v_3, v_4\} = \{p_3, p_4\}$, except for isomorphisms on $G(P)$.

(i) Suppose $\{v_3, v_4\} = \{p_1, p_4\}$. Then we can say $v_3 = p_1$ and $v_4 = p_4$ without loss of generality. Then $G(P)$ and its dual graph (that is, the edge graph of $P$) are graphs shown in Figure 7. Hence, $P$ is the quarter-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron.

(ii) Suppose $\{v_3, v_4\} = \{p_3, p_4\}$. Then we can say $v_3 = p_3$ and $v_4 = p_4$ without loss of generality. Then $G(P)$ and its dual graph (the edge graph of $P$) are graphs shown in Figure 8. Hence, $P$ is the half-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron. □

**Proposition 4.** *If $k = 3$, then $P$ is the $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron.*

*Proof.* Suppose $k = 3$. Since the edge $v_1 v_2$ is the longest edge in $G(P)$ and $|v_1 v_2| = 2\pi/3$, we have $v_2 = (2\pi/3, 0) = (\sqrt{3}/2, 0, -1/2)$ and

$$\| v_i v_j \| \in \{\pi/2, \ 2\pi/3\}$$

for $i$, $j\,(\ne i)$ in $\{1,2,3,4\}$. By an argument similar to the one in the proof of Proposition 2, we have

$$\{v_3, \ v_4\} \subset \{p_i : \ 1 \le i \le 6\},$$

where $p_i$ $(1 \le i \le 6)$ are points similarly defined in the proof of Proposition 2

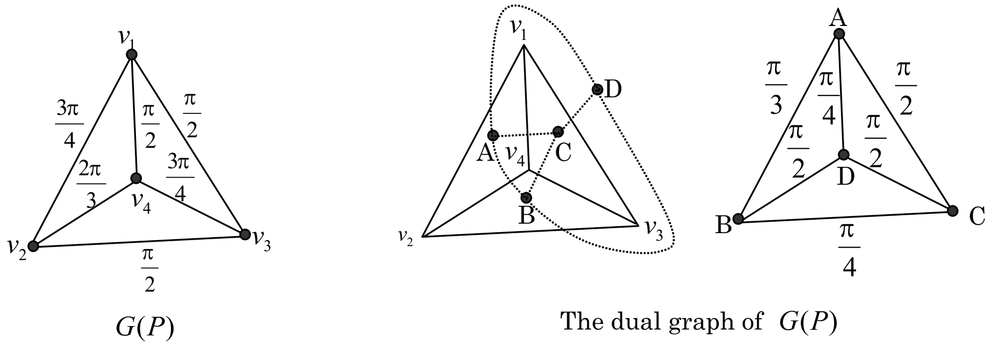



Figure 7: The sphere graph and its dual graph (the edge graph) with dihedral angles of the quarter-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron

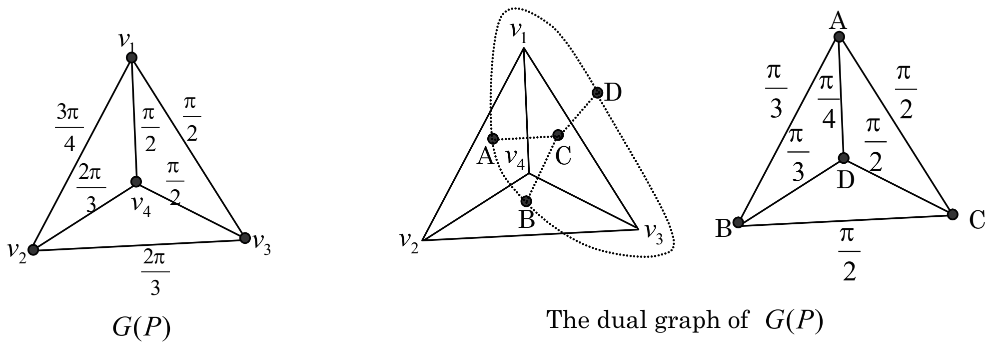



Figure 8: The sphere graph and its dual graph (the edge graph) with dihedral angles of the half-$(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron

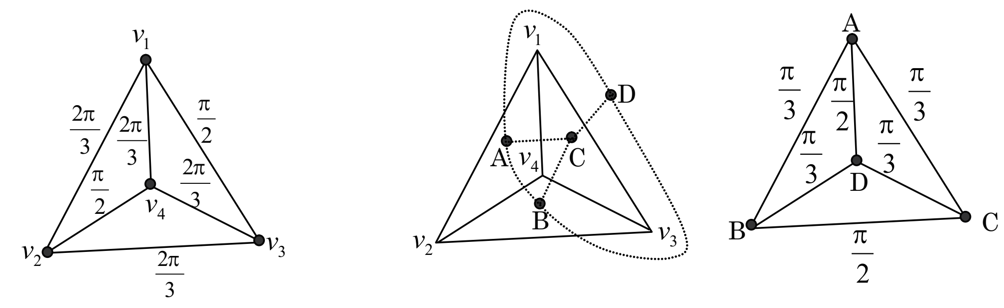
(Figure 6). Since $|v_2p_3| = 2\pi/3$ and $|v_2p_5| = \pi/2$, by calculating inner products we get

$$\cos 2\pi/3 = \sqrt{3}/2 \cos\alpha,\ \cos\pi/2 = 3/4\cos\beta + 1/4$$

and hence

$$\cos\alpha = -1/\sqrt{3},\ \ \sin\alpha = \sqrt{2/3},\ \ \sin\beta = 2\sqrt{2}/3.$$

These imply $\| p_1p_4 \| = (2\pi - \alpha) - \pi/2 = 3\pi/2 - \alpha \notin \{\pi/2,\ 2\pi/3\}$, $\| p_3p_4 \| = 2\pi - 2\alpha \notin \{\pi/2,\ 2\pi/3\}$, $\| p_3p_6 \| = \cos^{-1}(\sqrt{3}/2\cos\alpha\cos\beta - \sqrt{3}/2\sin\alpha\sin\beta) = \cos^{-1}(-1/2) = 2\pi/3$.




$G(P)$ The dual graph of $G(P)$

Figure 9: The sphere graph and its dual graph (the edge graph) with edge angles of the $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron

Hence $\{v_3, v_4\} = \{p_3, p_6\}$ and $\| v_3v_4 \| = 2\pi/3$. $G(P)$ and the edge graph of $P$, where $v_3 = p_3$, $v_4 = p_6$, are shown in Figure 9. Therefore $P$ is the $(\sqrt{3}, \sqrt{3}, 2)$-tetrahedron. □

**Proposition 5.** *If the sphere graph $G(P)$ of a reflective space-filling polyhedron $P$ has five vertices, then $P$ is a triangular right prism whose base is an equilateral triangle, an isosceles right triangle, or a right triangle with angles of $\pi/6$ and $\pi/3$.*

*Proof.* Let $P$ be a reflective space-filling polyhedron whose sphere graph $G(P)$ has five vertices. Then $G(P)$ has two vertices (namely, $v_1, v_2$) with degree three and three vertices (namely, $v_3, v_4, v_5$) with degree four shown in Figure 4. So the three vertices $v_3, v_4$, and $v_5$ are adjacent each other. Divide $S$ by the plane containing $\{v_3, v_4, v_5\}$. Since the interiors of any two edges in $G(P)$ are disjoint, two vertices $v_1, v_2$ are included in the distinct parts of $S$.

We show that the circle $C$ passing through $v_3$, $v_4$ and $v_5$ is a great circle of $S$. If not, one of $\{v_1, v_2\}$ (say, $v_1$) and the circle $C$ is included in the interior of a hemisphere of $S$, which contradicts $\| v_1v_i \| \geq \pi/2$ for all $i = 3, 4, 5$. Hence the circle $v_3v_4v_5$ is a great circle of $S$.

Therefore the set $\{\| v_3v_4 \|, \| v_3v_5 \|, \| v_4v_5 \|\}$ is

$$\{2\pi/3, 2\pi/3, 2\pi/3\}, \quad \{\pi/2, 3\pi/4, 3\pi/4\}, \quad \text{or} \quad \{\pi/2, 2\pi/3, 5\pi/6\}.$$

Since the length of edges are at least $\pi/2$, two vertices $v_1, v_2$ are anti-poles. By drawing the dual graphs of $G(P)$ (the edge graphs of $P$) corresponding to the set $\{2\pi/3, 2\pi/3, 2\pi/3\}$, $\{\pi/2, 3\pi/4, 3\pi/4\}$, or $\{\pi/2, 2\pi/3, 5\pi/6\}$, we see that $P$

is a triangular right prism whose base is an equilateral triangle, an isosceles right triangle, or a right triangle with angle of $\pi/6$ and $\pi/3$. $\square$

**Proposition 6.** *If the sphere graph $G(P)$ of a reflective space-filling polyhedron $P$ has six vertices, then $P$ is a cuboid.*

*Proof.* Let $P$ be a reflective space-filling polyhedron whose sphere graph $G(P)$ has six vertices. Since there are eight faces in $G(P)$ (Figure 4) and the arc length of each edge in $G(P)$ is at least $\pi/2$, all arc lengths of edges are exactly $\pi/2$. Therefore $P$ is a cuboid. $\square$

This completes the proof of the theorem.

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