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PASCAL TYPE MATRICES AND BERNOULLI NUMBERS

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Abstract: This paper introduces the notion of power sum matrix. The relationships between the power sum matrix, Pascal-type matrices, Stirling matrices, Vandermonde matrix and Bernoulli matrix are investigated, meanwhile some explicit factorizations of the power matrix are also obtained.

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1. Introduction

Let n be a positive integer. For each integer $k \geq 0$, let $Z(n,k)$ denote the sum of k-th powers of positive integers, i.e., $Z(n,k) = 1^k + 2^k + \cdots + n^k$, which is usually called the sum of integer powers, or simply the power sum. It is well-known that $Z(n,k)$ can be expressed as a polynomial in n with degree $k+1$ (see [5]), $Z(n,k) = a_{k,1}n + a_{k,2}n^2 + \cdots + a_{k,k+1}n^{k+1}$, where $a_{k,1}, a_{k,2}, \cdots, a_{k,k+1}$ are only depended on k. This polynomial is called the power sum polynomial. Using the Pascal type matrices and Stirling matrices, P. Maltais [7] showed that the coefficients of the power sum polynomial can be expressed in terms of the Stirling numbers. In this paper, we introduce the notion of the power sum matrix. The relationships between the power sum matrix, Pascal type matrices, Stirling matrices, Vandermonde matrix and Bernoulli matrix are investigated, meanwhile some explicit factorizations of the power matrix are also obtained.

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2. A Connection between the Power Sum Matrix and Pascal Type Matrices

The Pascal type matrices P, $\bar{P}$, and $\hat{P}$ are $(m+1)\times(m+1)$ matrices which are defined with the binomial coefficients by

$$P(i,j)=\begin{cases}\binom{i-1}{j-1}, & \text{if } i\ge j\ge 1,\\ 0, & \text{otherwise,}\end{cases}$$

$$\bar{P}(i,j)=\begin{cases}\binom{i}{j}, & \text{if } i\ge j\ge 1,\\ 0, & \text{otherwise,}\end{cases}$$

$$\hat{P}(i,j)=\begin{cases}\binom{i}{j-1}, & \text{if } i\ge j\ge 1,\\ 0, & \text{otherwise.}\end{cases}$$

The matrix P is the Pascal matrix [3], and $\bar{P}$ is the Pascal 1-eliminated matrix [2] which is obtained from the Pascal matrix by deleting its first row and column, while $\hat{P}$ is the 'reverse' of $\bar{P}$.

Using the coefficients of the polynomial of power sum $Z(n,k)=a_{k,1}n+a_{k,2}n^2+\ldots+a_{k,k+1}n^{k+1}$, $k=0,1,\cdots m$, we define the power sum matrix $R=[a_{i-1,j}]_{i,j=1}^{m+1}$. Let the vectors $\alpha=[Z(n,0),Z(n,1),\cdots,Z(n,m)]^T$, and $\beta=[n,n^2,\cdots,n^{m+1}]^T$, we have

$$[Z(n,0),Z(n,1),\cdots,Z(n,m)]^T=R\,[n,n^2,\cdots,n^{m+1}]^T,$$

i.e.,

$$\alpha=R\beta. \tag{1}$$

Since

$$\begin{aligned}\sum_{j=1}^{i}\binom{i}{j-1}Z(n,j-1)&=\sum_{j=0}^{i-1}\binom{i}{j}Z(n,j)=\sum_{j=0}^{i-1}\binom{i}{j}\sum_{r=1}^{n}r^j=\sum_{r=1}^{n}\sum_{j=0}^{i-1}\binom{i}{j}r^j\\&=\sum_{r=1}^{n}((r+1)^i-r^i)=(n+1)^i-1=\sum_{j=1}^{i}\binom{i}{j}n^j,\end{aligned}$$

we obtain

$$\hat{P}\alpha=\bar{P}\beta. \tag{2}$$

It is easy to check that ([3, 7]) $P^{-1}=JPJ$, $\bar{P}^{-1}=J\bar{P}J$, where $J=\operatorname{diag}(1,-1,\cdots,(-1)^{m+1})$.

Theorem 1. $\bar{P}^{-1}\hat{P}=J\hat{P}J$; $\hat{P}P^{-1}=J\hat{P}J$.

Proof. Using the identities $\binom{i}{t}\binom{t}{j}=\binom{i}{j}\binom{i-j}{t-j}$ and $\sum_{r=0}^{i}(-1)^{i-r}\binom{i}{r}=0$ we have

$$(\bar{P}^{-1}\hat{P})(i,j) = \sum_{r=j}^{i}(-1)^{i-r}\binom{i}{r}\binom{r}{j-1} = \sum_{r=j}^{i}(-1)^{i-r}\binom{i}{j-1}\binom{i-j+1}{r-j+1}$$
$$= \sum_{r=j-1}^{i}(-1)^{i-r}\binom{i}{j-1}\binom{i-j+1}{r-j+1} + (-1)^{i-j}\binom{i}{j-1}\binom{i-j+1}{0}$$
$$= \binom{i}{j-1}\sum_{r=j-1}^{i}(-1)^{i-r}\binom{i-j+1}{r-j+1} + (-1)^{i-j}\binom{i}{j-1}\binom{i-j+1}{0}$$
$$= (-1)^{i-j}\binom{i}{j-1} = (J\hat{P}J)(i,j),$$

$$(\hat{P}P^{-1})(i,j) = \sum_{r=1}^{i}(-1)^{r-j}\binom{i}{r-1}\binom{r-1}{j-1} = \sum_{r=j}^{i}(-1)^{r-j}\binom{i}{j-1}\binom{i-j+1}{r-j}$$
$$= \sum_{r=j}^{i+1}(-1)^{r-j}\binom{i}{j-1}\binom{i-j+1}{r-j} + (-1)^{i-j}\binom{i}{j-1}\binom{i-j+1}{i-j+1}$$
$$= \binom{i}{j-1}\sum_{r=j}^{i+1}(-1)^{r-j}\binom{i-j+1}{r-j} + (-1)^{i-j}\binom{i}{j-1}\binom{i-j+1}{i-j+1}$$
$$= (-1)^{i-j}\binom{i}{j-1} = (J\hat{P}J)(i,j). \quad \square$$

Let $\tilde{P} = J\hat{P}J$, then from (2) and Theorem 1, we have $\tilde{P} = \bar{P}^{-1}\hat{P} = \hat{P}P^{-1}$, and $\tilde{P}^{-1} = J\hat{P}^{-1}J = \hat{P}^{-1}\bar{P} = P\hat{P}^{-1}$. Thus, from (1) we obtain the following result.

Corollary 2. *The power sum matrix and Pascal type matrices are connected such as*

$$R = \tilde{P}^{-1} = \hat{P}^{-1}\bar{P} = P\hat{P}^{-1} = J\hat{P}^{-1}J. \tag{3}$$

3. A Connection between the Power Sum Matrix, Stirling Matrices and Vandermonde Matrix

Let n, k be nonnegative integers and $n \geq k$, the Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$ can be defined by: $(x)_n = \sum_{k=0}^{n} s(n,k)x^k$, and $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$, where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ for any integer $k > 0$, and $(x)_0 = 1$; $s(k,k) = S(k,k) = 1$ for $k \geq 0$, and $s(k,0) = S(k,0) = 0$ for $k > 0$. The Stirling matrix of the first kind S_1 and of the second kind S_2 are defined respectively by (see [4]) $S_1 = [s(i,j)]_{1\leq i,j\leq m+1}$, $S_2 = [S(i,j)]_{1\leq i,j\leq m+1}$, where $s(i,j) = 0, S(i,j) = 0$ if $i < j$.

Theorem 3. *Let* $\Lambda = \mathrm{diag}(1,2,\cdots,m+1)$ *be a diagonal matrix, then the Pascal type matrix* $\tilde{P}$ *can be factorized into the products of the Stirling matrices*

$$\tilde{P} = S_2 \Lambda S_1. \tag{4}$$

The power sum matrix R *can be factorized into the products of the Stirling matrices*

$$R = S_2 \Lambda^{-1} S_1. \tag{5}$$

Proof. Let the (i,j) entry of the matrix $S_2\Lambda S_1$ is $b_{i,j}$, that is, $b_{i,j} = \sum_{k=1}^{i} S(i,k)ks(k,j)$, the generating function for $b_{i,j}$ is

$$\begin{aligned}\sum_{j=1}^{i} b_{i,j}x^j &= \sum_{j=1}^{i}\sum_{k=1}^{i} S(i,k)ks(k,j)x^j = \sum_{k=1}^{i} S(i,k)k\sum_{j=1}^{k} s(k,j)x^j \\ &= \sum_{k=1}^{i} S(i,k)k(x)_k = \sum_{k=1}^{i} S(i,k)x(x)_k - \sum_{k=1}^{i} S(i,k)(x-k)(x)_k \\ &= x\sum_{k=1}^{i} S(i,k)(x)_k - x\sum_{k=1}^{i} S(i,k)(x-1)_k = x^{i+1} - x(x-1)^i.\end{aligned}$$

On the other hand, the generating function for $(-1)^{i-j}\binom{i}{j-1}$ is

$$\begin{aligned}\sum_{j=1}^{i}(-1)^{i-j}\binom{i}{j-1}x^j &= \sum_{r=0}^{i-1}(-1)^{i-r+1}\binom{i}{r}x^{r+1} = x^{i+1} + \sum_{r=0}^{i}(-1)^{i-r+1}\binom{i}{r}x^{r+1} \\ &= x^{i+1} + x\sum_{r=0}^{i}(-1)^{i-r+1}\binom{i}{r}x^{r} = x^{i+1} - x(x-1)^i.\end{aligned}$$

Therefore

$$(-1)^{i-j}\binom{i}{j-1} = \sum_{k=1}^{i} S(i,k)k\,s(k,j), \quad \text{and} \quad \tilde{P} = S_2\Lambda S_1.$$

Applying Theorem 1, we get $R = S_2\Lambda^{-1}S_1$. $\square$

Theorem 4. *The power sum matrix R can be factorized as*

$$R = V(P^T)^{-1}D^{-1}S_1, \tag{6}$$

where $V = [j^{i-1}]_{i,j=1}^{m+1}$ is the Vandermonde matrix, and D is a diagonal matrix, $D = \mathrm{diag}(1!, 2!, \cdots, (m+1)!)$.

Proof. Since

$$j^{i-1} = j^{-1}j^i = j^{-1}\sum_{r=1}^{i} S(i,r)r!\binom{j}{r} = \sum_{r=1}^{i} S(i,r)(r-1)!\binom{j-1}{r-1},$$

the Vandermonde matrix V can be factorized as $V = S_2\Delta P^T$, where $\Delta = diag(1, 1!, 2!, \cdots, m!)$. Hence $S_2 = V(P^T)^{-1}(\Delta)^{-1}$, and $R = S_2\Lambda^{-1}S_1 = V(P^T)^{-1}(\Delta)^{-1}\Lambda^{-1}S_1 = V(P^T)^{-1}D^{-1}S_1 = V(P^T)^{-1}D^{-1}S_1$. □

4. A Connection between the Power Sum Matrix and Bernoulli Matrix

The Bernoulli numbers B_0, B_1, B_2, $\cdots$ are given by $B_0 = 1$ and the recursion $\sum_{i=0}^{n}\binom{n+1}{i}B_i = 0$. For every nonnegative integer n, the n-th Bernoulli polynomial $B_n(x)$ is defined by $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k\, x^{n-k}$. Obviously, $B_n(0) = B_n$ for all integers $n \geq 0$.

The Bernoulli numbers B_n and the Bernoulli polynomials $B_n(x)$ are connected by [8]

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k, \quad n \geq 0; \tag{7}$$

$$x^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x), \quad n \geq 0. \tag{8}$$

The Bernoulli matrix B is defined by (see [1]) $B = \left[B_{i-j}\binom{i-1}{j-1}\right]_{i,j=1}^{m+1}$. The matrix representations for (7) and (8) are:

$$[B_0(x), B_1(x), \cdots, B_m(x)]^T = B\,[1, x, \cdots, x^m]^T, \tag{9}$$

$$[1, x, \cdots, x^m]^T = \Lambda^{-1}\hat{P}\,[B_0(x), B_1(x), \cdots, B_m(x)]^T. \tag{10}$$

Thus, $B = (\Lambda^{-1}\hat{P})^{-1} = \hat{P}^{-1}\Lambda$, and

$$\hat{P}^{-1} = B\Lambda^{-1}. \tag{11}$$

Applying Corollary 2 and (11) give the following theorem.

Theorem 5. *The power sum matrix R has the following factorization:*

$$R = B\Lambda^{-1}\bar{P}, \tag{12}$$

$$R = PB\Lambda^{-1}, \tag{13}$$

$$R = JB\Lambda^{-1}J. \tag{14}$$

The last means that the matrix R is obtained from the matrix $\hat{P}^{-1} = B\Lambda^{-1}$ with every other subdiagonal multiplied by a factor of negative one, but their odd lower subdiagonals are almost 0, so these two matrices are basically the same, except their first lower subdiagonal.

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References

[1] L. Aceto, D. Trigiante, The matrices of Pascal and other greats, *The American Mathematical Monthly*, **108** (2001), 232-245.

[2] M. Bayat, H. Teimoori, Pascal k-eliminated functional matrix and its property, *Linear Algebra Appl.*, **308** (2000), 65-75.

[3] R. Brawer, M. Pirovino, The linear algebra of the Pascal matrix, *Linear Algebra Appl.*, **174** (1992), 13-23.

[4] G.-S. Cheon, J.-S. Kim, Stirling matrix via Pascal matrix, *Linear Algebra Appl.*, **329** (2001), 49-59.

[5] L. Comtet, *Advanced Combinatories*, Reidel, Dordrecht (1974).

[6] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Second Edition, Addison Wesley Company (1994).

[7] P. Maltais, T.A. Gulliver, Pascal matrices and Stirling numbers, *Appl. Math. Lett.*, **11**, No. 2 (1998), 7-11.

[8] H. M. Srivastava, Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Applied Mathematics Letters*, **17** (2004), 375-380.