LOOPS OF POLYGONS

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Abstract: A loop of polygons is a planar arrangement of congruent non overlapping regular polygons in which each polygon has a common edge with exactly two other polygons. A loop encloses an equilateral polygon (not necessarily convex or equiangular but we will require it to be not self-intersecting and of positive area) that we will refer to as the window of the loop. We conduct an exhaustive computational search for all loops of polygons or \textit{n}-gons for $3 \leq n \leq 10$ and small values of the perimeter of their windows. The stages leading to the generation and display of the loops were implemented in Mathematica.

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1. Introduction

A loop of polygons is a planar arrangement of congruent non overlapping regular polygons in which each polygon has a common edge with exactly two other polygons. A loop encloses an equilateral polygon (not necessarily convex or equiangular but we will require it to be not self-intersecting and of positive area)
area) that we will refer to as the window of the loop. The question regarding to the existence of loops of polygons was posted in 1991 during a workshop for mathematics teachers by Gerry Price and Roger May [1, 2]. Let us call a regular polygon with \( n > 2 \) unit-length sides an \( n \)-gon and let \( \theta = \frac{2\pi}{n} \). As any \( n \)-gon has all its internal angles equal to \( \pi - \theta \), if a number \( m \) of \( n \)-gons can be fitted around a vertex then \( m(\pi - \theta) \leq 2\pi \), that is \( \frac{1}{m} \geq \frac{1}{2} - \frac{1}{n} \). As the value of \( n \) increases from 3 onwards, the corresponding sequence 6, 4, 3, 3, 2, \cdots of maximum values of \( m \) decreases towards the constant value of 2. Thus, if \( n > 6 \) then \( m \leq 2 \); that is, there exist at most two \( n \)-gons, for \( n > 6 \), that can meet at a vertex. This statement holds even for \( n > 5 \) if we omit the loop of 3 hexagons forming a window of area zero. Now, suppose we can produce a loop enclosing a window with \( p \) unit-length sides (or \( p \)-window) described by a sequence of \( p \) (or \( p \)-sequence) of \( m_i \) values at each vertex \( i \) belonging to the window traversed in an anti-clockwise fashion. Because any such window must have interior angle sum equal to \( (p - 2)\pi \), then

\[
\sum_{i=1}^{p} 2\pi - m_i(\pi - \theta) = (p - 2)\pi, \tag{1}
\]

where \( m_i \) is the number of \( n \)-gons fitted around vertex \( i \). This equation simplifies to

\[
\sum_{i=1}^{p} m_i = \frac{n(p + 2)}{n - 2}. \tag{2}
\]

Excluding the cases where the window results of area zero, the possible \( m_i \) values for equilateral triangles are \( \{2, 3, 4, 5\} \), for squares and pentagons \( \{1, 2, 3\} \) and \( \{1, 2\} \) for all higher values of \( n > 5 \). The value of \( m_i = 1 \) corresponds to the case where two consecutive sides, joined by vertex \( i \), of an \( n \)-gon are part of the window. For a given value of \( n \), we establish that the values of \( p \), for which it is not possible to express the integer \( \frac{n(p + 2)}{n - 2} \) as a sum of \( p \) of the allotted values of \( m_i \), are not feasible. Let \( k_n \) be the maximum number of \( n \)-gons we are allowed to fit around a vertex. Thus, \( p \leq \frac{n(p + 2)}{n - 2} \leq k_n p \). For instance, \( k_3 = 5 \) and \( k_n = 2 \) for \( n > 5 \). The first ten possible values of \( p \) for a given value of \( n \), running from 3 to 16 are shown in Table 1. Notice that each one of the sequences forms an arithmetic sequence with ratio equal to \( n - 2 \) if \( n \) is odd, and \( \frac{n}{2} - 1 \) if \( n \) is even. Moreover, if \( n > 7 \), \( n - 4 \) occurs as first or second element of the sequence (if \( n > 10 \), \( n - 4 \) occurs as first element if \( n \) is odd; otherwise, \( \frac{n}{2} - 3 \) occurs as first element if \( n \) is even).

Equation (2), however, is not a sufficient condition to guarantee feasible values of \( p \), because the closing condition for a loop is not enforced. When
Table 1: The first ten possible values of $p$ for $3 \leq n \leq 16$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p$</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>3 4 5 6 7 8 9 10 11 12</td>
</tr>
<tr>
<td>4</td>
<td>4 5 6 7 8 9 10 11 12 13</td>
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<tr>
<td>5</td>
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</tr>
<tr>
<td>6</td>
<td>6 8 10 12 14 16 18 20 22 24</td>
</tr>
<tr>
<td>7</td>
<td>8 13 18 23 28 33 38 43 48 53</td>
</tr>
<tr>
<td>8</td>
<td>4 7 10 13 16 19 22 25 28 31</td>
</tr>
<tr>
<td>9</td>
<td>5 12 19 26 33 40 47 54 61 68</td>
</tr>
<tr>
<td>10</td>
<td>6 10 14 18 22 26 30 34 38 42</td>
</tr>
<tr>
<td>11</td>
<td>7 16 25 34 43 52 61 70 79 88</td>
</tr>
<tr>
<td>12</td>
<td>3 8 13 18 23 28 33 38 43 48</td>
</tr>
<tr>
<td>13</td>
<td>9 20 31 42 53 64 75 86 97 108</td>
</tr>
<tr>
<td>14</td>
<td>4 10 16 22 28 34 40 46 52 58</td>
</tr>
<tr>
<td>15</td>
<td>11 24 37 50 63 76 89 102 115 128</td>
</tr>
<tr>
<td>16</td>
<td>5 12 19 26 33 40 47 54 61 68</td>
</tr>
</tbody>
</table>

$n = 4$, for example, the odd values of $p$ are unfeasible. Also, for $n = 7$, the loop of polygons associated to the value of $p = 13$ does not exist. It has been proved by McLean [2] that there exists a loop with a number $g$ of regular $n$-gons, where $n \geq 4$ is even, inducing a $p$-window for some value of $p$. If $n$ is a power of 2 then the number $g$ of $n$-gons, around a $p$-window, must be even. Otherwise, if $n$ is not a power of 2 and $g$ is odd then $g \geq q$, where $q$ is the least odd prime factor of $n$. However, these results do not seem to carry over to our search problem.

Given the values of $n$ and $p$, we are interested in obtaining all possible $p$-sequences of $m_i$ values. To this aim we compute all $p$-sequences of $m_i$ values adding up to $\frac{n(p+2)}{2}$, where the values of $m_i$ are taken from the set of integer values associated to the number of $n$-gons that can be fitted at vertex $i$. The following recursive function $\text{ms}[p, s, k]$ generates all $p$-sequences of $m_i$ values adding up to $s$ using the integers in the range 1 to $k$.

\[
\begin{align*}
\text{ms}[1, s, k] & := \text{If}[1 \leq s \leq k, \{\{s\}\}, \{\}] \\
\text{ms}[p, s, k] & := \text{ms}[p, s, k] = \text{Module}[\{i\}, \\
& \text{Join}@\text{Table}[\text{Prepend}[#, i] & \& @\text{ms}[p - 1, s - i, k], \{i, k\}]]
\end{align*}
\]

For our purposes, $s = \frac{n(p+2)}{2}$ and $k = k_n$. For example, in the case of heptagons forming 8-sequences of $m_i$ values, of all the $2^8 = 256$ possible lists
of \{1, 2\}-digits, only the sequences depicted in Table 1 sum to the required number of 14. It is easy to see that, if \(k=2\), those lists have \(2p-s\) 1’s and \(s-p\) 2’s, and there are \({2p\choose 2p-s}\) lists in total.

### 2. Equivalent Lists

Among the lists generated by function \texttt{ms}, or \(m\)-lists, we can regard as equivalent, or isomorphic, those equal under rotation and/or reflection. Given an \(m\)-list, we compute all those that are equivalent to it by constructing its class. This is accomplished by function \texttt{class} below.

\[
\text{class}[m_.] := \text{Module}\left[\{i = \text{Length}[m] - 1\}, \text{Join}[\text{NestList}[\text{RotateRight}, m, i], \text{NestList}[\text{RotateRight}, \text{Reverse}[m], i]]\right]
\]

For instance, the lists \{1, 2, 3, 4\}, \{4, 1, 2, 3\}, \{3, 4, 1, 2\}, \{2, 3, 4, 1\}, \{4, 3, 2, 1\}, \{1, 4, 3, 2\}, \{2, 1, 4, 3\}, \{3, 2, 1, 4\} form a class, and each list is equivalent to \{1, 2, 3, 4\} by rotation and/or reflection.

We can substantially trim the \(m\)-lists by invoking the function \texttt{class} and then eliminating from each class its members except for the first one, according to the new polymorphic function \texttt{ms} below.

\[
\text{ms}[n_.\ p_.] := \text{ms}[n, p] = \text{Module}\left[\{s = n(p + 2)/(n - 2), m, f, r\}, \text{If}[\text{IntegerQ}[s], m = \text{ms}[p, s, \text{Switch}[n, 3, 5, 4, 3, 5, 3, \_ 2]]; m = \text{Select}[m, (p\text{First}[[#] == \text{Total}[[#]]) \lor (\text{First}[[#]] \neq \text{Last}[[#]])\&]; r = \text{Reap}[\text{While}[m \neq \{\}, \text{Sow}[f = \text{First}[m]]; m = \text{Complement}[m, \text{class}[f]]]]; \text{If}[\text{Last}[r] == \{\}, \{\}, r[[2,1]], \{\}]\right]
\]
Thus, in the case of heptagons, the 43758 18-sequences generated by function \( \text{ms}[p,s,k] \) above reduce to 1282 non equivalent sequences generated by the function \( \text{ms}[n,p] \).

3. Constrained Runs of \( m_i \) Values

It turns out that the condition on the sum given by equation (2) and the elimination of isomorphic \( p \)-sequences must be supplemented by restrictions on the number of consecutive \( m_i \) values appearing on the sequences. It is easy to see that a valid sequence can have at most \( n - 3 \) consecutive 1’s. For particular values of \( n \), the runs of \( m_i \) values behave as follows. For triangles: at most 4 (consecutive) 2’s; unlimited 3’s; if \( p = 6 \), exactly 6 4’s, else at most 5 4’s; if \( p = 3 \), exactly 3 5’s, else at most 2 5’s. For squares: unlimited 2’s; if \( p = 4 \), exactly 4 3’s, else at most 2 3’s. For pentagons: if \( p = 10 \), exactly 10 2’s, else at most 9 2’s; at most 1 3’s. For hexagons: if \( p = 6 \), exactly 6 2’s, else at most 4 2’s. For heptagons: at most 3 2’s. For octagons: if \( p = 4 \), 4 2’s, else at most 2 2’s. For nonagons: at most 2 2’s. For decagons: if \( p = 3 \), 3 2’s else at most 2 2’s. These conditions are encapsulated at function \( \text{runsOk} \) below, which tests a given sequence corresponding to a given value of \( n \).

\[
\text{runs}[w_] := \text{Module}[\{c, s\}, \\
 s = \text{Union}[\{\text{First}[#], \text{Length}[#]\}&/@\text{Split}[w]]; \\
\text{Last}/@\text{Map} \\
[(c = \text{Cases}[s, \{#, \_\}]; \{#, \text{If}[c == \{\}, 0, \text{Max}[\text{Last}/@c]]\}&, \text{Range}[5]])
\]

\[
\text{runsOk}[n_, w_] := \text{Module}[\{p = \text{Length}[w]\}, \\
 s = \text{Switch}[n, 3, \{0, 4, 1000, \text{If}[p == 6, 6, 5], \text{If}[p == 3, 3, 2]\}, \\
4, \{1, 1000, \text{If}[p == 4, 4, 2], 0, 0\}, 5, \{2, \text{If}[p == 10, 10, 9], 1, 0, 0\}, \\
6, \{3, \text{If}[p == 6, 6, 4], 0, 0, 0\}, 7, \{4, 3, 0, 0, 0\}, 8, \\
\{5, \text{If}[p == 4, 4, 2], 0, 0, 0\}, 9, \{6, 2, 0, 0, 0\}, 10, \\
\{7, \text{If}[p == 3, 3, 2], 0, 0, 0\}, \_ \{n - 3, 1, 0, 0, 0\}]; \\
\text{And}@\text{Thread}[\#1 \leq \#2&[\text{runs}[w], s]]
\]

Using these restrictions, the 18-sequences of heptagons further reduce their number to 448.
4. Testing for Closure

We can interpret a $p$-sequence as a code describing the path forming the window of the loop. Not all sequences correspond to closed paths, as would be the case in describing a window, and thus we need to test for this condition. As we established previously, a window has its sides of unit length. Without loss of generality, assume the first side of the window goes from the origin to point 1. In the following, all points are described by complex numbers. In drawing the window, each segment is computed by rotating the previous one an angle depending on the given $p$-sequence by function `getPath` below.

```
getPath[n, p] := Module[{θ = 2.π/n, loop = {0, 1}},
  Map[AppendTo[loop, loop[[-1]] - (loop[[-1]] - loop[[-2]]) \[ExponentialE]i\[Pi]θ] &, p];
Chop[loop]]
```

In this context, the $p$-sequence corresponds to a list of interior angles of the window rather than a list of $m_i$ values. The closing condition is then coded in function `closedQ` as follows:

```
closedQ[w_] := (Take[w, -2] == {0, 1})
```

Notice that even some $p$-sequences produce a loop of polygons, the loop does not induce of a window, since its characteristics of not self-intersecting and of positive area do not hold. Consequently, there exist polygons forming loops which have either common edges or common edges and vertices with more than two polygons (see Figure 1(a) and (b), respectively); or the polygons result to be overlapped (see Figure 1(c)). We will deal with these anomalies in the next section.

Selecting those sequences for which the closing condition holds from the previous 448 18-sequences of heptagons, we obtain only 13 sequences that consist in closed paths.

5. Testing for Self-Intersection

A further condition imposed on $p$-sequences is their non self-intersection. Besides overlapping, we regard the coincidences of edges and vertices as self-intersection as they increase the number of neighbors allowed in loops. These three types of self-intersection are depicted in Figure 1. The former two types
Figure 1: Loops of polygons not inducing a window: Anomalies

Figure 2: Sample of loops of triangles

Figure 3: Sample of loops of squares

are tested by function notRepeatedQ, and the latter by function notSelfQ below. These functions are coded as follows:

\[
\begin{align*}
\text{intersectQ[} & \{a\_\_ , b\_\_\}, \{c\_\_ , d\_\_\}] := \text{Module}[\{\alpha , \beta \}, \\
& \text{If}[\text{Abs}[\text{Det}[\{d - c, b - a\}]] < 0.001, \{\}, \{\{\alpha , \beta \}\} = \\
& (\{\alpha , \beta \} /. \text{Solve}[\alpha (d - c) - \beta (b - a) == a - c, \{\alpha , \beta \}]); \\
& \text{If}[0 < \alpha < 1) \land (0 < \beta < 1), c + \alpha (d - c), \{\}]]] \\
\text{notSelfQ[w\_]} := \text{Module}[\{i , j\}, \\
\end{align*}
\]
Figure 4: Loops of pentagons

Figure 5: Sample of loops of hexagons
Figure 6: Loops of heptagons

(a) $p = 8$

(b) $p = 18$

(c) $p = 23$

Figure 7: Sample of loops of octagons

(a) $p = 4$

(b) $p = 10$

(c) $p = 16$

(d) $p = 22$

\[
\text{Flatten}\left[\text{Table}\left[\text{intersectQ}\left[\text{toXY}\left[\{w_{[i]}, w_{[(i+1)]}\}\right], \text{toXY}\left[\{w_{[j]}, w_{[(j+1)]}\}\right]\right], \{i, \text{Length}[w] - 3\}, \{j, i + 2, \text{Length}[w] - 1\}\right]\right] == \{}\right]
\]
Figure 8: Sample of loops of nonagons

Figure 9: Sample of loops of decagons

\[\text{notRepeatedQ}[w_\_] := \text{Module}[\{i, j\},\]
\[(\text{Cases}[\text{Flatten}[\text{Table}][\text{Chop}[w[[i]] - w[[j]]]],\]
\[\{i, \text{Length}[w] - 1\}, \{j, i + 1, \text{Length}[w]\}], 0] == \{0, 0\}]\]

Selecting those not self-intersecting from the previous 13 corresponding to 
\(n = 7, p = 18\), we obtain the six loops depicted at Figure 6(b).
Table 3: Number \( l_p^n \) of loops of \( n \)-gons inducing a \( p \)-window

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p/l_p^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3/1 4/1 5/1 6/4 7/4 8/13</td>
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</tr>
<tr>
<td>5</td>
<td>4/1 7/1 10/5 13/9</td>
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<td>9</td>
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<td>24/5</td>
</tr>
<tr>
<td>16</td>
<td>12/2 26/4</td>
</tr>
</tbody>
</table>

6. Generating and Displaying the Solutions

The final rendering of the \( p \)-sequences corresponding to loops of \( n \)-gons passing all the previous conditions is handled by function `getAll` below:

```math
getAll[n_, p_] := Module[{h, u},
  h = Select[ms[n, p], runsOk[n, #] &];
  Select[#, getPath[n, #]] & /@ h,
  (u = Last[#]; closedQ[u] \[And] notSelfQ[u] \[And] notRepeatedQ[u]) &]
```

In Table 3 we include our findings where for each \( n \) and feasible \( p \), we indicate the value of \( l_p^n \) enumerating the amount of non-isomorphic loops of \( n \)-gons inducing a \( p \)-window.

Figures 4 and 6 show all solutions for loops of pentagons and heptagons, outstanding for their beauty on the whole set of loops found by the authors so far. Whereas, Figures 2, 3, 5, and 7, 8, 9 depict a chosen loop of polygons from Table 3, corresponding to each value of \( n \) and \( p \) therein.

Further work can be done generalizing the concept of loop to include different regular polygons inducing a window and also altering the matching conditions on the edges such as those considered in the Penrose’s rhombuses.
References
