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ON THE CORRELATIONS, SELBERG INTEGRAL AND SYMMETRY OF SIEVE FUNCTIONS IN SHORT INTERVALS, II

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Abstract: We give estimates for the correlations, the Selberg integral and the symmetry integral of divisor sums.

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1. Introduction and Statement of the Results

We pursue our study of arithmetical functions f in short intervals, started in Coppola [1]; in particular, here we will confine ourselves with the case of "divisor sums" in specified ranges (all can be reduced to dyadic ranges):

$$f(n) = d_{A,B}(n) \stackrel{def}{=} \sum_{\substack{d|n \\ A<d\le B}} 1,$$

where we may assume $A, B \in \mathbb{N}$ and $A < B$. Also, we will set $d_Q(n) \stackrel{def}{=} \sum_{d|n, d\le Q} 1$ (instead of $d_{0,Q}$). In the sequel, we abbreviate $L \stackrel{def}{=} \log N$, where $N \to \infty$. In summations, $m \sim X$ is for dyadic $X < m \le 2X$.

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Definition. The *(auto-)correlation* of the divisor sum $d_{Q,2Q}$ is

$$\mathfrak{C}_Q(a) \stackrel{def}{=} \sum_{n\sim N} d_{Q,2Q}(n) d_{Q,2Q}(n-a).$$

We state, now, our main result for these correlations. (Hereon $(a,b)=1$ means a,b are *coprime*.)

Theorem 1. *Fix $\delta>0$ small. Assume $a\in\mathbb{Z}$, $a\neq 0$, $|a|=o(N)$ and $Q^2 \gg |a|N^{1-2\delta/3}$, $Q=o(N^{1-\delta})$, when $N\to\infty$. Then $\forall\varepsilon>0$*

$$\mathfrak{C}_Q(a) = N\sum_{\ell|a}\frac{1}{\ell}\sum_{q\sim\frac{Q}{\ell}}\frac{1}{q}\sum_{\substack{d\sim\frac{Q}{\ell}\\(d,q)=1}}\frac{1}{d} + \mathcal{O}_\varepsilon\left(N^\varepsilon(Q+|a|)\right)$$
$$+\mathcal{O}_\varepsilon\left(N^\varepsilon(Q^{3/2}N^\delta + Q^3N^{\delta-1} + N^{1-\delta})\right).$$

Definition. We will call $g:\mathbb{N}\to\mathbb{C}$ "*essentially bounded*" $\stackrel{def}{\Longleftrightarrow}$ $\forall\varepsilon>0$ $|g(n)|\ll_\varepsilon n^\varepsilon$ (esp., $d_{A,B}$ is such $\forall A,B$), writing

$$F \lll G \stackrel{def}{\Longleftrightarrow} \forall\varepsilon>0 \;\; F\ll_\varepsilon N^\varepsilon G, \quad \text{when} \quad N\to\infty.$$

An immediate consequence (hence, no explicit proof) of this first Theorem is the following

Corollary 1. *Fix $\delta>0$ small. Assume $a\in\mathbb{Z}$, $a\neq 0$, $|a|=o(N^{1-2\delta})$ and $\sqrt{|a|N^{1-\delta/2}} \ll Q \ll N^{2/3-2\delta}$, when $N\to\infty$. Then*

$$\left|\mathfrak{C}_Q(a) - N\sum_{\ell|a}\frac{1}{\ell}\sum_{q\sim\frac{Q}{\ell}}\frac{1}{q}\sum_{\substack{d\sim\frac{Q}{\ell}\\(d,q)=1}}\frac{1}{d}\right| \lll N^{1-\delta}.$$

Remark. This estimate is *uniform in a with* $0<|a|=o(N^{1-\delta})$.

It is clear that, from these asymptotic formulæ, we can get the correlations of more general divisor sums, namely $d_{A,B}$, provided that A has the right lower bound and B the right upper one; however, we will refrain from doing this, due to the technical complications involved.

Instead, we will study their consequences as regards the ("dyadic") *symmetry integral* of $d_{Q,2Q}$, i.e.

$$I_Q(N,h) \stackrel{def}{=} \int_N^{2N}\Big|\sum_{|n-x|\le h}\operatorname{sgn}(n-x)\sum_{\substack{d|n\\d\sim Q}}1\Big|^2 dx$$

and the ("dyadic") *Selberg integral* of $d_{Q,2Q}$, i.e.

$$J_Q(N,h) \stackrel{def}{=} \int_N^{2N} \Big| \sum_{0<|n-x|\le h} \sum_{\substack{d|n \\ d\sim Q}} 1 - M_Q(2h) \Big|^2 dx,$$

where

$$M_Q(2h) \stackrel{def}{=} 2h \sum_{d\sim Q} \frac{1}{d} = 2h\Big(\log 2 + \mathcal{O}\Big(\frac{1}{Q}\Big)\Big)$$

is the "mean-value" of $d_{Q,2Q}(n)$ in the short interval $[x-h, x+h]$. Hereon the *integer part* of $r \in \mathbb{R}$ will be indicated (as no confusion arises with the intervals notation), as usual, $[r] \stackrel{def}{=} \max_{n\in\mathbb{Z}, n\le r} n$.

From our previous result about (single) correlations, we obtain (averaging with suitable weights):

Theorem 2. *Fix $\delta > 0$ small and $\theta < \frac{1}{2}$. Let $h = [N^\theta]$, with $h \to \infty$ if $N \to \infty$. Assume that $Q^2 \gg hN^{1-2\delta/3}$, $Q = o(N^{1-\delta})$. Then*

$$I_Q(N,h) \lll Nh + Qh^2 + (Q^{3/2}N^\delta + Q^3 N^{\delta-1} + N^{1-\delta})h^2,$$

$$J_Q(N,h) \lll Nh + Qh^2 + (Q^{3/2}N^\delta + Q^3 N^{\delta-1} + N^{1-\delta})h^2.$$

Also in this case, we have immediately the

Corollary 2. *Fix $\delta > 0$ small and $\theta < \frac{1}{2}$. Let $h = [N^\theta]$, with $h \to \infty$ if $N \to \infty$. Assume that $\sqrt{hN^{1-\delta/2}} \ll Q \ll N^{2/3-2\delta}$, when $N \to \infty$. Then $I_Q(N,h) \lll Nh^2N^{-\delta}$, $\quad J_Q(N,h) \lll Nh^2N^{-\delta}$.*

These estimates are the same for both the Selberg and the symmetry integral, like in Coppola [1] results; by the way, these cover the "low range", i.e. $Q \ll \sqrt{hN^{1-\delta/2}}$, since this is the case of $\lambda < \frac{1+\theta}{2}$, compare the Corollary in Coppola [1].

For the "high range", $Q \gg N^{2/3-2\delta}$, we cannot follow the same lines, applying proof of Theorem 1 (actually, Theorem 2 follows from this) the Weil bound for Kloosterman sums (actually, no wonder: in the literature this method's limit is well known: the exponent 2/3, here). This limitation for both Theorems 1 and 2 is structural, since we want an individual asymptotic estimate for the correlations. Actually, for h large enough (i.e., intervals $[x-h, x+h]$ not too short!), the (elementary) methods of Coppola [1] are capable of producing non-trivial bounds beyond this limitation, exploiting the fact that any estimate for the two integrals is "*easier*", in some sense, than estimating correlations individually.

However, for the symmetry integral we may follow an elementary approach ("*flipping*" the divisors) in

Theorem 3. *Fix $\theta < \frac{1}{2}$. Let $h = [N^\theta]$ and $h \to \infty$ if $N \to \infty$. Then, uniformly $\forall D \in \mathbb{N}$, $I_D(N,h) \lll Nh$.*

Applying (to the inner $d-$sum) a *dissection into dyadic intervals* (they are $\mathcal{O}(L)$, with $D \ll Q$)

$$I_{d_Q}(N,h) \stackrel{def}{=} \int_N^{2N} \Big| \sum_{|n-x|\le h} \mathrm{sgn}(n-x) d_Q(n) \Big|^2 dx$$

$$= \int_N^{2N} \Big| \sum_{|n-x|\le h} \mathrm{sgn}(n-x) \sum_{d|n, d\le Q} 1 \Big|^2 dx$$

may be estimated (through Theorem 3) *uniformly* $\forall Q \in \mathbb{N}$ as into our

Corollary 3. *Fix $\theta < \frac{1}{2}$. Let $h = [N^\theta]$, with $h \to \infty$ if $N \to \infty$. Then, uniformly $\forall Q \in \mathbb{N}$,*

$$I_{d_Q}(N,h) \lll Nh.$$

(This result is used in Coppola [2], even if there the integral is from h to N; the adaption to that case is immediate.)

In the same way, we get the *uniform estimate* $I_{A,B}(N,h) \lll Nh$, *in all the ranges* $[A,B]$, *for the symmetry integral* $I_{A,B}(N,h)$ of the divisor sums $d_{A,B}(n)$, here (of course, in the same hypotheses).

Unfortunately, this is possible nor for the single correlations, neither for the Selberg integral: they both have "main terms" that have a bad behaviour, when applying the flipping of the divisors!

The paper is organized as follows:

— in next section we state and prove the Lemmas;

— in section 3 we prove our Theorems.

2. Statements and Proofs of the Lemmas

We start from a sort of "*flipping*" for the *symmetry sums* of $d_{A,B}$:

$$S^{\pm}_{A,B}(x) \stackrel{def}{=} \sum_{|n-x|\le h} \mathrm{sgn}(n-x) \sum_{\substack{d|n \\ A<d\le B}} 1 = \sum_{A<d\le B} \chi_d(x),$$

where, say,

$$\chi_q(x) \stackrel{def}{=} \sum_{\left|m-\frac{x}{q}\right|\le\frac{h}{q}} \mathrm{sgn}\left(m-\frac{x}{q}\right)$$

are "character-like" functions, which values are $-1,0,1$ and are $q-$periodic (but not multiplicative; actually, they vanish whenever $\left[\frac{x-h}{q},\frac{x+h}{q}\right]\cap\mathbb{N}=\emptyset$, here).

We "flip" the divisors d in $S^{\pm}_{A,B}$ as follows.

Lemma A. *Let* $N,h\in\mathbb{N}$ *with* $h\to\infty$ *and* $h=o(N)$ *when* $N\to\infty$. *Assume* $A,B\in\mathbb{R}$, $A<B$ *and* $A,B\to\infty$ *if* $N\to\infty$. *Then, uniformly* $\forall x\in[N,2N]$, we have

$$\frac{B}{A}>1+\frac{2h}{N-h} \quad \Rightarrow \quad S^{\pm}_{A,B}(x)=\sum_{\frac{x}{B}<q\le\frac{x}{A}}\chi_q(x)+\mathcal{O}\left(h\left(\frac{B}{N}+\frac{1}{h}+\frac{1}{A}+\frac{h}{N}\right)\right).$$

Proof. Of course, we assume $A,B\in\mathbb{N}$ (gives same error-terms):

$$S^{\pm}_{A,B}(x)=\sum_{\substack{m,\ A<d\le B\\ |md-x|\le h}}\sum \mathrm{sgn}(md-x)=\sum_{\frac{x-h}{B}\le m<\frac{x+h}{A}}\ \sum_{\substack{|d-\frac{x}{m}|\le\frac{h}{m}\\ A<d\le B}}\mathrm{sgn}\Big(d-\frac{x}{m}\Big)$$

$$=\Sigma(B)+\sum_{\frac{x+h}{B}<m<\frac{x-h}{A}}\chi_m(x)+\Sigma(A),$$

(using $\frac{B}{A}>1+\frac{2h}{N-h}$ to get $\frac{x+h}{B}<\frac{x-h}{A}$ uniformly $\forall x\in[N,2N]$), where, *say*,

$$\Sigma(B)\stackrel{def}{=}\sum_{\frac{x-h}{B}\le m\le\frac{x+h}{B}}\ \sum_{\substack{|d-\frac{x}{m}|\le\frac{h}{m}\\ A<d\le B}}\mathrm{sgn}\left(d-\frac{x}{m}\right)$$

$$=\sum_{\frac{x-h}{B}\le m\le\frac{x+h}{B}}\ \sum_{\frac{x-h}{m}<d\le B}\mathrm{sgn}\left(d-\frac{x}{m}\right)$$

and

$$\Sigma(A)\stackrel{def}{=}\sum_{\frac{x-h}{A}\le m<\frac{x+h}{A}}\ \sum_{\substack{|d-\frac{x}{m}|\le\frac{h}{m}\\ A<d\le B}}\mathrm{sgn}\left(d-\frac{x}{m}\right)$$

$$=\sum_{\frac{x-h}{A}\le m<\frac{x+h}{A}}\ \sum_{A<d\le\frac{x+h}{m}}\mathrm{sgn}\left(d-\frac{x}{m}\right).$$

However (and this will be useful in the sequel)

$$S^{\pm}_{A,B}(x) = \sum_{\frac{x}{B}<q\le\frac{x}{A}} \chi_q(x) + \Sigma(A) + \Sigma(B) + \mathcal{O}\left(\frac{h}{A}+1\right),$$

which we will now use together with the estimates

$$\Sigma(A) \ll \sum_{\frac{x-h}{A}\le m\le\frac{x+h}{A}} \left(\frac{h}{m}+1\right) \ll \frac{Ah}{N}\left(\frac{h}{A}+1\right) + \left(\frac{h}{A}+1\right) \ll h\left(\frac{h}{N}+\frac{A}{N}+\frac{1}{A}+\frac{1}{h}\right)$$

and

$$\Sigma(B) \ll \sum_{\frac{x-h}{B}\le m\le\frac{x+h}{B}} \left(\frac{h}{m}+1\right) \ll \frac{Bh}{N}\left(\frac{h}{B}+1\right) + \left(\frac{h}{B}+1\right) \ll h\left(\frac{h}{N}+\frac{B}{N}+\frac{1}{B}+\frac{1}{h}\right). \quad \square$$

Define $\overline{d}(\mathrm{mod}\, q)$, the *reciprocal* residue of $d(\mathrm{mod}\, q)$, $\forall (d,q)=1$, as $\overline{d}d \equiv 1(\mathrm{mod}\, q)$ and, $\forall a \in \mathbb{Z}$, $a \neq 0$,

$$R_Q(|a|) = R(|a|,g,Q,N) \overset{def}{=} -\sum_{\substack{\ell|a\\ b:=\frac{|a|}{\ell}}} \sum_{q\sim\frac{Q}{\ell}} g(\ell q) \sum_{\substack{d\sim\frac{Q}{\ell}\\ (d,q)=1}} g(\ell d)$$
$$\times\left(B_1\left(\frac{[2N/\ell d]-\overline{d}b}{q}\right) - B_1\left(\frac{[N/\ell d]-\overline{d}b}{q}\right)\right) - \sum_{\substack{\ell|a\\ b:=\frac{|a|}{\ell}}} \sum_{q\sim\frac{Q}{\ell}} g(\ell q) \sum_{\substack{d\sim\frac{Q}{\ell}\\ (d,q)=1}} g(\ell d)$$
$$\times\left(B_1\left(\frac{[2N/\ell d]+\overline{d}b}{q}\right) - B_1\left(\frac{[N/\ell d]+\overline{d}b}{q}\right)\right),$$

where we recall the *first Bernoulli function* (1−periodicized of the 1-st Bernoulli *polynomial*) definition :

$$B_1(\alpha) \overset{def}{=} \{\alpha\} - 1/2, \quad \forall \alpha \in \mathbb{R}\backslash\mathbb{Z}$$

(here $\{\alpha\} \overset{def}{=} \alpha - [\alpha]$ is the *fractional part* of $\alpha \in \mathbb{R}$), $B_1 = 0$ on $\mathbb{Z}$. Hereon we use $e(\beta) \overset{def}{=} e^{2\pi i\beta}$ and

$$\|\alpha\| \overset{def}{=} \min_{n\in\mathbb{Z}} |\alpha - n|,$$

the *distance from the integers*.

In what follows we will explicitly avoid the case $Q \ll |a|$ (assuming $\frac{Q}{\ell} \gg 1$

in following proof of Lemma), since

$$a \neq 0, \quad Q \ll |a|, \quad g \lll 1 \ \Rightarrow \ R_Q(|a|) \lll \sum_{\ell | a} \frac{a^2}{\ell^2} \lll a^2$$

can be considered a very good bound (recall we will use it when $|a| \ll h$, with $\theta < 1/2$, giving $R_Q(|a|) \ll N^{1-\delta}$).

However, this will be not explicitly remarked in our

Lemma B. *Fix* $\delta > 0$ *enough small (say* $\delta < \frac{1}{100}$*). Let* $N \in \mathbb{N}$ *and* $a \in \mathbb{Z}$, $a \neq 0$, $|a| = o(N)$; *assume* $g : \mathbb{N} \to \mathbb{R}$ *supported into* $]Q, 2Q]$, *where* $Q \in \mathbb{N}$ *with* $Q^2 \gg |a| N^{1-2\delta/3}$ *and* $Q = o(N^{1-\delta})$ *when* $N \to \infty$. *Also, assume that* g *is essentially bounded. Then* $\forall \varepsilon > 0$

$$R_Q(|a|) = 2 \sum_{\substack{\ell | a \\ b := \frac{|a|}{\ell}}} \sum_{q \sim \frac{Q}{\ell}} \frac{g(\ell q)}{q} \sum_{\substack{d \sim \frac{Q}{\ell} \\ (d,q)=1}} g(\ell d) \sum_{t | q} \sum_{\substack{j \le \frac{tJ}{q} \\ (j,t)=1}} \cot \frac{\pi j}{t}$$
$$\times \left(\sin \frac{2\pi [2N/\ell d] j}{t} - \sin \frac{2\pi [N/\ell d] j}{t} \right) \cos \frac{2\pi j \overline{d} b}{t} + \mathcal{O}_{\varepsilon}\left(N^{1-\delta+\varepsilon} \right),$$

where $J = J(\ell, d, q, \delta, N) \overset{def}{=} [\ell d q / N^{1-\delta}]$ *(see that* $J \to \infty$ *and* $J = o(Q/\ell)$ *from our assumptions for* Q*).*

Proof. The *finite Fourier expansion* (see, esp., Lemma 3 in the file example on my webpage):

$$B_1\left(\frac{n}{q}\right) = -\frac{1}{q} \sum_{j \le \frac{q}{2}} \cot \frac{\pi j}{q} \sin \frac{2\pi j n}{q} \qquad \forall q \in \mathbb{N}, \ \forall n \in \mathbb{Z}$$

(see that, whenever $q = 1$, the sum is empty and in fact $B_1 = 0$) gives (now on $b = a/\ell$ in the sums)

$$R_Q(|a|) = 2 \sum_{\ell | a} \sum_{q \sim \frac{Q}{\ell}} \frac{g(\ell q)}{q} \sum_{\substack{d \sim \frac{Q}{\ell} \\ (d,q)=1}} g(\ell d) \sum_{j \le \frac{q}{2}} F\left(\frac{j}{q}\right),$$

with $F\left(\frac{j}{q}\right) = F_{b\overline{d}, N/\ell d}\left(\frac{j}{q}\right)$ defined as $F_{b\overline{d}, N/\ell d}\left(\frac{j}{q}\right) \overset{def}{=}$

$$\overset{def}{=} \cot \frac{\pi j}{q} \left(\sin \frac{2\pi [2N/\ell d] j}{q} - \sin \frac{2\pi [N/\ell d] j}{q} \right) \cos \frac{2\pi j \overline{d} b}{q}$$

Then, *split* the sum over j at $J = [\ell d q / N^{1-\delta}]$ and use the fact that F

depends only on j/q :

$$\sum_{j\le \frac{q}{2}} F\left(\frac{j}{q}\right) = \sum_{j\le J} F\left(\frac{j}{q}\right) + \sum_{J<j\le \frac{q}{2}} F\left(\frac{j}{q}\right)$$

and

$$\sum_{j\le J} F\left(\frac{j}{q}\right) = \sum_{t|q} \sum_{\substack{j'\le \frac{J}{t}\\ (j',\frac{q}{t})=1}} F\left(\frac{j'}{q/t}\right) = \sum_{t|q} \sum_{\substack{j\le \frac{tJ}{q}\\ (j,t)=1}} F\left(\frac{j}{t}\right)$$

(where we *flipped* the divisors t into q/t), with non-empty sum on $j>J$, due to our hypotheses.

Applying *partial summation* (Tenenbaum [7]) to $\sum_n e(n\alpha) \ll \frac{1}{\|\alpha\|}$ (Davenport [5, Chapter 26]) for the exponentials $e_q(\theta) \stackrel{def}{=} e(\theta/q), \forall\theta\in\mathbb{R}$ (Vinogradov [8]):

$$\sum_{J<j\le \frac{q}{2}} F\left(\frac{j}{q}\right)$$
$$\ll q\int_J^{\frac{q}{2}} \left|\sum_{J<j\le v} e_q(j([2N/\ell d]\pm \overline{d}b))\right| + \left|\sum_{J<j\le v} e_q(j([N/\ell d]\pm \overline{d}b))\right| \frac{dv}{v^2}$$
$$\ll \frac{q}{J}\left(\frac{1}{\left\|\frac{[2N/\ell d]\pm \overline{d}b}{q}\right\|} + \frac{1}{\left\|\frac{[N/\ell d]\pm \overline{d}b}{q}\right\|}\right),$$

whenever none of these four distances from the integers vanish, i.e. $q \nmid d\left[\frac{cN}{\ell d}\right]\pm b$ $\forall c=1,2$, while

$$q \,\Big|\, d\left[\frac{cN}{\ell d}\right]\pm b \Rightarrow \sum_{J<j\le \frac{q}{2}} F\left(\frac{j}{q}\right) \lll q,$$

from the trivial

$$\sum_{J<j\le v} e(j\alpha) \ll v, \quad \forall v\in[J,q/2].$$

Thus, it remains to estimate, say,

$$\Sigma = \Sigma(|a|,g,J) \stackrel{def}{=} 2\sum_{\ell|a}\sum_{q\sim\frac{Q}{\ell}} \frac{g(\ell q)}{q} \sum_{\substack{d\sim\frac{Q}{\ell}\\ (d,q)=1}} g(\ell d) \sum_{J<j\le\frac{q}{2}} F\left(\frac{j}{q}\right) \lll$$

$$\lll \frac{N^{1-\delta}}{Q^2}\sum_{\ell|a}\ell \sum_{\substack{d,q\sim\frac{Q}{\ell}\\ d\left[\frac{cN}{\ell d}\right]\not\equiv\pm b(q)}}\sum \frac{1}{\left\|\frac{[cN/\ell d]\mp\overline{d}b}{q}\right\|} + \sum_{\ell|a}\sum_{\substack{d,q\sim\frac{Q}{\ell}\\ q|d\left[\frac{cN}{\ell d}\right]\pm b}}\sum 1$$

where we mean, with this right-hand-side, the more complicate expression

$$\frac{N^{1-\delta}}{Q^2}\sum_{\ell|a}\ell\left(\sum_{\substack{d,q\sim\frac{Q}{\ell}\\ d\left[\frac{2N}{\ell d}\right]\not\equiv b(q)}}\sum \frac{1}{\left\|\frac{\left[\frac{2N}{\ell d}\right]-\overline{d}b}{q}\right\|} + \sum_{\substack{d,q\sim\frac{Q}{\ell}\\ d\left[\frac{2N}{\ell d}\right]\not\equiv -b(q)}}\sum \frac{1}{\left\|\frac{\left[\frac{2N}{\ell d}\right]+\overline{d}b}{q}\right\|}\right.$$

$$\left. + \sum_{\substack{d,q\sim\frac{Q}{\ell}\\ d\left[\frac{N}{\ell d}\right]\not\equiv b(q)}}\sum \frac{1}{\left\|\frac{\left[\frac{N}{\ell d}\right]-\overline{d}b}{q}\right\|} + \sum_{\substack{d,q\sim\frac{Q}{\ell}\\ d\left[\frac{N}{\ell d}\right]\not\equiv -b(q)}}\sum \frac{1}{\left\|\frac{\left[\frac{N}{\ell d}\right]+\overline{d}b}{q}\right\|}\right)$$

$$+\sum_{\ell|a}\sum_{d\sim\frac{Q}{\ell}}\left(\sum_{q|d\left[\frac{2N}{\ell d}\right]-b}1 + \sum_{q|d\left[\frac{2N}{\ell d}\right]+b}1 + \sum_{q|d\left[\frac{N}{\ell d}\right]-b}1 + \sum_{q|d\left[\frac{N}{\ell d}\right]+b}1\right)$$

and (obviously) we pursue that abbreviation; from the above, using: $d[cN/\ell d]\pm b=0 \Rightarrow d|b$,

$$\Sigma \lll \frac{N^{1-\delta}}{Q^2}\sum_{\ell|a}\ell\sum_{q\sim\frac{Q}{\ell}}\sum_{0<|r|\le\frac{q}{2}}\frac{q}{|r|}\sum_{\substack{d\sim\frac{Q}{\ell}\\ d[cN/\ell d]\pm b\equiv rd(q)}}1$$

$$+\sum_{\ell|a}\left(\sum_{\substack{d\sim\frac{Q}{\ell}\\ d\nmid b}}\sum_{q|d\left[\frac{cN}{\ell d}\right]\pm b}1+\sum_{d|b}\frac{Q}{\ell}\right)$$

and, finally, $d[cN/\ell d]\pm b = rd \Rightarrow d|b$ (whatever is the integer $r\neq 0$) and $n\neq 0 \Rightarrow \sum_{m|n}1 \lll 1$ give

$$\Sigma \lll \frac{N^{1-\delta}}{Q^2}\sum_{\ell|a}\ell\left(\sum_{d\sim\frac{Q}{\ell},d\nmid b}\frac{Q}{\ell}+\sum_{d|b}\left(\frac{Q}{\ell}\right)^2\right)+Q \lll N^{1-\delta}. \qquad \square$$

Lemma C. *Let* $h\to\infty$ *and* $h=o(N)$ *when* $N\to\infty$. *Assume that* $1\le A<B$ *are real numbers depending eventually on* N *and* h *(say,* $A\to\infty$

and $B \ll N$). *Then*

$$\int_N^{2N} \Big| \sum_{\frac{x}{B}<q\le\frac{x}{A}} \chi_q(x)\Big|^2 dx \ll NhL^4 + NhL^5\left(\frac{N}{A^2}\right).$$

Proof. We recall (see Coppola et al [3])

$$\chi_q(x) = \sum_{\substack{d|q\\ d>1}} \frac{d}{q} {\sum_{j\le d}}^{*} c_{j,d}e_d(jx), \text{ with } {\sum_{j\le d}}^{*} |c_{j,d}|^2 \le \sum_{j\le d} |c_{j,d}|^2 \ll \frac{h}{d}$$

(hereon $\sum^*$ is over reduced classes, i.e. over $(j,d)=1$) whence, using a dyadic dissection of $d-$range,

$$\Big|\sum_{\frac{x}{B}<q\le\frac{x}{A}} \chi_q(x)\Big| = \Big|\sum_{1<d\le\frac{x}{A}} \Big(\sum_{\frac{x}{dB}<k\le\frac{x}{dA}} \frac{1}{k}\Big) {\sum_{j\le d}}^{*} c_{j,d}e_d(jx)\Big|$$
$$\ll L \max_{D\ll\frac{N}{A}} \Big|\sum_{d\sim D} \alpha_d(x) {\sum_{j\le d}}^{*} c_{j,d}e_d(jx)\Big|,$$

say, with $\alpha_d(x) \ll L$ and abbreviate $X_1 \overset{def}{=} \max(N, d_1k_1A, d_2k_2A)$, $X_2 \overset{def}{=} \min(2N, d_1k_1B, d_2k_2B)$, indicating I the integral to bound,

$$I \ll L^4 \max_{D\ll\frac{N}{A}} \max_{k_1,k_2\ll\frac{N}{AD}} \sum\sum_{\substack{(j_1,d_1)=1\\(j_2,d_2)=1}} |c_{j_1,d_1}||c_{j_2,d_2}| \left|\int_{X_1}^{X_2} e\left(\left(\frac{j_1}{d_1}-\frac{j_2}{d_2}\right)x\right)dx\right|,$$

having expanded the square and exchanged the sums with the integral (that "doesn't see" points):

$$\int_{X_1}^{X_2} e\left(\left(\frac{j_1}{d_1}-\frac{j_2}{d_2}\right)x\right)dx \ll \min\left(X_2-X_1, \frac{1}{\left\|\frac{j_1}{d_1}-\frac{j_2}{d_2}\right\|}\right)$$

(compare Davenport [5, Chapter 26]); this gives

$$I \ll L^4 \max_{D\ll\frac{N}{A}} \sum_{d_1\sim D}\sum_{d_2\sim D} {\sum_{j_1\le d_1}}^{*} {\sum_{j_2\le d_2}}^{*} |c_{j_1,d_1}||c_{j_2,d_2}| \min\left(N, \frac{1}{\left\|\frac{j_1}{d_1}-\frac{j_2}{d_2}\right\|}\right).$$

Since

$$c_{j,d} := \frac{1}{d}\sum_{|s|\le h} \operatorname{sgn}(s)e_d(js) \;\Rightarrow\; C_{j/d} := dc_{j,d} \text{ depends only on } j/d$$

we have

$$\sum\sum_{d_1,d_2\sim D} {\sum_{j_1\le d_1}}^{*} {\sum_{j_2\le d_2}}^{*} |c_{j_1,d_1}|\cdot|c_{j_2,d_2}| \min\left(N, \frac{1}{\left\|\frac{j_1}{d_1}-\frac{j_2}{d_2}\right\|}\right)$$

$$\ll \frac{N}{D^2} \sum_{\kappa \in \mathcal{F}} |C_\kappa|^2 + \frac{1}{D^2} \sum\sum_{\substack{\kappa_1,\kappa_2 \in \mathcal{F} \\ \kappa_1 \neq \kappa_2}} \frac{|C_{\kappa_1}| \cdot |C_{\kappa_2}|}{\|\kappa_1 - \kappa_2\|}$$

with $\mathcal{F} \stackrel{def}{=} \{j/d : (j,d) = 1, d \sim D\}$ *Farey fractions* with denominators $d \sim D$; then,

$$N \sum_{\kappa \in \mathcal{F}} |C_\kappa|^2 + \sum\sum_{\substack{\kappa_1,\kappa_2 \in \mathcal{F} \\ \kappa_1 \neq \kappa_2}} \frac{|C_{\kappa_1}| \cdot |C_{\kappa_2}|}{\|\kappa_1 - \kappa_2\|} \ll (N + D^2 L) \sum_{\kappa \in \mathcal{F}} |C_\kappa|^2$$

$$\ll (N + D^2 L) \sum_{d \sim D} d^2 \sum_{j \le d}{}^{*} |c_{j,d}|^2,$$

compare Coppola et al [3], Lemma 2; (from $c_{j,d}$ bounds) this is $\ll (N + D^2 L) h D^2$, whence, gathering last bounds,

$$\int_N^{2N} \Big| \sum_{\frac{x}{B} < q \le \frac{x}{A}} \chi_q(x) \Big|^2 dx \ll L^4 h \left(N + \frac{N^2 L}{A^2} \right). \qquad \square$$

3. Proofs of the Theorems

A kind of "elementary dispersion"(see Coppola [1]) gives, for all essentially bounded functions $f : \mathbb{N} \to \mathbb{R}$, $\forall \varepsilon > 0$,

$$J_f(N,h) = \sum_{a \neq 0} S(a) \mathfrak{C}_f(a) - M_f^2 (2h) N + \mathcal{O}_\varepsilon \left(N^\varepsilon \left(Nh + h^3 + Qh^2 \right) \right)$$

and

$$I_f(N,h) = \sum_{a \neq 0} W(a) \mathfrak{C}_f(a) + \mathcal{O}_\varepsilon \left(N^\varepsilon \left(Nh + h^3 \right) \right),$$

where $Q = \max$ of $g \stackrel{def}{=} f * \mu$ support (i.e., $f(n) = \sum_{d|n, d \le Q} g(d)$; for details, Coppola [1]). Thus, we start proving Theorem 1.

Proof of Theorem 1. Use $\mathfrak{C}_Q(a)$ def.: $a > 0 \Rightarrow \mathfrak{C}_Q(-a) = \mathfrak{C}_Q(a) + \mathcal{O}_\varepsilon(N^\varepsilon a)$,

$$a \neq 0 \ \Rightarrow \ \mathfrak{C}_Q(a) = \frac{\mathfrak{C}_Q(a) + \mathfrak{C}_Q(-a)}{2} + \mathcal{O}_\varepsilon \left(N^\varepsilon |a| \right)$$

and we first want to get the *main term* of this correlation ($a > 0$):

$$\mathfrak{C}_Q(a) = \sum\sum_{d,q \sim Q} \sum_{\substack{n \sim N \\ n \equiv 0(d) \\ n \equiv a(q)}} 1 = \sum_{\ell | a} \sum\sum_{\substack{d,q \sim \frac{Q}{\ell} \\ (d,q) = \ell}} \sum_{\substack{k \sim \frac{N}{d} \\ kd \equiv a(q)}} 1$$

$$= \sum_{\substack{\ell|a \\ b:=\frac{a}{\ell}}} \sum_{\substack{d,q\sim\frac{Q}{\ell} \\ (d,q)=1}} \sum_{\substack{m\sim\frac{N}{d\ell} \\ m\equiv\overline{d}b(q)}} 1 := \mathcal{M}_Q(a)+\mathcal{E}_Q(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon Q\right)$$

(now on $\ell \in \mathbb{N}$), *say*, with error due to $q\,\big|\,d\left[\frac{cN}{\ell d}\right]-b$:

$$\sum_{\substack{\ell|a \\ b:=\frac{a}{\ell}}} \sum_{d\sim\frac{Q}{\ell}} \sideset{}{^*}\sum_{q\sim\frac{Q}{\ell},q|d\left[\frac{cN}{\ell d}\right]-b} 1 := \sum_{\ell|a} \sum_{d\sim\frac{Q}{\ell}} \Bigg(\sideset{}{^*}\sum_{\substack{q\sim\frac{Q}{\ell} \\ q|d\left[\frac{N}{\ell d}\right]-\frac{a}{\ell}}} 1 + \sideset{}{^*}\sum_{\substack{q\sim\frac{Q}{\ell} \\ q|d\left[\frac{2N}{\ell d}\right]-\frac{a}{\ell}}} 1\Bigg)$$

$$\ll \sum_{\substack{\ell|a \\ b:=\frac{a}{\ell}}} \sum_{d\sim\frac{Q}{\ell}} \sum_{q\sim\frac{Q}{\ell},q|d\left[\frac{cN}{\ell d}\right]-b} 1 \ll \sum_{\ell|a} \left(\frac{Q}{\ell} \sum_{d|b} 1 + \sum_{d\sim\frac{Q}{\ell}, d\nmid b} N^\varepsilon \right) \lll Q,$$

and we define the following ("*main term*")

$$\mathcal{M}_Q(a) \overset{def}{=} \sum_{\ell|a} \sum_{q\sim\frac{Q}{\ell}} \sideset{}{^*}\sum_{d\sim\frac{Q}{\ell}} \frac{1}{q} \left(\left[\frac{2N}{\ell d}\right] - \left[\frac{N}{\ell d}\right] \right)$$

$$= N \sum_{\ell|a} \frac{1}{\ell} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sideset{}{^*}\sum_{d\sim\frac{Q}{\ell}} \frac{1}{d} + \mathcal{O}\left(\sum_{\ell|a} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sum_{d\sim\frac{Q}{\ell}} 1 \right)$$

(fractional parts give $\mathcal{O}_\varepsilon\left(N^\varepsilon Q\right)$ as above), with ("*error term*")

$$\mathcal{E}_Q(a) \overset{def}{=} -\sum_{\substack{\ell|a \\ b:=\frac{a}{\ell}}} \sum_{q\sim\frac{Q}{\ell}} \sideset{}{^*}\sum_{d\sim\frac{Q}{\ell}} \left(B_1\!\left(\frac{[2N/\ell d]-\overline{d}b}{q} \right) - B_1\!\left(\frac{[N/\ell d]-\overline{d}b}{q} \right) \right)$$

$$\ll \sum_{\ell|a} \left(\frac{Q^2}{\ell^2} + 1 \right) \lll Q^2$$

a trivial estimate, which gives $\mathcal{O}_\varepsilon\left(N^{1-\delta+\varepsilon}\right)$, from $Q^2 \ll N^{1-\delta+\varepsilon}$. However, $Q^2 \gg |a|N^{1-\delta+\varepsilon}$, $Q=o(N^{1-\delta})$ into Lemma B give $\forall a \neq 0$

$$\mathcal{E}_Q(a)+\mathcal{E}_Q(-a) = R_Q(|a|) = 2 \sum_{\substack{\ell|a \\ b:=\frac{|a|}{\ell}}} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sideset{}{^*}\sum_{d\sim\frac{Q}{\ell}} \sum_{t|q} \sum_{\substack{j\le\frac{tJ}{q} \\ (j,t)=1}} \cot\frac{\pi j}{t}$$

$$\times \left(\sin\frac{2\pi[2N/\ell d]j}{t} - \sin\frac{2\pi[N/\ell d]j}{t} \right) \cos\frac{2\pi j\overline{d}b}{t} + \mathcal{O}_\varepsilon\left(N^{1-\delta+\varepsilon}\right)$$

(now on $b := \frac{|a|}{\ell}$); recall $J \in \mathbb{N}$, $J \to \infty$ and $J = o(Q/\ell)$ $\forall \ell|a$. Write $\sin\frac{2\pi[cN/\ell d]j}{t}$ $(c=1,2)$ both for $\sin\frac{2\pi[N/\ell d]j}{t}$, $\sin\frac{2\pi[2N/\ell d]j}{t}$:

$$\sin\frac{2\pi[cN/\ell d]j}{t} = \sin\frac{2\pi cNj}{\ell dt} + \mathcal{O}\left(\frac{J}{q}\right), \quad \forall c=1,2; \text{ whence}$$

$$\sum_{d\sim\frac{Q}{\ell}}^{*}\left(\sin\frac{2\pi[2N/\ell d]j}{t}-\sin\frac{2\pi[N/\ell d]j}{t}\right)\cos\frac{2\pi j\overline{d}b}{t}$$
$$=\sum_{d\sim\frac{Q}{\ell}}^{*}\left(\sin\frac{4\pi Nj}{\ell dt}-\sin\frac{2\pi Nj}{\ell dt}\right)\cos\frac{2\pi j\overline{d}b}{t}+\mathcal{O}\left(\frac{Q^2N^\delta}{\ell N}\right)$$

and this gives for, *say*, $E_Q(|a|)\overset{def}{=}\sum_{\ell|a}\sum_{q\sim\frac{Q}{\ell}}\frac{1}{q}\sum_{\substack{t|q\\ t\ge\frac{q}{J}}}\sum_{\substack{j\le\frac{tJ}{q}\\ (j,t)=1}}\cot\frac{\pi j}{t}\times$

$$\times\sum_{d\sim\frac{Q}{\ell}}^{*}\left(\sin\frac{2\pi[2N/\ell d]j}{t}-\sin\frac{2\pi[N/\ell d]j}{t}\right)\cos\frac{2\pi j\overline{d}b}{t}$$

which is $\frac{R_Q(|a|)}{2}$ main term ($\Rightarrow$ of $\frac{\mathcal{E}_Q(a)+\mathcal{E}_Q(-a)}{2}$, too), that

$$E_Q(|a|)=\sum_{\ell|a}\sum_{q\sim\frac{Q}{\ell}}\frac{1}{q}\sum_{\substack{t|q\\ t\ge\frac{q}{J}}}\sum_{\substack{j\le\frac{tJ}{q}\\ (j,t)=1}}\cot\frac{\pi j}{t}\sum_{d\sim\frac{Q}{\ell}}^{*}\left(\sin\frac{4\pi Nj}{\ell dt}-\sin\frac{2\pi Nj}{\ell dt}\right)\cos\frac{2\pi j\overline{d}b}{t}$$
$$+\mathcal{O}_\varepsilon\left(N^{\delta+\varepsilon}\frac{Q^3}{N}\right).$$

We *leave the negligible* $\lll\frac{Q^3}{N}N^\delta$*−terms and pass to bound* ($\forall c=1,2$)

$$\sum_{d\sim\frac{Q}{\ell}}^{*}\sin\frac{2\pi cNj}{\ell dt}\cos\frac{2\pi j\overline{d}b}{t}$$
$$\ll\frac{Nj}{Qt}\Big|\sum_{d\sim\frac{Q}{\ell}}^{*}\cos\frac{2\pi j\overline{d}b}{t}\Big|+\int_{\frac{Q}{\ell}}^{\frac{2Q}{\ell}}\Big|\sum_{\frac{Q}{\ell}<d\le v,(d,q)=1}\cos\frac{2\pi j\overline{d}b}{t}\Big|\frac{Nj}{\ell tv^2}dv$$
$$\lll\frac{Nj}{\ell t^2}\sqrt{(b,t)}\sqrt{t},$$

obtained (from $\frac{Q}{\ell}\gg q\gg t\Rightarrow\frac{Q}{\ell t}\gg 1$ and) using *partial summation* (Tenenbaum [7]) *with Weil bound* (Iwaniec et al [6, Chapter 11])

$$D_2'-D_1'\ll t\Rightarrow\sum_{\substack{D_1'<d\le D_2'\\ (d,t)=1}}\cos\frac{2\pi jb\overline{d}}{t}\lll\sqrt{(b,t)}\sqrt{t}\quad\forall j\in\mathbb{N},\ (j,t)=1$$

into the elementary bound, $\forall(j,t)=1$, where $\frac{Q}{\ell}\ll D_1,D_2\ll\frac{Q}{\ell}$:

$$\sum_{\substack{D_1<d\le D_2\\(d,t)=1\\(d,q/t)=1}} \cos\frac{2\pi j\overline{d}b}{t} = \sum_{\substack{k|\frac{q}{t}\\(k,t)=1}} \mu(k) \sum_{\substack{\frac{D_1}{k}<d\le\frac{D_2}{k}\\(d,t)=1}} \cos\frac{2\pi j\overline{k}b\overline{d}}{t}$$

$$\lll \frac{D_2-D_1}{t}\sqrt{(b,t)}\sqrt{t}.$$

Finally (use $J=o(Q/\ell)\ \forall\ell \Rightarrow J=o(q)\Rightarrow j=o(t)\Rightarrow \frac{j}{t}\cot\frac{\pi j}{t}\ll 1$)

$$E_Q(|a|) \lll N\sum_{\ell|a}\frac{1}{\ell}\sum_{q\sim\frac{Q}{\ell}}\frac{1}{q}\sum_{t|q,t\ge\frac{q}{J}}\frac{1}{t}\sum_{j\le\frac{Jt}{q}}\sqrt{(b,t)}\sqrt{t}$$

$$\lll QN^\delta\sum_{\ell|a}\frac{1}{\ell}\sum_{\frac{Q}{\ell J}\le t\le\frac{2Q}{\ell}}\frac{\sqrt{(b,t)}}{\sqrt{t}} \lll Q^{\frac{3}{2}}N^\delta,$$

since $J\ll\frac{Q^2N^\delta}{\ell N}\Rightarrow \displaystyle\sum_{\substack{q\sim\frac{Q}{\ell},q\le tJ\\q\equiv 0(t)}}\frac{1}{q}\sum_{j\le\frac{Jt}{q}}1\ll\frac{J}{t}\sum_{k\sim\frac{Q}{\ell t}}\frac{1}{k^2}\ll\frac{J\ell}{Q}\ll\frac{QN^\delta}{N}$, and

$$\sum_{T_1\le t\le T_2}\frac{\sqrt{(b,t)}}{\sqrt{t}} = \sum_{\beta|b}\sqrt{\beta}\sum_{T_1\le t\le T_2,(t,b)=\beta}\frac{1}{\sqrt{t}}$$

$$=\sum_{\beta|b}\sum_{\frac{T_1}{\beta}\le t'\le\frac{T_2}{\beta},(t',b/\beta)=1}\frac{1}{\sqrt{t'}} \lll \sqrt{T_2}+1. \quad \square$$

Proof of Theorem 2. We prove the easiest bound, for the symmetry integral.

As already pointed out (it follows from Coppola [1, Lemma 1] that)

$$I_Q(N,h)=\sum_{a\ne 0}W(a)\mathfrak{C}_Q(a)+\mathcal{O}_\varepsilon\left(N^\varepsilon(Nh+h^3)\right),$$

recalling that $d_{A,B}$ is essentially bounded (unif.ly in A,B), i.e. $d_{Q,2Q}\lll 1$, in this case (now on "$\forall\varepsilon>0$" is implicit, whenever the $\mathcal{O}_\varepsilon-$symbol appears). Here W is even (with zero mean-value)

$$W(a)\overset{def}{=}\begin{cases} 2h-3a & \text{if}\quad 0\le a\le h;\\ a-2h & \text{if}\quad h\le a\le 2h;\\ 0 & \text{if}\quad a\ge 2h.\end{cases}$$

Since $W(a)\ll h$, uniformly $\forall a\in\mathbb{Z}$ and the support of W is within $[-2h,2h]$ (whence $a\ll h$ in the sum),

$$\left| I_Q(N,h) - \sum_{a\neq 0} W(a) N \sum_{\ell|a} \frac{1}{\ell} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sum_{d\sim\frac{Q}{\ell}}^{*} \frac{1}{d} \right| \lll Nh + Qh^2 + (Q^{3/2}N^{\delta} + Q^3 N^{\delta-1} + N^{1-\delta})h^2$$

is immediate from Theorem 1 (use $\theta < \frac{1}{2} \Rightarrow h^3 \lll Nh$). Thus, it will suffice to bound the following:

$$N \sum_{a\neq 0} W(a) \sum_{\ell|a} \frac{1}{\ell} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sum_{d\sim\frac{Q}{\ell}}^{*} \frac{1}{d} = 2N \sum_{\ell\le 2h} \left(\sum_{b>0} W(\ell b) \right) \frac{1}{\ell} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sum_{d\sim\frac{Q}{\ell}}^{*} \frac{1}{d}$$

as $\lll Nh$, thanks to the bound $\sum_{b>0} W(\ell b) \ll h$, uniformly $\forall \ell \in \mathbb{N}$, from Coppola [1, Lemma 4].

We prove, then, the other bound, for the Selberg integral.

As before (it follows from Coppola [1, Lemma 2] that)

$$J_Q(N,h) = \sum_{a\neq 0} S(a)\mathfrak{C}_Q(a) - NM_Q^2(2h) + \mathcal{O}_{\varepsilon}\left(N^{\varepsilon}(Nh + h^3 + Qh^2)\right), \qquad (*)$$

again, from: $d_{A,B}$ especially bounded (unif. in A,B), i.e. $d_{Q,2Q} \lll 1$, now. Here S is even (again, but has non-zero mean-value!) and defined as $S(a) \overset{def}{=} \max(2h-|a|,0)$, $\forall a \in \mathbb{Z}$; we proceed as above for I_Q, with the errors from Theorem 1; however, we've $(*)$ main term:

$$2N \sum_{\ell\le 2h} \left(\sum_{b>0} S(\ell b) \right) \frac{1}{\ell} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sum_{d\sim\frac{Q}{\ell}}^{*} \frac{1}{d} = 4Nh^2 \sum_{\ell\le 2h} \frac{1}{\ell^2} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sum_{d\sim\frac{Q}{\ell}}^{*} \frac{1}{d} + \mathcal{O}_{\varepsilon}\big(N^{1+\varepsilon}h\big),$$

since, uniformly $\forall \ell \le 2h$,

$$\sum_{b>0} S(\ell b) = \sum_{b\le\frac{2h}{\ell}} (2h - \ell b) = 2h \left[\frac{2h}{\ell}\right] - \frac{\ell}{2}\left[\frac{2h}{\ell}\right]\left(\left[\frac{2h}{\ell}\right]+1\right) = \frac{2h^2}{\ell} + \mathcal{O}(h),$$

with

$$4Nh^2 \sum_{\ell\le 2h} \frac{1}{\ell^2} \sum_{q\sim\frac{Q}{\ell}} \frac{1}{q} \sum_{d\sim\frac{Q}{\ell}}^{*} \frac{1}{d} = N(2h)^2 \sum_{\ell\le 2h} \sum_{q'\sim Q} \frac{1}{q'} \sum_{\substack{d'\sim Q \\ (d',q')=\ell}} \frac{1}{d'}$$

$$= N(2h)^2 \sum_{\ell=1}^{\infty} \sum_{q\sim Q} \frac{1}{q} \sum_{\substack{d\sim Q \\ (d,q)=\ell}} \frac{1}{d} + \mathcal{O}_{\varepsilon}\left(N^{1+\varepsilon}h\right) = N(2h)^2 \sum_{q\sim Q} \frac{1}{q} \sum_{d\sim Q} \frac{1}{d} + \mathcal{O}_{\varepsilon}\left(N^{1+\varepsilon}h\right)$$

$$= N\left(2h\sum_{q\sim Q}\frac{1}{q}\right)^2 + \mathcal{O}_{\varepsilon}\left(N^{\varepsilon}Nh\right) = NM_Q^2(2h) + \mathcal{O}_{\varepsilon}\left(N^{\varepsilon}Nh\right). \quad \square$$

Proof of Theorem 3. First, we need the (local) definition of "symmetry sum"

$$S_D^{\pm}(x) := \sum_{|n-x|\le h}\operatorname{sgn}(n-x)\sum_{\substack{d|n \\ d\sim D}}1 \quad \left(\text{instead of } S_{D,2D}^{\pm}\right)$$

which has $\lll Nh$ mean-square, $\forall D \ll \sqrt{Nh}$ (Coppola [1, Theorem 1] and $\theta < \frac{1}{2} \Rightarrow h^3 \lll Nh$). When $D \gg \sqrt{N}$, we "flip" the divisors :

$$S_D^{\pm}(x) = \sum_{\frac{x}{2D}<q\le\frac{x}{D}}\chi_q(x) + \mathcal{O}\left(h\left(\frac{D}{N}+\frac{1}{h}+\frac{1}{D}+\frac{h}{N}\right)\right),$$

from Lemma A (which we may apply, due to $\frac{B}{A} = 2 > 1+\frac{2h}{N-h}$, being $h = o(N)$ here), the remainders contribute, $\forall D \ll N/h$, to the final mean-square

$$\lll Nh^2\left(\frac{1}{h^2}+\frac{1}{D^2}+\frac{h^2}{N^2}\right) \lll Nh$$

(last inequality from $\theta := \frac{\log h}{\log N} < \frac{1}{2}$), while from Lemma C (compare Coppola et al [3] and Coppola et al [4])

$$\int_N^{2N}\Big|\sum_{\frac{x}{2D}<q\le\frac{x}{D}}\chi_q(x)\Big|^2 dx \lll \left(N+\left(\frac{N}{D}\right)^2\right)h \lll Nh$$

(use $D \gg \sqrt{N}$, now).

We're left with the range $\frac{N}{h} \ll D \ll N$ (since $x \ll N$, all our divisors d into $I_{A,B}$, whatever A and B, are $d \ll N$, whence $D \ll N$). For this, our Lemma A doesn't suffice, as the single remainder $\mathcal{O}\left(h\frac{D}{N}\right)$ gets (as D approaches $\mathcal{O}(N)$, here) near the trivial bound $S_D^{\pm} = \mathcal{O}(h)$. However, see Lemma A proof, this "bad" remainder comes from $\Sigma(A)$ and $\Sigma(B)$, here with $A = D$ and $B = 2D$, actually (the A case is similar to the B case following):

$$\int_N^{2N}\Big|\Sigma(B)\Big|^2 dx = \int_N^{2N}\Big|\sum_{\frac{x-h}{B}\le m\le\frac{x+h}{B}}\ \sum_{\frac{x-h}{m}<d\le B}\operatorname{sgn}\Big(d-\frac{x}{m}\Big)\Big|^2 dx$$

$$= \int_N^{2N}\sum_{\frac{x-h}{B}\le m\le\frac{x+h}{B}}\Big|\sum_{\frac{x-h}{m}<d\le B}\operatorname{sgn}\Big(d-\frac{x}{m}\Big)\Big|^2 dx,$$

expanding the $m-$sum-square: here $h = N^\theta$, $\theta < \frac{1}{2} \Rightarrow h = o(\sqrt{N})$

$$B = 2D \gg \frac{N}{h}, h = o(\sqrt{N}) \Rightarrow h = o(B) \Rightarrow \left| \left[\frac{x-h}{B}, \frac{x+h}{B} \right] \right| < 1,$$

uniformly $\forall x \sim N$ (here $|\ |$ is the length), i.e. this interval can contain at most one integer m:

$$\sum_{\frac{x-h}{B} \le m_1, m_2 \le \frac{x+h}{B}} \sum = \sum_{\frac{x-h}{B} \le m \le \frac{x+h}{B}} \qquad (m \text{ if exists is unique!}),$$

something like a "squeezing" on the diagonal. Exchange $\sum$ and $\int$:

$$\int_N^{2N} \left| \Sigma(B) \right|^2 dx \ll \sum_{\frac{N-h}{B} \le m \le \frac{2N+h}{B}} h \left(\frac{h}{m} \right)^2 \ll h^3 \frac{B}{N} \ll h^3 \lll Nh. \qquad \square$$

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