International Journal of Pure and Applied Mathematics
Volume 58 No. 3 2010, 315-332

OTHER EIGENVALUES OF THE M/M/1 RETRIAL QUEUEING MODEL WITH SPECIAL RETRIAL TIMES

Yusup Ismayil[1], Geni Gupur[2 §]
[1,2]College of Mathematics and Systems Science
Xinjiang University
Urumqi, 830046, P.R. CHINA
[2]e-mails: genigupur@yahoo.cn, geni@xju.edu.cn

Abstract: We prove that for all $\theta \in (0,1)$, $\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}\theta$ are eigenvalues of the operator corresponding to the M/M/1 retrial queueing model with special retrial times and show: it is impossible that the time-dependent solution of the model exponentially converges to its steady state solution.

AMS Subject Classification: 47A10
Key Words: the M/M/1 retrial queueing model with special retrial times, eigenvalue, geometric multiplicity

1. Introduction

According to Gomez-Corral [1], the M/M/1 retrial queueing model with special retrial times can be described by the following system of equations:

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \beta \int_0^\infty Q_0(x,t)dx, \tag{1}$$

$$\frac{\partial p_n}{\partial t} + \frac{\partial p_n}{\partial x} = -(\lambda+\alpha)p_n(x,t), \quad \forall n \geq 1, \tag{2}$$

$$\frac{\partial Q_0}{\partial t} + \frac{\partial Q_0}{\partial x} = -(\lambda+\beta)Q_0(x,t), \tag{3}$$

$$\frac{\partial Q_n}{\partial t} + \frac{\partial Q_n}{\partial x} = -(\lambda+\beta)Q_n(x,t) + \lambda Q_{n-1}(x,t), \quad \forall n \geq 1, \tag{4}$$

Received: December 15, 2009

§Correspondence author

$$p_n(0,t) = \beta \int_0^\infty Q_n(x,t)dx \quad \forall n \geq 1, \tag{5}$$

$$Q_0(0,t) = \lambda p_0(t) + \alpha \int_0^\infty p_1(x,t)dx, \tag{6}$$

$$Q_n(0,t) = \lambda \int_0^\infty p_n(x,t)dx + \alpha \int_0^\infty p_{n+1}(x,t)dx, \quad \forall n \geq 1, \tag{7}$$

$$p_0(0) = 1, \quad p_n(x,0) = 0, \quad \forall n \geq 1, \quad Q_j(x,0) = 0, \quad \forall j \geq 0. \tag{8}$$

Here $p_0(t)$ represents the probability that at time t there is no customer and the server is idle; $p_n(x,t)dx(n \geq 1)$ represents the probability that at time t the server is idle and there are n customers in the system with elapsed retrial time lying in $[x, x+dx)$; $Q_n(x,t)dx$ represents the probability that at time t the server is busy, there are n customers in the system with elapsed service time of the customer undergoing service lying in $[x, x+dx)$; λ is arrival rate of customers; α is repeated attempt rate; β is service rate.

In 1999, Gomez-Corral [1] established the M/M/1 retrial queueing model with special retrial times by using supplementary variable technique and obtained the existence of its steady-state solution by using probability generating function. In 2005, by using C_0-semigroup theory of linear operators Geni Gupur [2] has proved that the model has a unique positive time-dependent solution which satisfies probability condition. In 2005, Zhang et al [8] studied the resolvent set of the operator corresponding to the M/M/1 retrial queueing model with special retrial times and obtained all points on the imaginary axis except for 0 belong to the resolvent set of the operator. In 2006, Jiang at al [3, 4] proved that 0 is an eigenvalue of the operator with geometric and algebraic multiplicity one and 0 is an eigenvalue of its adjoint operator. In 2006, Zhang [7] gave the asymptotic behavior of the time-dependent solution of this model. In 2009, Lv et al [5] researched on eigenvalue of the operator on the left half complex plane and obtained that the operator has one eigenvalue on the left half complex plane. Any other results have not been found about this model until now. This paper is an effort on this subject. We further research on eigenvalues of the operator on the left half complex plane and obtain that it has uncountably infinite many eigenvalues on the left half complex plane, thus we show that the C_0-semigroup generated by the operator is not compact, even not eventually compact. Our result also implies: It is impossible that the time-dependent solution of the model exponentially converges to its steady state solution and it is impossible to write discrete type asymptotic expansion of the time-dependent solution, which is useful for engineers.

In this paper we use the notations in Gupur [2]. Choose the state space as follows.

$$X = \{g = (p, Q) \mid \| g \| = \| p \| + \| Q \| < \infty\},$$

where

$$p \in R \times L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots,$$
$$Q \in R \times L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots,$$
$$\|p\| = |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)}, \quad \|Q\| = \sum_{n=0}^{\infty} \|Q_n\|_{L^1[0,\infty)}.$$

It is obvious that X is a Banach space. We denote

$$\Gamma_0 = \begin{pmatrix} e^{-x} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta & 0 & 0 & \cdots \\ 0 & 0 & \beta & 0 & \cdots \\ 0 & 0 & 0 & \beta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$\Gamma_2 = \begin{pmatrix} \lambda e^{-x} & \alpha & 0 & 0 & 0 & \cdots \\ 0 & \lambda & \alpha & 0 & 0 & \cdots \\ 0 & 0 & \lambda & \alpha & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In the following we define operators and their domains

$$A(p,Q)(x) = \left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -\frac{d}{dx} & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \end{pmatrix} \right),$$

$$D(A) = \left\{ (p,Q) \in X \,\middle|\, \begin{array}{l} \frac{dp_i(x)}{dx} \in L^1[0,\infty)\ ,(i \geq 1), \frac{dQ_n(x)}{dx} \in L^1[0,\infty)\ (n \geq 1), \\ p_i(x) \text{ and } Q_n(x) \text{ are absolutely continuous functions and satisfy} \\ p(0) = \int_0^\infty \Gamma_0 p(x)dx + \int_0^\infty \Gamma_1 Q(x)dx,\ Q(0) = \int_0^\infty \Gamma_2 p(x)dx \end{array} \right\},$$

$$U(p,Q)(x)=\left(\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & -(\lambda+\alpha) & 0 & 0 & \cdots \\ 0 & 0 & -(\lambda+\alpha) & 0 & \cdots \\ 0 & 0 & 0 & -(\lambda+\alpha) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -(\lambda+\beta) & 0 & 0 & 0 & \cdots \\ \lambda & -(\lambda+\beta) & 0 & 0 & \cdots \\ 0 & \lambda & -(\lambda+\beta) & 0 & \cdots \\ 0 & 0 & \lambda & -(\lambda+\beta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix}\right),$$

$$E(p,Q)(x)=\begin{pmatrix} \int_0^\infty \beta Q_0(x)dx \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$$D(U)=X, \quad D(E)=X.$$

Then the above equations (1)–(8) can be written as an abstract Cauchy problem in the Banach space X

$$\frac{d(p,Q)(t)}{dt}=(A+U+E)(p,Q)(t), \quad \forall t\in(0,\infty), \tag{9}$$

$$(p,Q)(0)=(p(0),Q(0)), \tag{10}$$

$$p(0)=(1,0,0,\cdots), \quad Q(0)=(0,0,0,\cdots).$$

Geni Gupur [2] obtained the following results.

Theorem 1. *$A+U+E$ generates a positive contraction C_0-semigroup $T(t)$. $T(t)$ is isometric for the initial value. So the system (9)-(10) has a unique positive time-dependent solution which satisfies probability condition.*

Zhang et al [7, 8], Jiang et al [3, 4] got the following results.

Theorem 2. *If $\frac{\lambda(\alpha+\lambda)}{\alpha\beta}<1$, then 0 is an eigenvalue of $(A+U+E)$ and $(A+U+E)^*$ with geometric multiplicity one.*

$$\left\{\gamma\in C \;\middle|\; \begin{array}{l} \sup\left\{\frac{\lambda}{|\gamma+\lambda|},\frac{\lambda\alpha|\gamma+\lambda+\beta|(Re\gamma+\lambda+\alpha+\beta)}{(Re\gamma+\lambda+\alpha)(Re\gamma+\lambda+\beta)(|\gamma+\lambda+\alpha||\gamma+\lambda+\beta|-\alpha\beta)}\right. \\ \left.\quad+\frac{\lambda}{Re\gamma+\lambda+\alpha},\frac{\lambda|\gamma+\lambda+\alpha||\gamma+\lambda+\beta|(Re\gamma+\lambda+\alpha+\beta)}{(Re\gamma+\lambda+\alpha)(Re\gamma+\lambda+\beta)(|\gamma+\lambda+\alpha||\gamma+\lambda+\beta|-\alpha\beta)}\right\}<1 \end{array}\right\}$$

belongs to the resolvent set of $(A+U+E)^*$. Especially, all points on the imaginary axis except for 0 belong to the resolvent set of $A+U+E$. Therefore, *the time-dependent solution of the system (9)-(10) strongly converges to its steady-state solution.*

Lv etc [5] deduced the following result.

Theorem 3. *If* $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < \frac{1}{4}$*, then* $\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}$ *is an eigenvalue of* $(A+U+E)$ *with geometric multiplicity one.*

2. Main Results

Theorem 4. *If* $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < \frac{1}{4}$*, then for all* $\theta \in (0,1)$, $\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}\theta$ *are eigenvalues of* $(A+U+E)$ *with geometric multiplicity one.*

Proof. For simplicity, we take $\eta = \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}$. Consider the equation $(A+U+E)(p,Q) = \eta\theta(p,Q)$, which is equivalent to

$$-(\lambda+\eta\theta)p_0 + \beta\int_0^\infty Q_0(x)dx = 0, \tag{11}$$

$$-\frac{dp_n(x)}{dx} - (\lambda+\eta\theta+\alpha)p_n(x) = 0, \quad \forall n \geq 1, \tag{12}$$

$$-\frac{dQ_0(x)}{dx} - (\lambda+\eta\theta+\beta)Q_0(x) = 0, \tag{13}$$

$$-\frac{dQ_{n(x)}}{dx} - (\lambda+\eta\theta+\beta)Q_n(x) + \lambda Q_{n-1}(x) = 0, \quad \forall n \geq 1, \tag{14}$$

$$p_n(0) = \beta\int_0^\infty Q_n(x)dx, \quad \forall n \geq 1, \tag{15}$$

$$Q_0(0) = \lambda p_0(t) + \alpha\int_0^\infty p_1(x)dx, \tag{16}$$

$$Q_n(0) = \lambda\int_0^\infty p_n(x)dx + \alpha\int_0^\infty p_{n+1}(x)dx, \quad \forall n \geq 1. \tag{17}$$

Solving (12), (13) and (14) we have

$$p_n(x) = a_n e^{-(\lambda+\eta\theta+\alpha)x}, \quad \forall n \geq 1, \tag{18}$$

$$Q_0(x) = b_0 e^{-(\lambda+\eta\theta+\beta)x}, \tag{19}$$

$$Q_n(x) = b_n e^{-(\lambda+\eta\theta+\beta)x} + \lambda e^{-(\lambda+\eta\theta+\beta)x}\int_0^x e^{(\lambda+\eta\theta+\beta)\tau}Q_{n-1}(\tau)d\tau, \quad \forall n \geq 1. \tag{20}$$

By inserting (19) into (11) and using

$$\begin{aligned}\lambda+\beta+\eta\theta &= \lambda+\beta+\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}\theta \\ &= \frac{4(\lambda+\beta)-(2\lambda+\alpha+\beta)\theta+\theta\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4} \\ &> \frac{4(\lambda+\beta)-(2\lambda+\alpha+\beta)\theta+(\alpha+\beta)\theta}{4} \\ &= \frac{2\lambda+4\beta+2\lambda(1-\theta)}{4}>0\end{aligned}$$

we obtain

$$\begin{aligned}(\lambda+\eta\theta)p_0 &= \beta\int_0^\infty Q_0(x)dx=\beta\int_0^\infty b_0e^{-(\lambda+\eta\theta+\beta)x}dx \\ &= \frac{\beta}{\lambda+\eta\theta+\beta}b_0 \Rightarrow b_0=\frac{\lambda+\eta\theta+\beta}{\beta}(\lambda+\eta\theta)p_0. \qquad (21)\end{aligned}$$

From (18), (19), (21), (16) and noting

$$\begin{aligned}\lambda+\alpha+\eta\theta &= \lambda+\alpha+\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}\theta \\ &> \lambda+\alpha+\frac{-(2\lambda+\alpha+\beta)+(\alpha+\beta)}{4}\theta \\ &= \frac{2\lambda+4\alpha+2\lambda(1-\theta)}{4}>0\end{aligned}$$

it follows that

$$\begin{aligned}b_0 &= Q_0(0)=\alpha\int_0^\infty p_1(x)dx+\lambda p_0=\alpha\int_0^\infty a_1e^{-(\lambda+\eta\theta+\alpha)x}dx+\lambda p_0 \\ &= \frac{\alpha}{\lambda+\eta\theta+\alpha}a_1+\lambda p_0 \\ &\Rightarrow \\ \frac{\alpha}{\lambda+\eta\theta+\alpha}a_1 &= b_0-\lambda p_0=\frac{\lambda+\eta\theta+\beta}{\beta}(\lambda+\eta\theta)p_0-\lambda p_0 \\ &\Rightarrow \\ a_1 &= \frac{(\lambda+\eta\theta+\alpha)[(\lambda+\eta\theta)^2+\beta\eta\theta]}{\alpha\beta}p_0. \qquad (22)\end{aligned}$$

By combining (15), (18), (19) and (20) with (21), (22) and $\lambda+\eta\theta+\beta>0$ we deduce

$$a_1=p_1(0)=\beta\int_0^\infty Q_1(x)dx$$

$$
\begin{aligned}
&= \beta \int_0^\infty \left[b_1 e^{-(\lambda+\eta\theta+\beta)x} + \lambda e^{-(\lambda+\eta\theta+\beta)x} \int_0^x e^{(\lambda+\eta\theta+\beta)\tau} Q_0(\tau) d\tau \right] dx \\
&= \frac{\beta}{\lambda+\eta\theta+\beta} b_1 + \lambda\beta \int_0^\infty e^{-(\lambda+\eta\theta+\beta)x} \int_0^x e^{(\lambda+\eta\theta+\beta)\tau} Q_0(\tau) d\tau dx \\
&= \frac{\beta}{\lambda+\eta\theta+\beta} b_1 + \lambda\beta \int_0^\infty e^{-(\lambda+\eta\theta+\beta)x} \int_0^x e^{(\lambda+\eta\theta+\beta)\tau} b_0 e^{-(\lambda+\eta\theta+\beta)\tau} d\tau dx \\
&= \frac{\beta}{\lambda+\eta\theta+\beta} b_1 + \lambda\beta b_0 \int_0^\infty x e^{-(\lambda+\eta\theta+\beta)x} dx \\
&= \frac{\beta}{\lambda+\eta\theta+\beta} b_1 \\
&\quad + \lambda\beta b_0 \left\{ -\frac{1}{\lambda+\eta\theta+\beta} x e^{-(\lambda+\eta\theta+\beta)x} \Big|_0^\infty + \frac{1}{\lambda+\eta\theta+\beta} \int_0^\infty e^{-(\lambda+\eta\theta+\beta)x} dx \right\} \\
&= \frac{\beta}{\lambda+\eta\theta+\beta} b_1 + \frac{\lambda\beta}{(\lambda+\eta\theta+\beta)^2} b_0 \\
&= \frac{\beta}{\lambda+\eta\theta+\beta} b_1 + \frac{\lambda\beta}{(\lambda+\eta\theta+\beta)^2} \times \frac{\lambda+\eta\theta+\beta}{\beta} (\lambda+\eta\theta) p_0 \\
&= \frac{\beta}{\lambda+\eta\theta+\beta} b_1 + \frac{\lambda(\lambda+\eta\theta)}{(\lambda+\eta\theta+\beta)} p_0 \\
&\Rightarrow \\
&\frac{\beta}{\lambda+\eta\theta+\beta} b_1 = a_1 - \frac{\lambda(\lambda+\eta\theta)}{(\lambda+\eta\theta+\beta)} p_0 \\
&= \frac{(\lambda+\eta\theta+\alpha)[(\lambda+\eta\theta)^2 + \beta\eta\theta]}{\alpha\beta} p_0 - \frac{\lambda(\lambda+\eta\theta)}{(\lambda+\eta\theta+\beta)} p_0 \\
&= \frac{(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)[(\lambda+\eta\theta)^2 + \beta\eta\theta] - \alpha\beta\lambda(\lambda+\eta\theta)}{\alpha\beta(\lambda+\eta\theta+\beta)} p_0 \\
&\Rightarrow
\end{aligned}
$$

$$
b_1 = \frac{(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)[(\lambda+\eta\theta)^2 + \beta\eta\theta] - \alpha\beta\lambda(\lambda+\eta\theta)}{\alpha\beta^2} p_0. \tag{23}
$$

Similarly, by (17), (18), (20), (22), (23) and $\lambda+\eta\theta+\alpha > 0$ we have

$$
\begin{aligned}
b_1 = Q_1(0) &= \lambda \int_0^\infty p_1(x) dx + \alpha \int_0^\infty p_2(x) dx \\
&= \lambda \int_0^\infty a_1 e^{-(\lambda+\eta\theta+\alpha)x} dx + \alpha \int_0^\infty a_2 e^{-(\lambda+\eta\theta+\alpha)x} dx \\
&= \frac{\lambda}{\lambda+\eta\theta+\alpha} a_1 + \frac{\alpha}{\lambda+\eta\theta+\alpha} a_2 \\
&\Rightarrow
\end{aligned}
$$

$$\begin{aligned}
\frac{\alpha}{\lambda+\eta\theta+\alpha}a_2 &= b_1 - \frac{\lambda}{\lambda+\eta\theta+\alpha}a_1 \\
&= \frac{(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)[(\lambda+\eta\theta)^2+\beta\eta\theta]-(\lambda+\eta\theta)\lambda\alpha\beta}{\alpha\beta^2}p_0 \\
&\quad - \frac{\lambda}{\lambda+\eta\theta+\alpha}\times\frac{(\lambda+\eta\theta+\alpha)[(\lambda+\eta\theta)^2+\beta\eta\theta]}{\alpha\beta}p_0 \\
&= \Bigg[\frac{(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)[(\lambda+\eta\theta)^2+\beta\eta\theta]-(\lambda+\eta\theta)\lambda\alpha\beta}{\alpha\beta^2} \\
&\quad -\frac{\lambda\beta[(\lambda+\eta\theta)^2+\beta\eta\theta]}{\alpha\beta^2}\Bigg]p_0 \\
&\Rightarrow \\
a_2 &= \frac{\lambda+\eta\theta+\alpha}{(\alpha\beta)^2}\big\{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta][(\lambda+\eta\theta)^2+\beta\eta\theta] \\
&\quad -\alpha\beta\lambda(\lambda+\eta\theta)\big\}. \qquad (24)
\end{aligned}$$

By using (15), (16), (17), (18), (19), (20), the Fubini Theorem and $\lambda+\eta\theta+\alpha > 0,\ \lambda+\eta\theta+\beta > 0$ we calculate

$$\begin{aligned}
b_{n+1} &= Q_{n+1}(0) = \lambda\int_0^\infty p_{n+1}(x)dx + \alpha\int_0^\infty p_{n+2}(x)dx \\
&= \lambda\int_0^\infty a_{n+1}e^{-(\lambda+\eta\theta+\alpha)x}dx + \alpha\int_0^\infty a_{n+2}e^{-(\lambda+\eta\theta+\alpha)x}dx \\
&= \frac{\lambda}{\lambda+\eta\theta+\alpha}a_{n+1} + \frac{\alpha}{\lambda+\eta\theta+\alpha}a_{n+2}, \quad \forall n\ge 1, \qquad (25)
\end{aligned}$$

$$\begin{aligned}
a_{n+1} &= p_{n+1}(0) = \beta\int_0^\infty Q_{n+1}(x)dx \\
&= \beta\int_0^\infty\left[b_{n+1}e^{-(\lambda+\eta\theta+\beta)x} + \lambda e^{-(\lambda+\eta\theta+\beta)x}\int_0^x e^{(\lambda+\eta\theta+\beta)\tau}Q_n(\tau)d\tau\right]dx \\
&= \frac{\beta}{\lambda+\eta\theta+\beta}b_{n+1} + \lambda\beta\int_0^\infty e^{(\lambda+\eta\theta+\beta)\tau}Q_n(\tau)\int_\tau^\infty e^{-(\lambda+\eta\theta+\beta)x}dxd\tau \\
&= \frac{\beta}{\lambda+\eta\theta+\beta}b_{n+1} + \frac{\lambda\beta}{\lambda+\eta\theta+\beta}\int_0^\infty Q_n(\tau)d\tau \\
&= \frac{\beta}{\lambda+\eta\theta+\beta}b_{n+1} + \frac{\lambda}{\lambda+\eta\theta+\beta}p_n(0) \\
&= \frac{\beta}{\lambda+\eta\theta+\beta}b_{n+1} + \frac{\lambda}{\lambda+\eta\theta+\beta}a_n, \quad \forall n\ge 1. \qquad (26)
\end{aligned}$$

By inserting (25) into (26) and rearranging it is immediately got

$$a_{n+2} = \frac{(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta}{\alpha\beta}a_{n+1} - \frac{\lambda(\lambda+\eta\theta+\alpha)}{\alpha\beta}a_n, \ \forall n \geq 1. \quad (27)$$

If we set

$$s+t = \frac{(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta}{\alpha\beta}, \quad st = \frac{\lambda(\lambda+\eta\theta+\alpha)}{\alpha\beta}, \quad (s \neq t),$$

then it is easy to calculate

$$\begin{aligned} s,t =& \frac{1}{2\alpha\beta}\Big\{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta] \\ & \pm \sqrt{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta]^2 - 4\alpha\beta\lambda(\lambda+\eta\theta+\alpha)}\Big\}. \end{aligned} \quad (28)$$

By the definition of η and $\theta \in (0,1)$ we deduce

$$\begin{aligned} & 4\lambda^2 + 4\lambda\alpha > 0 \\ & \Rightarrow 4\lambda^2 + 4\lambda\alpha + 4\lambda\beta + (\alpha+\beta)^2 > 4\lambda\beta + (\alpha+\beta)^2 \\ & \Rightarrow (2\lambda+\alpha+\beta)^2 > 4\lambda\beta + (\alpha+\beta)^2 \\ & \Rightarrow -(2\lambda+\alpha+\beta) + \sqrt{(\alpha+\beta)^2+4\lambda\beta} < 0, \\ & \Rightarrow \frac{-(2\lambda+\alpha+\beta) + \sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4} < 0, \\ & \Rightarrow \eta < 0, \ \eta\theta < 0, \ 0 < \lambda+\eta\theta < \lambda. \end{aligned} \quad (29)$$

From which together with $\theta \in (0,1)$ and $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < \frac{1}{4} \Rightarrow \lambda(\alpha+\lambda) < \frac{1}{4}\alpha\beta$ we know

$$\begin{aligned} & \frac{-(2\lambda+\alpha+\beta) + \sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2} < \eta\theta < 0 \\ & \Rightarrow \frac{-(\alpha+\beta) + \sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2} < \lambda+\eta\theta \\ & \Rightarrow 2(\lambda+\eta\theta) + \alpha + \beta > \sqrt{(\alpha+\beta)^2+4\lambda\beta} \\ & \Rightarrow 4(\lambda+\eta\theta)^2 + 4(\lambda+\eta\theta)(\alpha+\beta) + (\alpha+\beta)^2 > (\alpha+\beta)^2 + 4\lambda\beta \\ & \Rightarrow (\lambda+\eta\theta)^2 + (\lambda+\eta\theta)(\alpha+\beta) > \lambda\beta \\ & \Rightarrow (\lambda+\eta\theta)^2 + (\lambda+\eta\theta)(\alpha+\beta) - \lambda\beta + \alpha\beta > \alpha\beta \\ & \Rightarrow [(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta] > \alpha\beta > 0 \\ & \Rightarrow \end{aligned}$$

$$[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta]^2 - 4\alpha\beta\lambda(\lambda+\eta\theta+\alpha) > (\alpha\beta)^2 - 4\alpha\beta\frac{1}{4}\alpha\beta = 0,$$

$$[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta] > \sqrt{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\lambda+\eta\theta+\alpha)}. \tag{30}$$

From (29) and $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < \frac{1}{4}$ it follows that

$$\begin{aligned}
&[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta] \\
&\quad + \sqrt{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\lambda+\eta\theta+\alpha)} \\
&< [(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-(\lambda+\eta\theta)\beta] \\
&\quad + \sqrt{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-(\lambda+\eta\theta)\beta]^2-4\alpha\beta(\lambda+\eta\theta)(\lambda+\eta\theta+\alpha)} \\
&= [(\lambda+\eta\theta)(\lambda+\eta\theta+\alpha)+\alpha\beta] \\
&\quad + \sqrt{[(\lambda+\eta\theta)(\lambda+\eta\theta+\alpha)+\alpha\beta]^2-4\alpha\beta(\lambda+\eta\theta)(\lambda+\eta\theta+\alpha)} \\
&= [(\lambda+\eta\theta)(\lambda+\eta\theta+\alpha)+\alpha\beta] + \sqrt{[\alpha\beta-(\lambda+\eta\theta)(\lambda+\eta\theta+\alpha)]^2} \\
&= 2\alpha\beta.
\end{aligned} \tag{31}$$

$$\begin{aligned}
0 &< [(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta] \\
&\quad - \sqrt{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\lambda+\eta\theta+\alpha)} \\
&< [(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta] \\
&\quad + \sqrt{[(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\lambda+\eta\theta+\alpha)} \\
&< 2\alpha\beta.
\end{aligned} \tag{32}$$

(30), (31) and (32) give

$$0 < s < 1, \quad 0 < t < 1. \tag{33}$$

Comparing (27) with (28) we find

$$\begin{aligned}
a_{n+2} &= \frac{(\lambda+\eta\theta+\alpha)(\lambda+\eta\theta+\beta)-\lambda\beta}{\alpha\beta}a_{n+1} - \frac{\lambda(\lambda+\eta\theta+\alpha)}{\alpha\beta}a_n \\
&= (s+t)a_{n+1} - sta_n \\
&\Rightarrow \\
a_{n+2} - sa_{n+1} &= t(a_{n+1}-sa_n) = t^2(a_n - sa_{n-1}) = t^3(a_{n-1}-sa_{n-2}) \\
&= \cdots = t^n(a_2 - sa_1), \quad n \geq 1. \\
&\Rightarrow \\
a_{n+2} - sa_{n+1} &= t^n(a_2-sa_1), \\
sa_{n+1} - s^2a_n &= st^{n-1}(a_2-sa_1), \\
s^2a_n - s^3a_{n-1} &= s^2t^{n-2}(a_2-sa_1), \\
&\cdots \\
s^{n-3}a_5 - s^{n-2}a_4 &= s^{n-3}t^3(a_2-sa_1),
\end{aligned}$$

$$\begin{aligned} s^{n-2}a_4 - s^{n-1}a_3 &= s^{n-2}t^2(a_2 - sa_1), \\ s^{n-1}a_3 - s^n a_2 &= s^{n-1}t(a_2 - sa_1). \end{aligned}$$

By adding all the above equations we have

$$\begin{aligned} &(a_{n+2} - sa_{n+1}) + (sa_{n+1} - s^2 a_n) + (s^2 a_n - s^3 a_{n-1}) + \cdots \\ &\quad + (s^{n-3}a_5 - s^{n-2}a_4) + (s^{n-2}a_4 - s^{n-1}a_3) + (s^{n-1}a_3 - s^n a_2) \\ &= t^n(a_2 - sa_1) + st^{n-1}(a_2 - sa_1) + s^2 t^{n-2}(a_2 - sa_1) + \cdots \\ &\quad + s^{n-3}t^3(a_2 - sa_1) + s^{n-2}t^2(a_2 - sa_1) + s^{n-1}t(a_2 - sa_1) \\ &\Rightarrow \end{aligned}$$

$$\begin{aligned} a_{n+2} - s^n a_2 &= (t^n + st^{n-1} + s^2 t^{n-2} + \cdots + s^{n-3}t^3 + s^{n-2}t^2 + s^{n-1}t)(a_2 - sa_1) \\ &= \left(\frac{s^{n+1} - t^{n+1}}{s - t} - s^n \right)(a_2 - sa_1) \\ &\Rightarrow \end{aligned}$$

$$a_{n+2} = s^{n+1}a_1 + \frac{s^{n+1} - t^{n+1}}{s - t}(a_2 - sa_1), \quad \forall n \geq 1. \tag{34}$$

Which implies

$$a_{n+1} = s^n a_1 + \frac{s^n - t^n}{s - t}(a_2 - sa_1), \tag{35}$$

$$a_n = s^{n-1}a_1 + \frac{s^{n-1} - t^{n-1}}{s - t}(a_2 - sa_1). \tag{36}$$

Combining (25) with (35) and (36) we determine

$$\begin{aligned} b_n &= \frac{\lambda}{\lambda + \eta\theta + \alpha} a_n + \frac{\alpha}{\lambda + \eta\theta + \alpha} a_{n+1} \\ &= \frac{\lambda}{\lambda + \eta\theta + \alpha} \left[s^{n-1}a_1 + \frac{s^{n-1} - t^{n-1}}{s - t}(a_2 - sa_1) \right] \\ &\quad + \frac{\alpha}{\lambda + \eta\theta + \alpha} \left[s^n a_1 + \frac{s^n - t^n}{s - t}(a_2 - sa_1) \right] \\ &= \frac{\lambda s^{n-1}}{\lambda + \eta\theta + \alpha} a_1 + \frac{\alpha s^n}{\lambda + \eta\theta + \alpha} a_1 \\ &\quad + \frac{\lambda}{\lambda + \eta\theta + \alpha} \times \frac{s^{n-1} - t^{n-1}}{s - t}(a_2 - sa_1) + \frac{\alpha}{\lambda + \eta\theta + \alpha} \times \frac{s^n - t^n}{s - t}(a_2 - sa_1) \\ &= \frac{s^{n-1}(\lambda + \alpha s)}{\lambda + \eta\theta + \alpha} a_1 + \frac{s^{n-1}(\lambda + \alpha s) - t^{n-1}(\lambda + \alpha t)}{(\lambda + \eta\theta + \alpha)(s - t)}(a_2 - sa_1), \quad \forall n \geq 1. \end{aligned} \tag{37}$$

From which together with (18), (19), (20), (21), (22), (24), (36), (37) and (33) and noting $\lambda+\eta\theta+\alpha>0, \lambda+\eta\theta+\beta>0$ we estimate

$$\begin{aligned}
\|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} = |p_0| + \sum_{n=1}^{\infty} \int_0^{\infty} |p_n(x)| dx \\
&= |p_0| + \sum_{n=1}^{\infty} \int_0^{\infty} |a_n e^{-(\lambda+\eta\theta+\alpha)x}| dx = |p_0| + \frac{1}{\lambda+\eta\theta+\alpha} \sum_{n=1}^{\infty} |a_n| \\
&= |p_0| + \frac{1}{\lambda+\eta\theta+\alpha} \left(|a_1| + \sum_{n=1}^{\infty} |a_{n+1}| \right) \\
&= |p_0| + \frac{1}{\lambda+\eta\theta+\alpha} \left[|a_1| + \sum_{n=1}^{\infty} \left| s^n a_1 + \frac{s^n - t^n}{s-t} (a_2 - s a_1) \right| \right] \\
&\le |p_0| + \frac{1}{\lambda+\eta\theta+\alpha} \left(|a_1| + \sum_{n=1}^{\infty} |s^n a_1| + |(a_2 - s a_1)| \sum_{n=1}^{\infty} \left| \frac{s^n - t^n}{s-t} \right| \right) \\
&\le |p_0| + \frac{1}{\lambda+\eta\theta+\alpha} \\
&\times \left(|a_1| + |a_1| \sum_{n=1}^{\infty} s^n + \frac{|(a_2 - s a_1)|}{|s-t|} \sum_{n=1}^{\infty} s^n + \frac{|(a_2 - s a_1)|}{|s-t|} \sum_{n=1}^{\infty} t^n \right) \\
&= |p_0| + \frac{1}{\lambda+\eta\theta+\alpha} \\
&\times \left(\frac{1}{1-s} |a_1| + \frac{|(a_2 - s a_1)|}{|s-t|} \times \frac{s}{1-s} + \frac{|(a_2 - s a_1)|}{|s-t|} \times \frac{t}{1-t} \right) \\
&< \infty.
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\|Q_n\|_{L^1[0,\infty)} = \int_0^{\infty} |Q_n(x)| dx \\
&\le \int_0^{\infty} |b_n| e^{-(\lambda+\eta\theta+\beta)x} dx + \lambda \int_0^{\infty} e^{-(\lambda+\eta\theta+\beta)x} \int_0^x e^{(\lambda+\eta\theta+\beta)\tau} |Q_{n-1}(\tau)| d\tau dx \\
&= \frac{1}{\lambda+\eta\theta+\beta} |b_n| + \lambda \int_0^{\infty} |Q_{n-1}(\tau)| e^{(\lambda+\eta\theta+\beta)\tau} \int_{\tau}^{\infty} e^{-(\lambda+\eta\theta+\beta)x} dx d\tau \\
&= \frac{1}{\lambda+\eta\theta+\beta} |b_n| + \frac{\lambda}{\lambda+\eta\theta+\beta} \int_0^{\infty} |Q_{n-1}(\tau)| d\tau \\
&\le \frac{1}{\lambda+\eta\theta+\beta} |b_n| + \frac{\lambda}{\lambda+\eta\theta+\beta} \int_0^{\infty} |b_{n-1}| e^{-(\lambda+\eta\theta+\beta)\tau} d\tau \\
&\quad + \frac{\lambda^2}{\lambda+\eta\theta+\beta} \int_0^{\infty} e^{-(\lambda+\eta\theta+\beta)\tau} \int_0^{\tau} |Q_{n-2}(\rho)| e^{(\lambda+\eta\theta+\beta)\rho} d\rho d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda+\eta\theta+\beta}|b_n| + \frac{\lambda}{(\lambda+\eta\theta+\beta)^2}|b_{n-1}| \\
&\quad + \frac{\lambda^2}{\lambda+\eta\theta+\beta}\int_0^\infty |Q_{n-2}(\rho)| e^{(\lambda+\eta\theta+\beta)\rho}\int_\rho^\infty e^{-(\lambda+\eta\theta+\beta)\tau} d\tau d\rho \\
&= \frac{1}{\lambda+\eta\theta+\beta}|b_n| + \frac{\lambda}{(\lambda+\eta\theta+\beta)^2}|b_{n-1}| + \frac{\lambda^2}{(\lambda+\eta\theta+\beta)^2}\int_0^\infty |Q_{n-2}(\rho)| d\rho \\
&\leq \frac{1}{\lambda+\eta\theta+\beta}|b_n| + \frac{\lambda}{(\lambda+\eta\theta+\beta)^2}|b_{n-1}| + \frac{\lambda^2}{(\lambda+\eta\theta+\beta)^3}|b_{n-2}| \\
&\quad + \frac{\lambda^3}{(\lambda+\eta\theta+\beta)^2}\int_0^\infty e^{-(\lambda+\eta\theta+\beta)\rho}\int_0^\rho |Q_{n-3}(\tau)| e^{(\lambda+\eta\theta+\beta)\tau} d\tau d\rho \\
&\leq \cdots \\
&\leq \frac{1}{\lambda+\eta\theta+\beta}|b_n| + \frac{\lambda}{(\lambda+\eta\theta+\beta)^2}|b_{n-1}| + \frac{\lambda^2}{(\lambda+\eta\theta+\beta)^3}|b_{n-2}| + \cdots \\
&\quad + \frac{\lambda^{n-2}}{(\lambda+\eta\theta+\beta)^{n-1}}|b_2| \\
&\quad + \frac{\lambda^{n-1}}{(\lambda+\eta\theta+\beta)^{n-2}}\int_0^\infty e^{-(\lambda+\eta\theta+\beta)x}\int_0^x |Q_1(\rho)| e^{(\lambda+\eta\theta+\beta)\rho} d\rho dx \\
&= \frac{1}{\lambda+\eta\theta+\beta}|b_n| + \frac{\lambda}{(\lambda+\eta\theta+\beta)^2}|b_{n-1}| + \frac{\lambda^2}{(\lambda+\eta\theta+\beta)^3}|b_{n-2}| + \cdots \\
&\quad + \frac{\lambda^{n-2}}{(\lambda+\eta\theta+\beta)^{n-1}}|b_2| \\
&\quad + \frac{\lambda^{n-1}}{(\lambda+\eta\theta+\beta)^{n-1}}\int_0^\infty |Q_1(\rho)| d\rho \\
&= \frac{1}{\lambda+\eta\theta+\beta}|b_n| + \frac{\lambda}{(\lambda+\eta\theta+\beta)^2}|b_{n-1}| + \frac{\lambda^2}{(\lambda+\eta\theta+\beta)^3}|b_{n-2}| + \cdots \\
&\quad + \frac{\lambda^{n-2}}{(\lambda+\eta\theta+\beta)^{n-1}}|b_2| \\
&\quad + \frac{\lambda^{n-1}}{(\lambda+\eta\theta+\beta)^{n}}|b_1| \\
&\quad + \frac{\lambda^{n}}{(\lambda+\eta\theta+\beta)^{n}}\int_0^\infty |Q_0(\rho)| d\rho \\
&= \frac{1}{\lambda+\eta\theta+\beta}|b_n| + \frac{\lambda}{(\lambda+\eta\theta+\beta)^2}|b_{n-1}| + \cdots + \frac{\lambda^{n-1}}{(\lambda+\eta\theta+\beta)^{n}}|b_1| \\
&\quad + \frac{\lambda^{n}}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0|
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{\lambda^{k-1}}{(\lambda+\eta\theta+\beta)^k}|b_{n-k+1}| + \frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0| \\
&= \sum_{k=1}^{n} \frac{\lambda^{k-1}}{(\lambda+\eta\theta+\beta)^k}\Big|\frac{s^{n-k}(\lambda+\alpha s)}{\lambda+\eta\theta+\alpha}a_1 + \frac{s^{n-k}(\lambda+\alpha s)-t^{n-k}(\lambda+\alpha t)}{(\lambda+\eta\theta+\alpha)(s-t)}(a_2-sa_1)\Big| \\
&\quad + \frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0| \\
&\le \sum_{k=1}^{n} \frac{\lambda^{k-1}s^{n-k}}{(\lambda+\eta\theta+\beta)^k}\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big||a_1| \\
&\quad + \sum_{k=1}^{n} \frac{\lambda^{k-1}s^{n-k}}{(\lambda+\eta\theta+\beta)^k}\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big| \\
&\quad + \sum_{k=1}^{n} \frac{\lambda^{k-1}t^{n-k}}{(\lambda+\eta\theta+\beta)^k}\Big|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big| + \frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0| \\
&= s^n\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big||a_1|\sum_{k=1}^{n}\frac{\lambda^{k-1}}{[s(\lambda+\eta\theta+\beta)]^k} + s^n\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big| \\
&\quad \times \sum_{k=1}^{n}\frac{\lambda^{k-1}}{[s(\lambda+\eta\theta+\beta)]^k} \\
&\quad + t^n\Big|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big|\sum_{k=1}^{n}\frac{\lambda^{k-1}}{[t(\lambda+\eta\theta+\beta)]^k} + \frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0| \\
&= s^n\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big||a_1|\frac{\frac{1}{s(\lambda+\eta\theta+\beta)}\left[1-\left(\frac{\lambda}{s(\lambda+\eta\theta+\beta)}\right)^n\right]}{1-\frac{\lambda}{s(\lambda+\eta\theta+\beta)}} \\
&\quad + s^n\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big|\frac{\frac{1}{s(\lambda+\eta\theta+\beta)}\left[1-\left(\frac{\lambda}{s(\lambda+\eta\theta+\beta)}\right)^n\right]}{1-\frac{\lambda}{s(\lambda+\eta\theta+\beta)}} \\
&\quad + t^n\Big|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big|\frac{\frac{1}{t(\lambda+\eta\theta+\beta)}\left[1-\left(\frac{\lambda}{t(\lambda+\eta\theta+\beta)}\right)^n\right]}{1-\frac{\lambda}{t(\lambda+\eta\theta+\beta)}} \\
&\quad + \frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0| \\
&= \Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\frac{|a_1|}{s(\lambda+\eta\theta+\beta)-\lambda}\left[s^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right] \\
&\quad + \Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big|\frac{1}{s(\lambda+\eta\theta+\beta)-\lambda}\left[s^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right]
\end{aligned}
$$

$$+\Big|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big|\frac{1}{t(\lambda+\eta\theta+\beta)-\lambda}\left[t^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right]$$
$$+\frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0|. \tag{39}$$

If we denote

$$M_n=\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\frac{|a_1|}{s(\lambda+\eta\theta+\beta)-\lambda}\left[s^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right],$$
$$N_n=\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big|\frac{1}{s(\lambda+\eta\theta+\beta)-\lambda}\left[s^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right],$$
$$L_n=\Big|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\Big|\Big|\frac{a_2-sa_1}{s-t}\Big|\frac{1}{t(\lambda+\eta\theta+\beta)-\lambda}\left[t^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right],$$

then (39) is equivalent to

$$\|Q_n\|_{L^1[0,\infty)}\le M_n+N_n+L_n+\frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0|,\quad \forall n\ge 1 \tag{40}$$

$$\Rightarrow$$

$$\|Q\|=\sum_{n=0}^{\infty}\|Q_n\|_{L^1[0,\infty)}=\|Q_0\|+\sum_{n=1}^{\infty}\|Q_n\|_{L^1[0,\infty)}$$
$$\le\int_0^\infty|b_0|e^{-(\lambda+\eta\theta+\beta)x}dx+\sum_{n=1}^{\infty}\left(M_n+N_n+L_n+\frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0|\right). \tag{41}$$

The condition $\frac{\lambda(\lambda+\alpha)}{\alpha\beta}<\frac{1}{4}<1$ implies $0<\lambda^2<\alpha(\beta-\lambda)$, thus $\beta>\lambda$ and

$$\begin{aligned}\beta+\eta\theta&=\beta+\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}\theta\\&>\beta+\frac{-\theta(2\lambda+\alpha+\beta)+\theta(\alpha+\beta)}{4}\\&=\frac{2\beta+2(\beta-\lambda\theta)}{4}>\frac{2\beta+2(\lambda-\lambda\theta)}{4}>0\\&\Rightarrow 0<\frac{\lambda}{\lambda+\eta\theta+\beta}<1.\end{aligned}$$

From which together with (33) we estimate

$$\sum_{n=1}^{\infty}M_n=\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\frac{|a_1|}{s(\lambda+\eta\theta+\beta)-\lambda}\sum_{n=1}^{\infty}\left[s^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right]$$
$$=\Big|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\Big|\frac{|a_1|}{s(\lambda+\eta\theta+\beta)-\lambda}\sum_{n=1}^{\infty}s^n$$

$$
\begin{aligned}
&-\left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\frac{|a_1|}{s(\lambda+\eta\theta+\beta)-\lambda}\sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\\
&=\left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\frac{|a_1|}{s(\lambda+\eta\theta+\beta)-\lambda}\left(\frac{s}{1-s}\right)\\
&\quad-\left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\frac{|a_1|}{s(\lambda+\eta\theta+\beta)-\lambda}\left(\frac{\lambda}{\eta\theta+\beta}\right)\\
&<\infty. \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}N_n &= \left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{s(\lambda+\eta\theta+\beta)-\lambda}\\
&\quad\times\sum_{n=1}^{\infty}\left[s^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right]\\
&=\left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{s(\lambda+\eta\theta+\beta)-\lambda}\sum_{n=1}^{\infty}s^n\\
&\quad-\left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{s(\lambda+\eta\theta+\beta)-\lambda}\sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\\
&=\left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{s(\lambda+\eta\theta+\beta)-\lambda}\left(\frac{s}{1-s}\right)\\
&\quad-\left|\frac{\lambda+\alpha s}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{s(\lambda+\eta\theta+\beta)-\lambda}\\
&\quad\times\left(\frac{\lambda}{\eta\theta+\beta}\right)<\infty. \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}L_n &= \left|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{t(\lambda+\eta\theta+\beta)-\lambda}\\
&\quad\times\sum_{n=1}^{\infty}\left[t^n-\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\right]\\
&=\left|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{t(\lambda+\eta\theta+\beta)-\lambda}\sum_{n=1}^{\infty}t^n\\
&\quad-\left|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{t(\lambda+\eta\theta+\beta)-\lambda}\sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda+\eta\theta+\beta}\right)^n\\
&=\left|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2-sa_1}{s-t}\right|\frac{1}{t(\lambda+\eta\theta+\beta)-\lambda}\left(\frac{t}{1-t}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \left|\frac{\lambda+\alpha t}{\lambda+\eta\theta+\alpha}\right|\left|\frac{a_2 - s a_1}{s-t}\right|\frac{1}{t(\lambda+\eta\theta+\beta)-\lambda} \\
&\times\left(\frac{\lambda}{\eta\theta+\beta}\right) < \infty. \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0| = \frac{|b_0|}{\lambda+\eta\theta+\beta}\times\left(\frac{\frac{\lambda}{\lambda+\eta\theta+\beta}}{1-\frac{\lambda}{\lambda+\eta\theta+\beta}}\right) \\
&= \frac{1}{\lambda+\eta\theta+\beta}\times\frac{\lambda+\eta\theta+\beta}{\beta}(\lambda+\eta\theta)|p_0|\times\left(\frac{\lambda}{(\lambda+\eta\theta+\beta)-\lambda}\right) \\
&= \frac{\lambda(\lambda+\eta\theta)}{\beta(\beta+\eta\theta)}|p_0| < \infty. \qquad (45)
\end{aligned}
$$

Combining (41) with (42), (43), (44) and (45) we obtain

$$
\begin{aligned}
\|Q\| &= \sum_{n=0}^{\infty}\|Q_n\|_{L^1[0,\infty)} = \|Q_0\| + \sum_{n=1}^{\infty}\|Q_n\|_{L^1[0,\infty)} \\
&\le \frac{1}{\lambda+\eta\theta+\beta}|b_0| + \sum_{n=1}^{\infty}\left(M_n+N_n+L_n+\frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0|\right) \\
&= \frac{1}{\lambda+\eta\theta+\beta}|b_0| + \sum_{n=1}^{\infty}M_n + \sum_{n=1}^{\infty}N_n + \sum_{n=1}^{\infty}L_n + \sum_{n=1}^{\infty}\frac{\lambda^n}{(\lambda+\eta\theta+\beta)^{n+1}}|b_0| \\
&< \infty. \qquad (46)
\end{aligned}
$$

From (38) and (46) we deduce the desired result

$$
\|(p,Q)\| = \|p\| + \|Q\| < \infty. \qquad (47)
$$

Which shows that for all $\theta \in (0,1)$, $\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}\theta$ are eigenvalues of $A+U+E$. Moreover, from (36), (37), (18), (19), (20), (21), (22), (23) and (24) we know that geometric multiplicity of them is one. □

Theorem 4 shows that $A+U+E$ has uncountable infinite many eigenvalues on the left half complex plane, therefore $T(t)$ is not compact, even not eventually compact. In addition, by Nagel [6] Theorem 4 implies "It is impossible that the time-dependent solution of the system (9)-(10) exponentially converges to its steady-state solution. It is impossible to write (discrete type) asymptotic expansion of the time-dependent solution of the system (9)-(10)." Which provide theoretical evidence for engineers.

There are some differences between the condition of Theorem 2 and the condition of Theorem 4, that it to say, the condition of Theorem 4 is stronger than the condition of Theorem 2. So, our next work is to research on the result of Theorem 4 under the condition $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < 1$.

Acknowledgments

The authors were supported by the Natural Science Foundation of China (No: 10861011).

References

[1] A. Gomez-Corral, Stochastic analysis of a single server queue with general retrial times, *Naval Research Logistics*, **46** (1999), 561-581.

[2] Geni Gupur, Semigroup method for M/G/1 retrial queue with general retial times, *International Journal of Pure and Applied Mathematics*, **18** (2005), 405-429.

[3] Jiang Mei, *An Eigenvalue of M/M/1 Retrial Queuing Model with Special Retrial Times and Its Application*, MS Thesis of Xinjiang University, Urumqi (2006)

[4] Jiang Mei, Geni Gupur, An eigenvalue of M/M/1 retrial queueing model with special retrial times, *Acta Analysis Functionalis Applicata*, **9** (2007), 193-203.

[5] Lv Yicheng, Geni Gupur, Another eigenvalue of the M/M/1 retrial queueing model with special retrial times, *Acta Analysis Functionalis Applicata*, **11** (2009), 123-133.

[6] Rainer Nagel, *One-Parameter Semigroups of Positive Operators*, LNM 1184, Springer, Berlin, (1986).

[7] Zhang Furong, *Asymptotic Property of the Time-Dependent Solution of M/M/1 Retrial Queuing Model with Special Retrial Times*, MS Thesis of Xinjiang University, Urumqi (2006).

[8] Zhang Furong, Geni Gupur, The resolvent set of M/M/1 retrial queueing model, *International Journal of Pure and Applied Mathematics*, **20** (2005), 449-469.