International Journal of Pure and Applied Mathematics
Volume 58 No. 3 2010, 299-308

ON OPERATOR INEQUALITIES IN TERMS OF GEOMETRIC MEAN

C.-S. Lin
Department of Mathematics
Bishop's University
2600, College Street, Sherbrooke, Quebec, J1M 1Z7, CANADA
e-mail: plin@ubishops.ca

Abstract: Two lemmas for operator inequalities in terms of geometric mean in Hilbert space are given in this paper, from which some operator inequalities in terms of geometric mean are given and proved.

AMS Subject Classification: 47A63
Key Words: logarithm of a positive and invertible operator, chaotic operator order, geometric mean of operators, Löwner-Heinz inequality, satellite theorems, grand Furuta inequality, and extended grand Furuta inequality

1. Introduction

The capital letters throughout this paper denote bounded linear operators on a Hilbert space, and O is the zero operator. A positive operator T is denoted by $T \geq O$, while a positive and invertible operator A is by $A > O$. For $A > O$, the logarithm of A is denoted by log A, which is defined as usual by log $A = \lim_{\alpha \to +0} \frac{A^\alpha - I}{\alpha}$. For $A, B > O$, we write $A \gg B$ in short for log $A \geq$ log B, which is a standard chaotic operator order and is weaker than the usual perator order $A \geq B$. Let us recall the well-known classical Löwner-Heinz inequality as it is used frequently in this paper.

Theorem LH. *If $S \geq T \geq O$, then $S^\alpha \geq T^\alpha$ for $\alpha \in [0, 1]$.*

Essentially, it is known that the Löwner-Heinz inequality does not hold in

Received: December 1, 2009

general if $\alpha > 1$. Also, that $A \gg B$ does not imply $A \geq B$ in general. Let us recall the Furuta inequality [3], which is an excellent generalization of the Löwner-Heinz inequality.

Theorem F. *If $A \geq B \geq O$, then for each $r \geq 0$,*

(i) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$ *and*

(ii) $(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Recall that the α-geometric mean of operators A and B introduced by Kubo-Ando [9] is given by

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2} B A^{-1/2})^\alpha A^{1/2} \text{ for } \alpha \in [0,1].$$

Correspondingly, the binary operation $\natural_\beta$ for A and B is defined same as the binary operation $\sharp_\alpha$ for any real number β, i.e., $A \natural_\beta B = A^{1/2}(A^{-1/2} B A^{-1/2})^\beta$ $A^{1/2}$. Relating to the α-geometric mean, two papers by Kamei [7, 8] are interesting. He considered the so called satellite theorems of the Furuta inequality (it is indeed operator inequalities in terms of geometric mean), one under the operator order [7], and the other under chaotic operator order [8]. Both results were obtained due to Theoren CF in the section two. More precisely,

For $A \geq B > O$ and for all $r \geq 0$ and $p \geq 1$, we have

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p., \text{ see [7]}, \quad (*)$$

For $A \gg B$, $A, B > O$, and for all $r \geq 0$ and $p \geq 1$, we have

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \ll A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p, \text{ see [8]}. \quad (**)$$

Notice that since $A \geq B > O$ implies $A \gg B$, then by (**) above we have $A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B$ and $A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p$ for all $r \geq 0$ and $p \geq 1$, and hence (*) holds. In other words, if (**) holds true, then so does (*).

In this paper we give two lemmas; one is about operator inequalities implying geometric mean operator inequalities (Lemma 1), and the other is about operator equalities involving geometric means (Lemma 2). Then we use them to give and prove some operator inequalities in terms of geometric mean. Remark that there are many such inequalities expressed in different ways and used different proofs in the literature, such as in [1], [2], [6], [7], [8] and perhaps others more. Ours are resulted directly from lemmas with the aid of well-known operator inequalities and equality, which will be stated in the next section with reference numbers for the convenience of readers.

2. Some Well-Known Operator Inequalities

The following results are developed from Theorem F, and this is the reason we state Theorem F in the section one first.

Theorem GF. (see [4]) *If $A \geq B > O$, then for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$,*

(a) $A^{1-t+r} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1-t+r}{(p-t)s+r}}$.

Theorem CF. (see [8]) *For all $p \geq 0$, $r \geq 0$ and $A, B > O$, $A \gg B$ if and only if*

(b) $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{r}{p+r}}$.

Theorem CGF. (see [1]) *For $p \geq 0$, $r \geq 0$, $s \geq 1$, $t \leq 0$ and $A, B > O$, $A \gg B$ if and only if*

(c) $A^{r-t} \geq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}}$.

Theorem EGF. (see [10]) *If $A \geq B \geq C > O$, then for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$,*

(d) $A^{1-t+r} \geq [A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1-t+r}{(p-t)s+r}}$.

The next result is a very usefull tool in operator equalities and inequalities.

Lemma F. (see [4]) *For any real number r, and $A, B > O$,*
$(BAB)^r = BA^{1/2}(A^{1/2}B^2A^{1/2})^{r-1}A^{1/2}B.$

3. Two Lemmas

The proofs of our main results in Section 4 are based on Lemma 1 and 2.

Lemma 1. *For $A, B, X, Y > O$ and for any real numbers a, b, c and d, the following hold.*

(1) *If $(B^{b/2}X^{-a}B^{b/2})^{c-1} \leq B^d$, then $X^a \natural_c B^b \leq B^{b+d}$.*

(2) *If $(A^{b/2}Y^{-a}A^{b/2})^{c-1} \geq A^d$, then $A^{b+d} \leq Y^a \natural_c A^b$.*

(3) *In above if $b + d = 1$ in particular and if $B \leq A$ ($B \ll A$), then*
$X^a \natural_c B^b \leq B \leq A \leq Y^a \natural_c A^b$ ($X^a \natural_c B^b \leq B \ll A \leq Y^a \natural_c A^b$).

Proof. (1) $X^a \natural_c B^b = X^{a/2}(X^{-a/2}B^bX^{-a/2})^cX^{a/2}$ by definition
$= X^{a/2}X^{-a/2}B^{b/2}(B^{b/2}X^{-a}B^{b/2})^{c-1}B^{b/2}X^{-a/2}X^{a/2}$ by Lemma F
$= B^{b/2}(B^{b/2}X^{-a}B^{b/2})^{c-1}B^{b/2}$
$\leq B^{b+d}$ by assumption,

and we have (1).

The proof of (2) follows similarly by definition, Lemma F and assumption (starting with $Y^a \natural_c A^b$), and should be omitted. That (3) is obvious. □

In order to have more complicated inequalities in terms of geometric mean we need the next result. Notice that (4) in Lemma 2 below is a well-known tool in geometric mean of operators.

Lemma 2. *For $A, B, C > O$ and for any real numbers a, b, c, d and e, we have*

(4) $A^a \natural_c B^b = B^b \natural_{1-c} A^a$.

(5) $B^{b-d} \natural_e (B^b \natural_c A^a) = B^{b/2}[B^{-d} \natural_e (B^{-b/2}A^aB^{-b/2})^c]B^{b/2}$.

(6) $C^{b-d} \natural_e C^{b/2}B^{-b/2}(B^b \natural_c A^a)B^{-b/2}C^{b/2} = C^{b/2}[C^{-d} \natural_e (B^{-b/2}A^aB^{-b/2})^c] C^{b/2}$.

Proof. (4) is a straightforward computation due to the geometric mean and Lemma F; and (5) is a special case of (6) (let $C = B$ in (6)). We shall use the definition of the geometric mean to show (6). The left side of (6) is equal to

$$C^{\frac{b-d}{2}}[C^{\frac{d-b}{2}}C^{b/2}B^{-b/2}(B^b \natural_c A^a)B^{-b/2}C^{b/2}C^{\frac{d-b}{2}}]^eC^{\frac{b-d}{2}}$$

$$= C^{\frac{b-d}{2}}[C^{d/2}B^{-b/2}B^{b/2}(B^{-b/2}A^aB^{-b/2})^cB^{b/2}B^{-b/2}C^{d/2}]^eC^{\frac{b-d}{2}}$$

$$= C^{\frac{b-d}{2}}[C^{d/2}(B^{-b/2}A^aB^{-b/2})^cC^{d/2}]^eC^{\frac{b-d}{2}}.$$

The right side $= C^{b/2}C^{-d/2}[C^{d/2}(B^{-b/2}A^aB^{-b/2})^cC^{d/2}]^eC^{-d/2}C^{b/2}$, and the proof of the lemma is finished. □

4. Some Operator Inequalities in Terms of Geometric Mean

In this section we assume that $A, B, C > O$, and Theorem LH will be frequently used without mentioning it. Firstly, from Theorem GF we have

Theorem 1. *For $p \geq 1$, $r, s \geq 1$ and $t \in [0, 1]$, the following hold.*

If $B \leq A$ ($B \ll A$), then:

$$B^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \leq B \leq A \leq A^r \sharp_{\frac{r-1}{(p-t)s+r}} (A^{t/2}B^{-p}A^{t/2})^s,$$

$$(B^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \leq B \ll A \leq A^r \sharp_{\frac{r-1}{(p-t)s+r}} (A^{t/2}B^{-p}A^{t/2})^s).$$

Proof. Since $A \geq B$ if and only if $B^{-1} \geq A^{-1}$, (a) in Theorem GF implies two inequalities: $B^{-(1-t+r)} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{-\frac{1-t+r}{(p-t)s+r}}$ and

$$A^{-(1-t+r)} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{-\frac{1-t+r}{(p-t)s+r}}$$

for $s \geq 1$, $p \geq 1$ and $r \geq t \in [0,1]$. Then:

$$B^{1-r} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1-r}{(p-t)s+r}},$$

$$A^{1-r} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1-r}{(p-t)s+r}},$$

as $0 \leq \frac{r-1}{1-t+r} < 1$ for $r \geq 1$ and $t \in [0,1]$. From the two inequalities obtained above, let $X = B^{-t/2}A^pB^{-t/2}$, $Y = A^{-t/2}B^pA^{-t/2}$, $-a = s$, $b = r$, $d = 1-r$ and $c-1 = \frac{1-r}{(p-t)s+r}$ (then $c = \frac{1+(p-t)s}{(p-t)s+r}$, $0 < c \leq 1$ and $0 \leq 1-c < 1$) in Lemma 1. Then

$$(B^{t/2}A^{-p}B^{t/2})^s \sharp_{\frac{1+(p-t)s}{(p-t)s+r}} B^r \leq B \leq A \leq (A^{t/2}B^{-p}A^{t/2})^s \sharp_{\frac{1+(p-t)s}{(p-t)s+r}} A^r,$$

and by (4) in Lemma 2 the desired inequalities follow. □

In the next theorem we consider a different value of r in Theorem 1, and obtain a different result.

Theorem 2. *For $p, s \geq 1$, $r \geq 1+t$, and $t \in [0,1]$, the following hold.*

If $B \leq A$ ($B \ll A$), then:

$$B^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{-t} \natural_s A^{-p}) \leq B \leq A \leq A^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (A^{-t} \natural_s B^{-p})$$

$$(B^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{-t} \natural_s A^{-p}) \leq B \ll A \leq A^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (A^{-t} \natural_s B^{-p})).$$

Proof. In the proof of Theorem 1,

$$B^{-(1-t+r)} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{-\frac{1-t+r}{(p-t)s+r}}$$

and

$$A^{-(1-t+r)} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{-\frac{1-t+r}{(p-t)s+r}}$$

for all $s \geq 1$, $p \geq 1$ and $r \geq t \in [0,1]$, which yield

$$B^{1+t-r} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1+t-r}{(p-t)s+r}},$$

$$A^{1+t-r} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1+t-r}{(p-t)s+r}},$$

as $0 \leq \frac{r-t-1}{1-t+r} < 1$ for $r \geq 1+t$ and $t \in [0,1]$. From the two inequalities obtained above, let $X = B^{-t/2}A^pB^{-t/2}$, $Y = A^{-t/2}B^pA^{-t/2}$, $d = 1+t-r$, $b = r$, $-a = s$ and $c-1 = \frac{1+t-r}{(p-t)s+r}$ (then $c = \frac{1+t+(p-t)s}{(p-t)s+r}$, $0 < c \leq 1$ and $0 \leq 1-c < 1$) in Lemma 1. Then

$$(B^{t/2}A^{-p}B^{t/2})^s \sharp_{\frac{1+t+(p-t)s}{(p-t)s+r}} B^r \leq B^{1+t},$$

and

$$A^{1+t} \leq (A^{t/2}B^{-p}A^{t/2})^s \sharp_{\frac{1+t+(p-t)s}{(p-t)s+r}} A^r.$$

It follows by (4) in Lemma 2 that $B^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \leq B^{1+t}$ and

$A^{1+t} \le A^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (A^{t/2}B^{-p}A^{t/2})^s$. So,

$B^{-t/2}[B^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s]B^{-t/2} \le B$ and

$A \le A^{-t/2}[A^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (A^{t/2}B^{-p}A^{t/2})^s]A^{-t/2}$.

Finally, apply (5) in Lemma 2 to rewrite the left side and the right side of the two inequalities above, respectively, with $b = -t$, $-d = r$, $e = \frac{r-1-t}{(p-t)s+r}$ (then $0 \le e < 1$), $a = -p$, and $c = s$. □

From Theorem EGF we have

Theorem 3. *For* $p, s \ge 1$, $t \in [0,1]$, *and* $r \ge 1$, *the following hold.*

If $C \le B \le A$ $(C \ll B \ll A)$, *then*

$C^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \le C \le B \le A \le A^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}C^{-p}B^{t/2})^s$

$(C^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \le C \ll B \ll A \le A^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}C^{-p}B^{t/2})^s)$.

Proof. As $C \le B \le A$ if and only if $A^{-1} \le B^{-1} \le C^{-1}$, and by (d) in Theorem EGF we have $C^{-(1-t+r)} \ge [C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{-(1-t+r)}{(p-t)s+r}}$ and $[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{-(1-t+r)}{(p-t)s+r}} \ge A^{-(1-t+r)}$ for $p, s \ge 1$ and $r \ge t \in [0,1]$. Then

$C^{1-r} \ge [C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{1-r}{(p-t)s+r}}$ and

$[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1-r}{(p-t)s+r}} \ge A^{1-r}$,

as $0 \le \frac{r-1}{1-t+r} < 1$ for $r \ge 1$ and $t \in [0,1]$. From the two inequalities obtained above, let $X = B^{-t/2}A^pB^{-t/2}$, $Y = B^{-t/2}C^pB^{-t/2}$, $b = r$, $-a = s$, $d = 1-r$, and $c-1 = \frac{1-r}{(p-t)s+r}$ (then $c = \frac{1+(p-t)s}{(p-t)s+r}$, $0 < c \le 1$ and $0 \le 1-c < 1$) in Lemma 1. Then

$(B^{t/2}A^{-p}B^{t/2})^s \sharp_{\frac{1+(p-t)s}{(p-t)s+r}} C^r \le C$, and $A \le (B^{t/2}C^{-p}B^{t/2})^s \sharp_{\frac{1+(p-t)s}{(p-t)s+r}} A^r$,

which by (4) in Lemma 2 yields the required result. □

From Theorem CGF we have

Theorem 4. *Let* $p \ge 0$, $r \ge 1$, $s \ge 1$ *and* $t \le 0$, *the following hold.*

If $B \ll A$ $(B \le A)$, *then*

$B^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \le B \ll A \le A^r \sharp_{\frac{r-1}{(p-t)s+r}} (A^{t/2}B^{-p}A^{t/2})^s$

$(B^r \sharp_{\frac{r-1}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \le B \le A \le A^r \sharp_{\frac{r-1}{(p-t)s+r}} (A^{t/2}B^{-p}A^{t/2})^s)$.

Proof. Since $B \ll A$ if and only if $A^{-1} \ll B^{-1}$, $B^{-(r-t)} \ge$

$[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{-(r-t)}{(p-t)s+r}}$ and $A^{-(r-t)} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^s A^{r/2}]^{\frac{-(r-t)}{(p-t)s+r}}$ for $p, r \geq 0$, $s \geq 1$ and $t \leq 0$ by Theorem CGF. Then

$B^{1-r} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1-r}{(p-t)s+r}}$ and

$A^{1-r} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1-r}{(p-t)s+r}}$

as $0 \leq \frac{r-1}{r-t} < 1$ for $r \geq 1$ and $t \leq 0$. From the two inequalities obtained above, let $X = B^{-t/2}A^pB^{-t/2}$, $Y = A^{-t/2}B^pA^{-t/2}$, $-a = s$, $b = r$, $d = 1-r$, and $c - 1 = \frac{1-r}{(p-t)s+r}$ (then $c = \frac{1+(p-t)s}{(p-t)s+r}$, $0 < c \leq 1$ and $0 \leq 1-c < 1$) in Lemma 1. Then

$(B^{t/2}A^{-p}B^{t/2})^s \sharp_{\frac{1+(p-t)s}{(p-t)s+r}} B^r \leq B$ and $A \leq (A^{t/2}B^{-p}A^{t/2})^s \sharp_{\frac{1+(p-t)s}{(p-t)s+r}} A^r$.

Now, apply (4) in Lemma 2 to get the result. $\square$

The next result shows that a different value of α in Theorem LH makes a different result from Theorem 4. By Theorem CGF we have

Theorem 5. *Let $p \geq 0$, $r \geq 1$, $s \geq 1$ and $t \leq 0$, the following hold.*

If $B \ll A$ $(B \leq A)$, then

$B^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{-t} \natural_s A^{-p}) \leq B \ll A \leq A^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (A^{-t} \natural_s B^{-p})$

$(B^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{-t} \natural_s A^{-p}) \leq B \leq A \leq A^{r-t} \sharp_{\frac{r-1-t}{(p-t)s+r}} (A^{-t} \natural_s B^{-p})$.

Proof. Since $B^{-(r-t)} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{-(r-t)}{(p-t)s+r}}$ and $A^{-(r-t)} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{-(r-t)}{(p-t)s+r}}$ for $p, r \geq 0$, $s \geq 1$ and $t \leq 0$ by the proof in Theorem 4. Then

$B^{1+t-r} \geq [B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1+t-r}{(p-t)s+r}}$ and

$A^{1+t-r} \leq [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1+t-r}{(p-t)s+r}}$

as $0 \leq \frac{r-t-1}{r-t} < 1$ for $r \geq 1$ and $t \leq 0$. From the two inequalities obtained above, let $X = B^{-t/2}A^pB^{-t/2}$, $Y = A^{-t/2}B^pA^{-t/2}$, $-a = s$, $b = r$, $d = 1+t-r$ and $c - 1 = \frac{1+t-r}{(p-t)s+r}$ (then $c = \frac{1+t+(p-t)s}{(p-t)s+r}$, $0 < c \leq 1$ and $0 \leq 1-c < 1$) in Lemma 1. Then

$(B^{t/2}A^{-p}B^{t/2})^s \sharp_{\frac{1+t+(p-r)s}{(p-t)s+r}} B^r \leq B^{1+t}$ and $A^{1+t} \leq (A^{t/2}B^{-p}A^{t/2})^s \sharp_{\frac{1+t+(p-r)s}{(p-t)s+r}} A^r$,

so that, by (4) in Lemma 2,

$B^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \leq B^{1+t}$ and $A^{1+t} \leq A^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (A^{t/2}B^{-p}A^{t/2})^s$.

Clearly, $B^{-t/2}[B^r\sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s]B^{-t/2} \leq B$ and $A \leq A^{-t/2}[A^r\sharp$

$\frac{r-1-t}{(p-t)s+r}\ (A^{t/2}B^{-p}A^{t/2})^s]A^{-t/2}$.

The required inequalities follows by (5) in Lemma 2 with $b=-t$, $-d=r$, $e=\frac{r-1-t}{(p-t)s+r}$ (then $0\le e<1$), $a=-p$, and $c=s$. □

Incidentally, notice that Theorem CF is a special case of Theorem CGF (if $s=1$ and $t=0$). Thus, one might expect that (*) and (**) in Section 1 are special cases of Theorem 4 or Theorem 5. This is indeed the case as next result shows. First, we may rephrase (*) and (**) as follows:

(★) For $B\ll A$ $(B\le A)$, $r\ge 1$ and $p\ge 0$, the following hold.

$A^{-p}\ \sharp_{\frac{1+p}{p+r}}\ B^r\le B\ll A\le B^{-p}\ \sharp_{\frac{1+p}{p+r}}\ A^r$

$(A^{-p}\ \sharp_{\frac{1+p}{p+r}}\ B^r\le B\le A\le B^{-p}\ \sharp_{\frac{1+p}{p+r}}\ A^r)$.

Because (*) and (**) are based on Theorem CF, and in Theorem CF the roles of p and r may be interchanged as $r\ge 0$ and $p\ge 0$. In other words, Theorem CF in section 2 may be rephrased as : *For all* $p\ge 0$, $r\ge 0$ *and* $A,B>O$, $A\gg B$ *if and only if* $A^p\ge (A^{p/2}B^rA^{p/2})^{\frac{p}{p+r}}$. Now,

Corollary. *(★) in above is a special case of Theorem 4 or Theorem 5.*

Proof. In Theorem 4 or Theorem 5 let $s=1$ and $t=0$. Then

$B^r\ \sharp_{\frac{r-1}{p+r}}\ A^{-p}\le B\ll A\le A^r\ \sharp_{\frac{r-1}{p+r}}\ B^{-p}$

$(B^r\ \sharp_{\frac{r-1}{p+r}}\ A^{-p}\le B\le A\le A^r\ \sharp_{\frac{r-1}{p+r}}\ B^{-p})$.

Use (4) in Lemma 2 to get (★). □

Our final result makes use of (6) in Lemma 2. From Theorem EGF we have

Theorem 6. *For* $p,s\ge 1$, $t\in[0,1]$, *and* $r\ge 1+t$, *the following hold.*

If $C\le B\le A$ $(C\ll B\ll A)$, *then*

$C^{r-t}\ \sharp_{\frac{r-1-t}{(p-t)s+r}}\ C^{-t/2}B^{t/2}(B^{-t}\ \natural_s\ A^{-p})B^{t/2}C^{-t/2}\le C\le B\le A$

$\le A^{r-t}\ \sharp_{\frac{r-1-t}{(p-t)s+r}}\ A^{-t/2}B^{t/2}(B^{-t}\ \natural_s\ C^{-p})B^{t/2}A^{-t/2}$

$(C^{r-t}\ \sharp_{\frac{r-1-t}{(p-t)s+r}}\ C^{-t/2}B^{t/2}(B^{-t}\ \natural_s\ A^{-p})B^{t/2}C^{-t/2}\le C\ll B\ll A$

$\le A^{r-t}\ \sharp_{\frac{r-1-t}{(p-t)s+r}}\ A^{-t/2}B^{t/2}(B^{-t}\ \natural_s\ C^{-p})B^{t/2}A^{-t/2})$.

Proof. By the proof of Theorem 3, $C^{-(1-t+r)}\ge [C^{r/2}(B^{-t/2}A^pB^{-t/2})^s C^{r/2}]^{\frac{-(1-t+r)}{(p-t)s+r}}$ and $[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{-(1-t+r)}{(p-t)s+r}}\ge A^{-(1-t+r)}$ for $p,s\ge 1$, and $r\ge t\in[0,1]$. It follows that

$C^{1+t-r}\ge [C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{1+t-r}{(p-t)s+r}}$ and

$$[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1+t-r}{(p-t)s+r}} \geq A^{1+t-r}$$

as $0 \leq \frac{r-t-1}{1-t+r} < 1$ for $r \geq 1+t$ and $t \in [0,1]$. From the two inequalities obtained above, let $X = B^{-t/2}A^pB^{-t/2}$, $Y = B^{-t/2}C^pB^{-t/2}$, $d = 1+t-r$, $b = r$, $-a = s$ and $c-1 = \frac{1+t-r}{(p-t)s+r}$ (then $c = \frac{1+t+(p-t)s}{(p-t)s+r}$, $0 < c \leq 1$ and $0 \leq 1-c < 1$) in Lemma 1. Then

$$(B^{t/2}A^{-p}B^{t/2})^s \sharp_{\frac{1+t+(p-t)s}{(p-t)s+r}} C^r \leq C^{1+t}$$

and

$$A^{1+t} \leq (B^{t/2}C^{-p}B^{t/2})^s \sharp_{\frac{1+t+(p-t)s}{(p-t)s+r}} A^r,$$

or

$C^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s \leq C^{1+t}$ and $A^{1+t} \leq A^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}C^{-p}B^{t/2})^s$ by (4) in Lemma 2. So, $C^{-t/2}[C^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}A^{-p}B^{t/2})^s]C^{-t/2} \leq C$ and $A \leq A^{-t/2}[A^r \sharp_{\frac{r-1-t}{(p-t)s+r}} (B^{t/2}C^{-p}B^{t/2})^s]A^{-t/2}$.

Finally, apply (6) in Lemma 3 with $b = -t$, $-d = r$, $e = \frac{r-1-t}{(p-t)s+r}$, $a = -p$, and $c = s$, to get the result. □

In conclusion we mention that due to Lemma 1 and 2 the process of the proof of each theorem is quite similar to one another. It is also the author's belief, unlike geometric mean operator inequalities in the literature, that our proofs are simple and easy to follow.

References

[1] M. Fujii, E. Kamei, R. Nakamoto, On a question of Furuta on chaotic order, *Linear Alg. and Its Appl.*, **341** (2002), 119-127.

[2] M. Fujii, E. Kamei, Difference between chaotic order and usual order in generalized Furuta inequalities, *Sci. Math. Japon.*, **59** (2004), 561-567.

[3] T. Furuta, $A \geq B \geq O$ assures $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.*, **101** (1987), 85-88.

[4] T. Furuta, Extension of the Furuta inequality and Ando-Hiai Log majorization, *Linear Alg. Appl.*, **219** (1995), 139-155.

[5] T. Furuta, *Invitation To Linear Operators*, ISBN: 0-415-26799-4, Taylor and Francis (2001).

[6] T. Furuta, $A \geq B \geq O$ essures $A^{1+r-t} \geq [A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{1+r-t}{(p-t)s+r}}$ for $t \in [0,1]$, $r \geq t$, $p \geq 1$, $s \geq 1$ and related inequalities, *Archiv. Inequal. Appl.*, **2** (2004), 141-158.

[7] E. Kamei, A satellite to Furuta's inequality, *Math. Japon.*, **33** (1988), 883-886.

[8] E. Kamei, Chaotic order and Furuta inequality, *Sci. Math. Japon.*, **53** (2001), 289-293.

[9] F. Kubo, T. Ando, Means of positive linear operators, *Math. Ann.*, **246** (1980), 205-224.

[10] M. Uchiyama, Criteria for operator means, *J. Math. Soc. Japan*, **55** (2003), 197-207.