International Journal of Pure and Applied Mathematics
Volume 58 No. 3 2010, 309-314

AN ALTERNATIVE DEFINITION OF CONTINUOUS COHOMOLOGY AND A VANISHING THEOREM

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Abstract: Given a locally compact group G and a continuous representation ρ of G on a real or complex Banach space V we obtain the corresponding cohomology groups $H^n(G, V, \rho)$ using a recursive construction adapted from [1]. As a consequence we prove under certain conditions (equivalent with the existence of a non-trivial simultaneous fixed point of the associated affine map) that all cohomology groups vanish.

AMS Subject Classification: 22D12, 22E41, 57T10
Key Words: continuous cohomology of groups and its vanishing, continuous representation, affine map, fixed point, short and long exact sequence, n-cochain, n-coboundary

1. Introduction

This note concerns an alternative definition of cohomology of groups and their continuous representations. Here we adopt the following notation: G is a locally compact second countable group, with (V, ρ) a continuous Banach G-space. We denote by V^G the G-fixed vectors of V, by Id the identity map on V and by $Z(G)$ the center of G. Also, we will use without distinction the notations $\rho(g)(v)$, or $g \cdot v$ ($g \in G$ and $v \in V$).

Given a locally compact second countable group G and a continuous k-linear ($k = \mathbb{R}$ or $\mathbb{C}$) representation ρ on a resp. real or complex Banach space V, the standard definition of group cohomology $H^n(G, V)$ is due to Hochschild and Mostow and is found in [3], or Borel and Wallach in [2]. We will use

Received: December 10, 2009

the notation of [3]. Here $F^n(G,V)$ stands for the space of n-cochains, i.e. all continuous functions from $G^n \to V$. $F^n(G,V)$ is a G-space under the action of G by right translation. An important feature of the cohomology of [3] is that given a short exact sequence of continuous Banach G-spaces,

$$(0) \to W \to V \to U \to (0),$$

one obtains a long exact sequence of the corresponding cohomology groups,

$$... \to H^n(G,W) \to H^n(G,V) \to H^n(G,U) \to H^{n+1}(G,W) \to ... .$$

Our objective here is to adapt an idea of Atiyah and Wall [1] to our situation and give a definition of the cohomology groups which we prove is equivalent to that of [3] and which has the advantage of being *recursive*. We will then use this recursive definition to prove a vanishing theorem which has important applications (particularly when G is nilpotent).

2. The Recursive Definition of the Cohomology

The recursive definition of the cohomology is as follows:

The 0-dimensional cohomology group of G with coefficients in V is $H^0(G,V) = V^G$. We define $H^1(G,V)$ to be the quotient group $\mathcal{Z}^1/\mathcal{B}^1$, where $\mathcal{Z}^1$ is the space of the crossed homomorphisms (or 1-cocycles)

$$\varphi : G \longrightarrow V \ : \ \varphi(gh) = \varphi(g) + g\varphi(h),$$

and $\mathcal{B}^1$ consists of those φ (or 1-coboundaries) having the form $\varphi(g) = g \cdot v_0 - v_0$, for some v_0 in V and all g in G.

To define the higher cohomological groups $H^n(G,V)$, let

$$V' = \{\varphi : G \longrightarrow V, \ \varphi \ \text{ continuous}\}.$$

Here V' is a k-vector space when equipped with the usual pointwise operations. It becomes a G-space with G acting on V' by

$$g \cdot \varphi : G \longrightarrow V \ : \ h \mapsto g \cdot \varphi(h).$$

Now we consider the embedding of G-Banach spaces $\varepsilon : V \hookrightarrow V'$ defined by

$$v \mapsto \varepsilon_v \ : \ G \longrightarrow V \ : \ g \mapsto g \cdot v.$$

The space $\varepsilon(V)$ is a closed subspace of V. In order to see this we recall the topology on $F^n(G,V)$ is that of uniform convergence on compacta of G^n, which implies pointwise convergence. Let $\varphi \in V'$ be the limit of a sequence $\epsilon(v_n)$, where v_n is an arbitrary sequence in V. Because of the definition of the map ϵ fixing a $g \in G$, $g \cdot v_n$ converges to $\varphi(g)$ for n sufficiently large. Now

$g \cdot V = V$ and therefore each $g \cdot V$ is closed in V. Hence $\varphi(g) = g \cdot v$ for some $v \in V$. Hence we can take the quotient space $V^\sharp = V/\varepsilon(V)$, which is also a Banach space. We now define $H^n(G,V)$ inductively for $n \geq 2$ by setting $H^n(G,V) = H^{n-1}(G,V^\sharp)$.

In order to show this definition is equivalent to the standard one, we shall first prove V' is acyclic, that is $H^n(G,V') = (0)$, for each $n > 0$.

Lemma 2.1. *$H^n(G,V') = (0)$, for all $n > 0$.*

Proof. Consider the n-cochain $f \in F^n(G,V')$. This is a map

$$f : \underbrace{G \times ... \times G}_{n-\text{times}} \longrightarrow V'.$$

Since the elements of the space V' are themselves maps from G to V, we can regard f as a map

$$f : \underbrace{G \times ... \times G \times G}_{(n+1)-\text{times}} \longrightarrow V.$$

We write

$$(f(g_1,...,g_n))(g_0) = f(g_0,g_1,...,g_n).$$

Now the coboundary operators,

$$\partial^n : F^n(G,V') \longrightarrow F^{n+1}(G,V')$$

give us (see [3], or [2])

$$[(\partial^n f)(g_1,...,g_{n+1})](g_0) = [(-1)^{n+1} f(g_1,...,g_n) + g_1 f(g_2,...,g_{n+1}) + \sum_{i=0}^{n} (-1)^i f(g_1,...,g_i g_{i+1},...g_{n+1})](g_0).$$

Hence

$$\partial^n f(g_0,...,g_{n+1}) = \sum_{i=0}^{n} (-1)^i f(g_0,g_1,...,g_i g_{i+1},...,g_{n+1}) + (-1)^{n+1} f(g_0,...,g_n).$$

Now, consider the maps

$$d^n : F^n(G,V') \longrightarrow F^{n-1}(G,V'),$$

defined by

$$(d^n f)(g_0,...,g_{n-1}) = f(1_G,g_0,...,g_{n-1}).$$

Then,

$$[d^{n+1}(\partial^n f)](g_0,...,g_n) = (\partial^n f)(1_G,g_0,...,g_n)$$
$$= f(g_0,...,g_n) - f(1_G,g_0 g_1,...,g_n) + ... + (-1)^n f(1_G,g_0,g_1,...,g_{n-1}g_n)$$

$$+ (-1)^{n+1} f(1_G, g_0, g_1, ..., g_{n-1}).$$

On the other hand,

$$\begin{aligned}
&[\partial^{n-1}(d^n f)](g_0, ..., g_n) \\
&\quad = (d^n f)(g_0 g_1, ..., g_n) + ... + (-1)^{n-1}(d^n f)(g_0, g_1, ..., g_{n-1}g_n) \\
&\qquad + (-1)^n (d^n f)(g_0, g_1, ..., g_{n-1}) = f(1_G, g_0 g_1, ..., g_n) \\
&\quad + ... + (-1)^{n+1} f(1_G, g_0, g_1, ..., g_{n-1}g_n) + (-1)^n f(1_G, g_0, g_1, ..., g_{n-1}).
\end{aligned}$$

Therefore, letting Id stand for $Id_{F^n(G,V')}$ we get,

$$[d^{n+1}(\partial^n f) + \partial^{n-1}(d^n f)](g_0, ..., g_n) = f(g_0, ..., g_n),$$

that is,

$$d^{n+1}\partial^n + \partial^{n-1} d^n = Id.$$

Hence, if f is a n-cocycle in $\mathcal{Z}^n(G, V')$, the above relation shows that it is also a n-coboundary, i.e.

$$\partial^{n-1}(d^n f) = f.$$

Therefore,

$$H^n(G, V') = (0), \text{ for each } n > 0. \qquad \square$$

This acyclicity now yields the equivalence of the two definitions of cohomology:

Theorem 2.2. *$H^n(G, V) \cong H^{n-1}(G, V^\sharp)$, for all $n > 1$.*

Proof. As we saw V' and $V^\sharp$ are G-Banach spaces and we have the following exact sequence:

$$(0) \longrightarrow V \longrightarrow V' \longrightarrow V^\sharp \longrightarrow (0).$$

This in turn gives us the long exact sequence,

$$... \longrightarrow H^{n-1}(G, V') \longrightarrow H^{n-1}(G, V^\sharp) \longrightarrow H^n(G, V) \longrightarrow H^n(G, V') \longrightarrow ... .$$

Since V' is acyclic this long exact sequence gives isomorphisms $H^n(G, V) \cong H^{n-1}(G, V^\sharp)$ for all $n > 1$. $\square$

3. An Application

We now give an application of the equivalence of the two definitions of cohomology. Consider the continuous linear representation $\rho : G \longrightarrow GL(V)$ and let φ be a 1-cocycle. Define the affine map,

$$\rho_\varphi : G \longrightarrow \mathrm{Aff}(V) := G \ltimes GL(V),$$

given by

$$\rho_\varphi(g) : V \longrightarrow V \quad \text{such that} \quad \rho_\varphi(g)(v) := \rho(g)(v) + \varphi(g).$$

Because of the cocycle identity this map is a homomorphism. Suppose the affine map ρ_φ had a fixed point, that is there is a v_0 in V with $\rho_\varphi(g)(v_0) = v_0$, for every g in G. Then $\rho(g)(v_0) + \varphi(g) = v_0$ so that φ is a coboundary.

The following lemma is left to the reader, where here $A \in GL(V)$ and $b \in V$.

Lemma 3.1. *An affine map $x \mapsto Ax + b$ has a fixed point in V if and only if $Id - A$ is invertible.*

Our application of Theorem 2.2 is:

Corollary 3.2. *Let G be a locally compact second countable group, and ρ be a representation of G on a Banach space V without non-trivial fixed points. If there is a z_0 in $Z(G)$ with $Id - \rho(z_0)$ invertible, then $H^n(G, V) = (0)$, $n \geq 0$.*

Proof. First, $H^0(G, V) := V^G = (0)$ because ρ by assumption has no non-trivial fixed points. Now, let $\phi : G \to V$ be a 1-cocycle. Consider the affine map ρ_ϕ defined above. Because of the lemma if $Id - \rho(z_0)$ is invertible, then the affine map $\rho_\varphi(z_0)$ has a fixed point. Namely, $v_0 = [Id - \rho(z_0)]^{-1}(\phi(z_0))$.

Now v_0 is fixed under each $\rho_\phi(g)$, $g \in G$. This is because $z_0 \in Z(G)$ and ρ_φ is a homomorphism. Hence $\rho_\phi(z_0 g) = \rho_\phi(g z_0) = \rho_\phi(g)(\rho_\phi(z_0)) = \rho_\phi(z_0)(\rho_\phi(g))$. Therefore, $\rho_\phi(g)(\rho_\phi(z_0))(v_0) = \rho_\phi(g)(v_0) = \rho_\phi(z_0)(\rho_\phi(g)(v_0))$. It follows that $\rho_\phi(g)(v_0) = \rho_\phi(z_0)(\rho_\phi(g)(v_0))$. In other words, $\rho_\phi(g)(v_0)$ is a fixed point of the map $\rho_\phi(z_0)$. Since the unique fixed point of this map is v_0, $\rho_\phi(g)(v_0) = v_0$ for each $g \in G$. That is, $\phi(g) = v_0 - \rho(g)(v_0)$, $g \in G$ and thus $H^1(G, V) = (0)$.

To see $H^n(G, V) = (0)$ for $n \geq 2$, we use Theorem 2.2, that is, $H^{n+1}(G, V) = H^n(G, V^\sharp)$. If $Id - \rho(z_0)$ is invertible on V, the corresponding representation on V' will satisfy the same condition, as will the associated representation on $V^\sharp$. By the above, $H^1(G, V^\sharp) = (0)$. Therefore, $H^2(G, V) = H^1(G, V^\sharp) = (0)$. By induction on n we get our conclusion. □

The definition of irreducibility is the usual one. A representation ρ is called irreducible if the only closed ρ-invariant subspace of V is (0), or V itself.

The following important extension of Schur's Lemma (to arbitrary continuous representations on a Banach space) is proved in Warner, p. 239 of [4].

If ρ is an irreducible representation of a locally compact second countable group G on a *complex* Banach space V, then the algebra of intertwining operators for ρ consists of only scalar multiples of the identity. Since the center, $Z(G)$, acts by scalar multiples of the identity, if $\rho|_{Z(G)}$ is non-trivial, $Id - \rho(z_0)$ is invertible for some $z_0 \in Z(G)$. We therefore have the following corollary of Theorem 3.2.

Corollary 3.3. *Let ρ be an irreducible representation of a locally compact second countable group on a complex Banach space V with $\rho|_{Z(G)}$ non-trivial, then $H^n(G, V) = (0)$ for all $n > 0$.*

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