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GLOBAL EXISTENCE AND LARGE TIME BEHAVIOR OF SOLUTIONS FOR PARABOLIC SYSTEMS WITH NONLINEAR GRADIENTS TERMS

Abeer AL-Elaiw[1], Slim Tayachi[2 §]

[1]Department of Mathematics
Faculty of Girls Education
King Faisal University
P.O. Box 789, Al-Hofuf, 31982, KINGDOM OF SAUDI ARABIA
e-mail: a.alelaiw@gmail.com

[2]Department of Mathematics
Faculty of Science
University "El-Manar" of Tunis
Tunis, 1060, TUNISIA
e-mail: slim.tayachi@fst.rnu.tn

Abstract: In this paper we study the global existence of mild solutions for the nonlinear parabolic system: $\partial_t u = \Delta u + a|\nabla v|^p$, $\partial_t v = \Delta v + b|\nabla u|^q$, $t > 0$, $x \in \mathbb{R}^N$, where $a,\ b \in \mathbb{R}$, $N \geq 1$ and $1 < p \leq q < 2$. Under the condition $pq > \frac{q}{N+1} + \frac{N+2}{N+1}$ and suitable smallness conditions on the initial values we prove the existence of global solutions. We study also the large time behavior for some of these global solutions. We prove that if the initial values $u(0,x) \sim \omega_1\,(x/|x|)\,|x|^{-\frac{2-p(q-1)}{pq-1}}$, $v(0,x) \sim \omega_2\,(x/|x|)\,|x|^{-\frac{2-q(p-1)}{pq-1}}$ as $|x| \to \infty$ and under suitable conditions on ω_1, ω_2 the resulting solutions are asymptotically self-similar. The asymptotic behavior is established in the $W^{1,\infty}$–norm and is stable under some small perturbations.

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§Correspondence author

1. Introduction

In this paper we study the global existence and the asymptotic behavior of solutions for the following parabolic system with nonlinear gradients terms:

$$\begin{cases} \partial_t u = \Delta u + a|\nabla v|^p, \\ \partial_t v = \Delta v + b|\nabla u|^q, \end{cases} \tag{1.1}$$

with initial value

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x), \tag{1.2}$$

where $u = u(t,x)$, $v = v(t,x)$ are real valued functions, $t > 0$, $x \in \mathbb{R}^N$, a, $b \in \mathbb{R}$. N is a positive integer and p and q are two real numbers such that

$$1 < p \leq q < 2 \tag{1.3}$$

and

$$pq > \frac{q}{N+1} + \frac{N+2}{N+1}. \tag{1.4}$$

The initial value $\Phi = (\varphi_1, \varphi_2)$ is small with respect to the norm $\mathcal{N}$ defined below by (1.9). In particular, we consider φ_1, $\varphi_2 \in C_0(\mathbb{R}^n)$ such that $\varphi_1(x) \sim c|x|^{-\alpha}$ and $\varphi_2(x) \sim c|x|^{-\beta}$ as $|x| \to \infty$, in some appropriate sense, $|c|$ is a small constant. We define the real numbers α and β by:

$$\alpha = \frac{2 - p(q-1)}{pq - 1} \tag{1.5}$$

and

$$\beta = \frac{2 - q(p-1)}{pq - 1}. \tag{1.6}$$

It follows by (1.3)–(1.4) that $0 < \alpha < N$, $0 < \beta < N$. Also, these φ_1, φ_2 are in $L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and in general are not in $L^1(\mathbb{R}^n)$.

The system (1.1) was introduced in [1]. In [1, 10] the authors studied the system (1.1) with a, $b < 0$ and with nonnegative initial values. In particular, they have studied the decay of the L^1–norm for nonnegative solutions under some appropriate conditions on the initial values. In [2] the global existence of solutions to a semilinear parabolic system similar to (1.1) was studied, but under conditions on initial values different from those of the present paper. In particular, it is supposed in [2] that initial values are small in $W^{1,\infty}(\mathbb{R}^N)$. In the present paper initial values are supposed to be small in some Besov spaces with negative order, see Theorem 3.1 and Remark 3.3 below. To our knowledge, there is no previous study of the existence of self-similar solutions and asymptotically self-similar solutions for the system (1.1). The aim of this paper is to fill this gap and to study the existence of asymptotically self-similar solution for the system (1.1)–(1.2) with neither a restriction on the sign of the initial data nor on a or b.

A recent method for studying the existence of self-similar solutions of the Navier-Stokes system was introduced by the papers [3, 4, 9]. Later, the ideas of these papers have been developed and applied to the semilinear heat equation [5, 13, 21], to the nonlinear heat equation with a nonlinear gradient term [19, 20], to the nonlinear Schrödinger equation [5, 6, 7, 8, 12, 14, 22], to the nonlinear wave equation [11, 15], to the damped wave equation [18] and to semilinear parabolic systems, but without nonlinear gradients terms [16, 17]. The new feature of the present paper is to develop these ideas to a parabolic system with nonlinear gradients terms.

The method introduced by [3, 4, 9] uses the integral formulation of the semilinear equation or system under study. The first step is to prove the existence and uniqueness of global solutions for the integral formulation of the problem in some functional spaces related to the problem under study, and allowing to use a contraction mapping argument on the whole time interval $(0, \infty)$ directly. The norm used to measure the size of the initial data is invariant under the dilations which leave the problem unchanged and weak enough so that homogenous functions have finite norm. The existence of self-similar solutions results then from the uniqueness and scaling properties of the equations, systems and norms are obtained. The asymptotic behavior for more general solutions is established using continuous dependance on the initial data with respect to a more refined norm. This method relies neither on the maximum principle nor on the positivity of solutions.We refer the reader to the introduction of [5] and to that of [21] for a more historical account of this method.

In this paper we prove global existence of mild solutions for the initial value problem (1.1)–(1.2). That is we prove global existence of solutions for the integral system associated to (1.1)–(1.2) which is given by

$$u(t) = e^{t\Delta}\varphi_1 + a\int_0^t e^{(t-\sigma)\Delta}\left(|\nabla v(\sigma)|^p\right)\mathrm{d}\sigma, \tag{1.7}$$

$$v(t) = e^{t\Delta}\varphi_2 + b\int_0^t e^{(t-\sigma)\Delta}\left(|\nabla u(\sigma)|^q\right)\mathrm{d}\sigma. \tag{1.8}$$

(see Theorems 3.1–3.4 below). In particular, we show that the global existence and the continuous dependence for the system (1.7)–(1.8) occur for small initial value $\Phi = (\varphi_1, \varphi_2)$ with respect to the norm $\mathcal{N}$ defined by

$\mathcal{N}(\Phi) :=$

$$\sup_{t>0}\left[t^{\alpha_1}||e^{t\Delta}\varphi_1||_r, t^{\alpha_1+1/2}||\nabla e^{t\Delta}\varphi_1||_r, t^{\beta_1}||e^{t\Delta}\varphi_2||_s, t^{\beta_1+1/2}||\nabla e^{t\Delta}\varphi_2||_s\right]. \quad (1.9)$$

Here and in the rest of the paper, $||\cdot||_m$ denotes the norm in $L^m\left(\mathbb{R}^N\right)$, $e^{t\Delta}$ is the linear heat semigroup, $r > N/\alpha$ and $s > N/\beta$ are two Lebesgue numbers satisfying some conditions and specified in Lemma 2.3 below. α_1 and β_1 are two positive real numbers defined by

$$\alpha_1 = \frac{\alpha}{2} - \frac{N}{2r}, \quad (1.10)$$

$$\beta_1 = \frac{\beta}{2} - \frac{N}{2s}. \quad (1.11)$$

Moreover, if the initial value Φ decays as $c(|x|^{-\alpha}, |x|^{-\beta})$ when $|x| \to \infty$ (c is a small constant), then the resulting solution of the system (1.7)–(1.8) is asymptotic, for large time, to the self–similar solution of (1.7)–(1.8) with initial value $c(|x|^{-\alpha}, |x|^{-\beta})$ (see Theorem 4.2 below). In the same theorem, we give an estimate for the rate at which an asymptotically self–similar solution (u, v) converges to a self–similar solution (u_s, v_s). In particular, we show that $||u(t) - u_s(t)||_{r'}$ and $\sqrt{t}||\nabla u(t) - \nabla u_s(t)||_{r'}$, $r' \in [r, \infty]$ and $||v(t) - v_s(t)||_{s'}$ and $\sqrt{t}||\nabla v(t) - \nabla v_s(t)||_{s'}$, $s' \in [s, \infty]$ decrease as a negative power of t faster than the decay of the self–similar solution (u_s, v_s) by itself. The asymptotic in the $W^{1,\infty}$–norm is obtained by iterative argument using some idea of [21].

The condition (1.4) enables us to construct suitable globally decaying solutions and to characterize their "slowly" decaying rate in time. Qualitatively, if a smooth initial value decays as $c(|x|^{-\alpha}, |x|^{-\beta})$ then the resulting solution (u, v) satisfies, for large t

$$d_1 t^{-\alpha/2} \leq ||u(t)||_\infty \leq d_2 t^{-\alpha/2}, \qquad d_1 t^{-(\alpha+1)/2} \leq ||\nabla u(t)||_\infty \leq d_2 t^{-(\alpha+1)/2},$$

and

$$d_1' t^{-\beta/2} \leq ||v(t)||_\infty \leq d_2' t^{-\beta/2}, \qquad d_1' t^{-(\beta+1)/2} \leq ||\nabla v(t)||_\infty \leq d_2' t^{-(\beta+1)/2},$$

where d_1, d_2, d_1' and d_1' are positive canstants.

We notice that the conditions (1.3)–(1.4) imply that $q > (N+2)/(N+1)$. Note also that, since $p,\ q < 2$ then the condition (1.4) can be written in the following equivalent form

$$N\frac{pq-1}{2-q(p-1)} > 1.$$

One should notice that, in the particular case $p = q$, the condition (1.4) becomes

$$p > \frac{N+2}{N+1},$$

which is used in [20] for a similar study to the present paper but for the the scalar equation

$$\partial_t u = \Delta u + a|\nabla u|^p, \qquad t > 0, \quad x \in \mathbb{R}^N.$$

The rest of this paper is organized as follows. In Section 2, we establish some preliminary lemmas which will be needed later in the proofs of the theorems. In Section 3, the existence of global solutions and continuous dependence for a system more general than (1.1) are established (see system (3.1)–(3.2) below). Finally, in Section 4, we show the asymptotic behavior results. In this paper, we will denote by C a positive constant which can be different at various places and also we denote it by C_δ to indicate that it depends on real number δ. We sometimes denote $u(t,.)$ by $u(t)$.

2. Preliminaries

In this section we shall state some basic facts and obtain auxiliary results which will be used later in the proofs and statements of the main theorems. First of all, let $e^{t\Delta}$ be the linear heat semigroup which is defined by

$$\left(e^{t\Delta}\varphi\right)(x) = \left(E(t,.) \star \varphi\right)(x),$$

where $E(t,.)$ is the heat kernel given by

$$E(t,x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^N, \tag{2.1}$$

and $\star$ denotes the convolution product. We recall the following smoothing properties of the heat semigroup

$$||e^{t\Delta}u||_{m_2} \leq \mathcal{H} t^{-\frac{N}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)} ||u||_{m_1}, \tag{2.2}$$

$$||\nabla e^{t\Delta}u||_{m_2} \leq \mathcal{H} t^{-\frac{1}{2}-\frac{N}{2}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)} ||u||_{m_1}, \tag{2.3}$$

for $t > 0$ and $u \in L^{m_1}\left(\mathbb{R}^N\right)$, $\mathcal{H} > 0$ is a positive constant and $1 \leq m_1 \leq m_2 \leq \infty$. We denote by $\mathcal{S}'$ the space of tempered distributions.

Now, we will give some preliminary lemmas which will be needed in the proofs of the main theorems.

Lemma 2.1. *Let p and q be two real numbers such that $1 < p \leq q < 2$. Let α and β be given by* (1.5) *and* (1.6) *respectively. Let us define k by*

$$k = \frac{q+1}{p+1}. \tag{2.4}$$

Then we have the following properties:

(*i*) $pk - 1 = q - k > 0$; (*ii*) $pk \leq q$;

$$(iii)\quad \frac{k}{\beta}\le\frac{1}{\alpha};\qquad (iv)\quad q-k<\frac{1}{\alpha}<\frac{q}{\beta};\qquad (v)\quad \frac{q}{\beta}\le\frac{pk}{\alpha}.$$

Proof. The proof of the lemma can be done through elementary calculations from the expression of α, β and k also by using (1.3). □

Lemma 2.2. *Let N be a positive integer and let p and q be two real numbers such that $1<p\le q<2$. Suppose that*

$$N\frac{pq-1}{2-q(p-1)}>1.$$

Let α, β and k be given by (1.5), (1.6) *and* (2.4) *respectively. Then there exists a real number $r>1$ satisfying the following conditions:*

$$(i)\quad pk<r,\ q<r;\qquad (ii)\quad N(pk-1)<r,\ N(q-k)<r;$$

$$(iii)\quad \frac{Nk}{\beta}<r,\ \frac{N}{\alpha}<r;\quad (iv)\quad r<\frac{Npk}{\alpha},\ r<\frac{Nq}{\beta}.$$

Proof. Using Lemma 2.1, it suffice to prove the existence of r satisfying:

$$(i)'\quad q<r;\qquad (ii)'\quad N(q-k)<r;$$

$$(iii)'\quad \frac{N}{\alpha}<r;\quad (iv)'\quad r<\frac{Nq}{\beta}.$$

We remark that a real number r exists if and only if the left–hand sides of inequalities (i)′–(iii)′ are less than the right–hand side of inequality (iv)′. This results from (1.3), (1.4) and Lemma 2.1. Thus the lemma is proved. □

Lemma 2.3. *Let N be a positive integer and let p and q be two real numbers such that $1<p\le q<2$. Suppose that*

$$N\frac{pq-1}{2-q(p-1)}>1.$$

Let α and β be given by (1.5) *and* (1.6) *respectively. Let α_1 and β_1 be given by* (1.10) *and* (1.11) *respectively. Then there exist two real numbers r and s satisfying the following conditions:*

$$(i)\quad 1<\frac{s}{p}<r;\qquad (ii)\quad 1<\frac{r}{q}<s;$$

$$(iii)\quad N\left(\frac{p}{s}-\frac{1}{r}\right)<1;\quad (iv)\quad N\left(\frac{q}{r}-\frac{1}{s}\right)<1.\quad (v)\quad \alpha_1>0,\ \beta_1>0;$$

$$(vi)\quad (\alpha_1+\frac{1}{2})q<1,\ \left(\beta_1+\frac{1}{2}\right)p\ <1;$$

$$(vii)\quad \alpha_1-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)-\left(\beta_1+\frac{1}{2}\right)p+1=\frac{\alpha}{2}-\frac{1}{2}\left(\beta+1\right)p+1=0;$$

$$(viii)\quad \beta_1 - \frac{N}{2}\left(\frac{q}{r} - \frac{1}{s}\right) - \left(\alpha_1 + \frac{1}{2}\right) q + 1 = \frac{\beta}{2} - \frac{1}{2}(\alpha + 1)\, q + 1 = 0.$$

Proof. Let r be any real number satisfying the conditions of Lemma 2.2 and let

$$s = \frac{p+1}{q+1} r = \frac{r}{k}. \tag{2.5}$$

Then the proof of the lemma is a consequence of Lemmas 2.2 and 2.1. □

3. Global Existence

In this section we consider a more general system than the system (1.1)–(1.2). Let us consider the system

$$\begin{cases} \partial_t u = \Delta u + f(|\nabla v|), \\ \partial_t v = \Delta v + g(|\nabla u|), \end{cases} \tag{3.1}$$

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x), \tag{3.2}$$

where $u = u(t,x)$ and $v = v(t,x)$ are real valued functions, $t > 0$, $x \in \mathbb{R}^N$, $N \geq 1$ and f and g are two functions satisfying

$$f(0) = 0, \qquad |f(s_1) - f(s_2)| \leq \mathcal{C}\left(|s_1|^{p-1} + |s_2|^{p-1}\right)|s_1 - s_2|, \tag{3.3}$$

$$g(0) = 0, \qquad |g(s_1) - g(s_2)| \leq \mathcal{C}\left(|s_1|^{q-1} + |s_2|^{q-1}\right)|s_1 - s_2|, \tag{3.4}$$

with $s_1 > 0$, $s_2 > 0$, $\mathcal{C}$ a positive constant and $1 < p \leq q < 2$. The associated integral system is given by

$$u(t) = e^{t\Delta}\varphi_1 + \int_0^t e^{(t-\sigma)\Delta} f(|\nabla v(\sigma)|)\, \mathrm{d}\sigma, \tag{3.5}$$

$$v(t) = e^{t\Delta}\varphi_2 + \int_0^t e^{(t-\sigma)\Delta} g(|\nabla u(\sigma)|)\, \mathrm{d}\sigma. \tag{3.6}$$

We have obtained the following global existence result.

Theorem 3.1. (Global Existence) *Let the positive integer N and the real numbers p and q be such that $1 < p \leq q < 2$ and*

$$N \frac{pq - 1}{2 - q(p-1)} > 1.$$

Let α, β, α_1 and β_1 be given by (1.5), (1.6), (1.10) *and* (1.11) *respectively. Also, let r and s be two real numbers satisfying the conditions specified in Lemma* 2.3. *Let $M > 0$ be such that*

$$d = \max\left(d_0\, M^{p-1},\ d_0'\, M^{q-1}\right) < 1, \tag{3.7}$$

where d_0 *and* d_0' *are two positive constants given by* (3.24) *and* (3.25) *below. Choose* $R > 0$ *be such that*

$$R + dM \leq M. \tag{3.8}$$

Let $\Phi = (\varphi_1, \varphi_2)$ *be an element of* $\mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ *such that*

$$\mathcal{N}(\Phi) := \sup_{t>0} \left[t^{\alpha_1} ||e^{t\Delta}\varphi_1||_r, t^{\alpha_1+\frac{1}{2}} ||\nabla e^{t\Delta}\varphi_1||_r, t^{\beta_1} ||e^{t\Delta}\varphi_2||_s, t^{\beta_1+\frac{1}{2}} ||\nabla e^{t\Delta}\varphi_2||_s \right] \leq R. \tag{3.9}$$

Then there exists a unique global solution (u, v) *of the integral system* (3.5)–(3.6) *such that*

$$\sup_{t>0} \left[t^{\alpha_1} ||u(t)||_r, \, t^{\alpha_1+\frac{1}{2}} ||\nabla u(t)||_r, \, t^{\beta_1} ||v(t)||_s, \, t^{\beta_1+\frac{1}{2}} ||\nabla v(t)||_s \right] \leq M. \tag{3.10}$$

Furthermore:

(a) $u(t) - e^{t\Delta}\varphi_1 \in C\left([0, \infty), \, L^{\tau_1}(\mathbb{R}^N) \right)$ *for* $s/p \leq \tau_1 < N/\alpha$*;*

(b) $v(t) - e^{t\Delta}\varphi_2 \in C\left([0, \infty), \, L^{\tau_2}(\mathbb{R}^N) \right)$ *for* $r/q \leq \tau_2 < N/\beta$*;*

(c) $u(t) - e^{t\Delta}\varphi_1 \in L^\infty\left([0, \infty), \, L^{N/\alpha}(\mathbb{R}^N) \right)$ *and* $v(t) - e^{t\Delta}\varphi_2 \in L^\infty([0, \infty), L^{N/\beta}(\mathbb{R}^N))$*;*

(d) $\lim_{t \searrow 0} u(t) = \varphi_1$ *and* $\lim_{t \searrow 0} v(t) = \varphi_2$ *in the sense of tempered distributions.*

Moreover, the global solution (u, v) *satisfies*

$$\sup_{t>0} \left[t^{\alpha/2} ||u(t)||_\infty, \, t^{(\alpha+1)/2} ||\nabla u(t)||_\infty, \, t^{\beta/2} ||v(t)||_\infty, \, t^{(\beta+1)/2} ||\nabla v(t)||_\infty \right] < \infty. \tag{3.11}$$

Remark 3.2. As an example of initial values $\Phi = (\varphi_1, \varphi_2)$ satisfying (3.9) we may take $\varphi_1 \in L^{N/\alpha}(\mathbb{R}^N)$ and $\varphi_2 \in L^{N/\beta}(\mathbb{R}^N)$ with $||\varphi_1||_{N/\alpha}$ and $||\varphi_2||_{N/\beta}$ sufficiently small. See also Theorem 4.1 below for other examples.

Remark 3.3. It results from the smoothing properties of the heat semigroup that $\mathcal{N}(\Phi)$ is equivalent to the norm

$$|||\Phi||| = \sup_{t>0} \left[t^{\alpha_1} ||e^{t\Delta}\varphi_1||_r, t^{\beta_1} ||e^{t\Delta}\varphi_2||_s \right].$$

Then the norm defined by (3.9) is an equivalent version of the norm of the product of homogeneous Besov spaces with negative order

$$\dot{\mathcal{B}}_r^{-2\alpha_1,\infty}(\mathbb{R}^N) \times \dot{\mathcal{B}}_s^{-2\beta_1,\infty}(\mathbb{R}^N).$$

Proof of Theorem 3.1. We look for global solutions of the system (3.5)–(3.6) via a fixed point argument. Let us denote by $U = (u, v)$ and $\Phi = (\varphi_1, \varphi_2)$ where $\Phi \in \mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ satisfies (3.9). Let X be the set of Bochner measurable functions :

$$U : (0,\infty) \longrightarrow W^{1,r}\left(\mathbb{R}^N\right) \times W^{1,s}\left(\mathbb{R}^N\right),$$
$$t \longrightarrow (\, u(t), v(t)\,),$$

such that

$$||U||_X := \sup_{t>0}\left[t^{\alpha_1}||u(t)||_r, t^{\alpha_1+\frac{1}{2}}||\nabla u(t)||_r, t^{\beta_1}||v(t)||_s, t^{\beta_1+\frac{1}{2}}||\nabla v(t)||_s\right] < \infty, \quad (3.12)$$

where r and s are two positive real numbers which satisfy conditions specified in Lemma 2.3; α_1 and β_1 are respectively given by (1.10) and (1.11). Let M be a positive real number. Define

$$X_M = \{U \in X \,|\, ||U||_X \leq M\}.$$

X_M, endowed with the metric $d(U_1, U_2) = ||U_1 - U_2||_X$, is a complete metric space.

Now, consider the mapping $\mathcal{F}_\Phi$ defined by

$$\mathcal{F}_\Phi(U) = (\, F_\Phi(U), G_\Phi(U)\,), \quad (3.13)$$

where

$$F_\Phi(U)(t) = e^{t\Delta}\varphi_1 + \int_0^t e^{(t-\sigma)\Delta} f(\,|\nabla v(\sigma)|\,)\, \mathrm{d}\sigma, \quad (3.14)$$

$$G_\Phi(U)(t) = e^{t\Delta}\varphi_2 + \int_0^t e^{(t-\sigma)\Delta} g(\,|\nabla u(\sigma)|\,)\, \mathrm{d}\sigma. \quad (3.15)$$

We will prove that $\mathcal{F}_\Phi = (\, F_\Phi, G_\Phi\,)$ is a strict contraction mapping on X_M.

So let $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (\psi_1, \psi_2)$ belong to $\mathcal{S}'\left(\mathbb{R}^N\right) \times \mathcal{S}'\left(\mathbb{R}^N\right)$ and satisfy (3.9). Let $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ be two elements of X_M. Then, we have

$$t^{\alpha_1}||F_\Phi(U_1)(t) - F_\Psi(U_2)(t)||_r$$
$$\leq t^{\alpha_1}||e^{t\Delta}(\varphi_1 - \psi_1)\,||_r + t^{\alpha_1} \int_0^t ||e^{(t-\sigma)\Delta}\left[\, f(|\nabla v_1(\sigma)|) - f(|\nabla v_2(\sigma)|)\,\right]||_r \,\mathrm{d}\sigma.$$

Using the smoothing properties of the heat semigroup (2.2) where $(m_1, m_2) = (s/p, r)$, the part (i) of Lemma 2.3, and (3.3), we have

$$t^{\alpha_1}||F_\Phi(U_1)(t) - F_\Psi(U_2)(t)||_r \leq t^{\alpha_1}||e^{t\Delta}(\varphi_1 - \psi_1)\,||_r + \mathcal{C}\mathcal{H}t^{\alpha_1}$$
$$\times \int_0^t (t-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}||\left[\,\left(|\nabla v_1(\sigma)|^{p-1} + |\nabla v_2(\sigma)|^{p-1}\right)|\nabla v_1(\sigma) - \nabla v_2(\sigma)|\,\right]||_{s/p}\,\mathrm{d}\sigma.$$

Using the Hölder inequality, we obtain

$$t^{\alpha_1}||F_\Phi(U_1)(t) - F_\Psi(U_2)(t)||_r \leq t^{\alpha_1}||e^{t\Delta}(\varphi_1 - \psi_1)||_r + Ct^{\alpha_1}$$

$$\times \int_0^t (t-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)} \left(||\nabla v_1(\sigma)||_s^{p-1} + ||\nabla v_2(\sigma)||_s^{p-1}\right) ||\nabla v_1(\sigma) - \nabla v_2(\sigma)||_s \,\mathrm{d}\sigma.$$

Since U_1, U_2 belongs to X_M, so

$$\begin{aligned} t^{\alpha_1}||F_\Phi(U_1)(t) - F_\Psi(U_2)(t)||_r &\le t^{\alpha_1}||e^{t\Delta}(\varphi_1-\psi_1)||_r + 2CM^{p-1}t^{\alpha_1} \\ \times &\quad \left(\int_0^t (t-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma\right) ||U_1-U_2||_X \\ &\le t^{\alpha_1}||e^{t\Delta}(\varphi_1-\psi_1)||_r + 2CM^{p-1}t^{\alpha_1-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)-\left(\beta_1+\frac{1}{2}\right)p+1} \\ \times &\quad \left(\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma\right) ||U_1-U_2||_X. \end{aligned}$$

From Lemma 2.3 (vii), we obtain

$$\begin{aligned} t^{\alpha_1}\,&||F_\Phi(U_1)(t) - F_\Psi(U_2)(t)||_r \\ &\le\; t^{\alpha_1}||e^{t\Delta}(\varphi_1-\psi_1)||_r +\; d_1M^{p-1}||U_1-U_2||_X, \end{aligned} \tag{3.16}$$

where

$$d_1 =\; 2C\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma, \tag{3.17}$$

which is a finite positive constant from (iii) and (vi) of Lemma 2.3. Also, by using the estimate (2.3), Lemma 2.3 (i), and (3.3), we get

$$\begin{aligned} t^{\alpha_1+\frac{1}{2}}\,&||\nabla F_\Phi(U_1)(t) - \nabla F_\Psi(U_2)(t)||_r \le t^{\alpha_1+\frac{1}{2}}||\nabla e^{t\Delta}(\varphi_1-\psi_1)||_r + \mathcal{C}\mathcal{H}\,t^{\alpha_1+\frac{1}{2}} \\ &\times \int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}||\left[\left(|\nabla v_1(\sigma)|^{p-1} + |\nabla v_2(\sigma)|^{p-1}\right)|\nabla v_1 - \nabla v_2|\right]||_{s/p}\,\mathrm{d}\sigma \\ &\le t^{\alpha_1+\frac{1}{2}}||\nabla e^{t\Delta}(\varphi_1-\psi_1)||_r + 2C\,M^{p-1}t^{\alpha_1-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)-\left(\beta_1+\frac{1}{2}\right)p+1} \\ \times &\left(\int_0^1 (1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma\right) ||U_1-U_2||_X. \end{aligned}$$

Hence

$$\begin{aligned} t^{\alpha_1+\frac{1}{2}}&||\nabla F_\Phi(U_1)(t) - \nabla F_\Psi(U_2)(t)||_r \\ &\le t^{\alpha_1+\frac{1}{2}}||\nabla e^{t\Delta}(\varphi_1-\psi_1)||_r + d_2M^{p-1}||U_1-U_2||_X, \end{aligned} \tag{3.18}$$

where

$$d_2 =\; 2C\int_0^1 (1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma. \tag{3.19}$$

Using (iii) and (vi) of Lemma 2.3, d_2 is a finite positive constant.

Similarly, we obtain analogous estimates of $t^{\beta_1}||G_\Phi(U_1)(t) - G_\Psi(U_2)(t)||_s$ and $t^{\beta_1+1/2}||\nabla G_\Phi(U_1)(t) - \nabla G_\Psi(U_2)(t)||_s$. In fact, the integral equation (3.15) looks like the integral equation (3.14) with v, f and φ_1 respectively replaced by u, g and φ_2. Thus

$$t^{\beta_1}||G_\Phi(U_1)(t) - G_\Psi(U_2)(t)||_s \leq t^{\beta_1}||\nabla e^{t\Delta}(\varphi_2 - \psi_2)||_s + d_3\, M^{q-1}||U_1 - U_2||_X, \quad (3.20)$$

$$t^{\beta_1+\frac{1}{2}}||\nabla G_\Phi(U_1)(t) - \nabla G_\Psi(U_2)(t)||_s \leq t^{\beta_1+\frac{1}{2}}||\nabla e^{t\Delta}(\varphi_2 - \psi_2)||_s + d_4\, M^{q-1}||U_1 - U_2||_X, \quad (3.21)$$

where

$$d_3 = 2C\int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q}\, d\sigma, \quad (3.22)$$

$$d_4 = 2C\int_0^1 (1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{\left(\alpha_1+\frac{1}{2}\right)q}\, \mathrm{d}\sigma, \quad (3.23)$$

which are finite positive constants by using (iv) and (vi) of Lemma 2.3.

Now, from (3.17) and (3.19) we take

$$d_0 = \max(d_1, d_2) \quad (3.24)$$

and from (3.22) and (3.23) we take

$$d_0' = \max(d_3, d_4)\,. \quad (3.25)$$

So, inequalities (3.16), (3.18), (3.20) and (3.21) lead to

$$||\mathcal{F}_\Phi(U_1) - \mathcal{F}_\Psi(U_2)||_X \leq \mathcal{N}(\Phi - \Psi) + d\,||U_1 - U_2||_X, \quad (3.26)$$

where

$$d = \max\left(d_0 M^{p-1}, d_0' M^{q-1}\right).$$

Now, if we take $\Psi = 0$ and $U_2 \equiv 0$, the inequality (3.26) becomes

$$||\mathcal{F}_\Phi(U_1)||_X \leq \mathcal{N}(\Phi) + d\,||U_1||_X. \quad (3.27)$$

If we choose M and R such that (3.7) and (3.8) are satisfied, then by (3.27), $\mathcal{F}_\Phi$ maps X_M into itself. For $\Phi = \Psi$, (3.26) becomes

$$||\mathcal{F}_\Phi(U_1) - \mathcal{F}_\Phi(U_2)||_X \leq d\,||U_1 - U_2||_X.$$

Hence inequality (3.7) gives that $\mathcal{F}$ is a strict contraction mapping from X_M into itself. So $\mathcal{F}$ has a unique fixed point $U = (u, v)$ in X_M which is a solution of the integral system (3.5)–(3.6). This terminates the proof of the existence of a unique global solution of (3.5)–(3.6) in X_M.

Now, we will prove the statements (a)–(d). Let τ_1 be a positive real number satisfying

$$\frac{s}{p} \leq \tau_1 < \frac{N}{\alpha}. \quad (3.28)$$

Such a τ_1 exists by Lemma 2.3 (vi). Then by (2.2) and (3.3), we have

$$\begin{aligned}||u(t)-e^{t\Delta}\varphi_1||_{\tau_1} \leq\ & \int_0^t ||e^{(t-\sigma)\Delta} f\left(|\nabla v(\sigma)|\right)||_{\tau_1}\, \mathrm{d}\sigma \\ \leq\ & \mathcal{C}\mathcal{H}\int_0^t (t-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{\tau_1}\right)}||\nabla v(\sigma)||_s^p\, \mathrm{d}\sigma \\ \leq\ & C\, M^p t^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{\tau_1}\right)-\left(\beta_1+\frac{1}{2}\right)p+1} \\ & \times \int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{\tau_1}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\, \mathrm{d}\sigma.\end{aligned}$$

Using Lemma 2.3 (vii), the latter inequality gives

$$||u(t)-e^{t\Delta}\varphi_1||_{\tau_1} \leq\ C_0\, t^{\frac{N}{2\tau_1}-\frac{\alpha}{2}}, \tag{3.29}$$

where

$$C_0 =\ C\, M^p \int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{\tau_1}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\, d\sigma, \tag{3.30}$$

which is a positive constant. Owing to (3.28) and (vi) of Lemma 2.3, the constant C_0 is finite.

Similarly, we deduce for

$$\frac{r}{q} \leq \tau_2 < \frac{N}{\beta}, \tag{3.31}$$

the following inequality

$$||v(t)-e^{t\Delta}\varphi_2||_{\tau_2} \leq\ C_0'\, t^{\frac{N}{2\tau_2}-\frac{\beta}{2}}, \tag{3.32}$$

where C_0' is a positive constant given by

$$C_0' =\ C\, M^q \int_0^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{\tau_2}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q}\, d\sigma, \tag{3.33}$$

which is finite by (3.31) and (vi) of Lemma 2.3. Owing to the conditions (3.28) and (3.31), the right–hand sides of (3.29) and (3.32) converge to zero as $t \searrow 0$. Also, it is clear that (3.29) and (3.32) still holds if $\tau_1 = N/\alpha$ and $\tau_2 = N/\beta$, which proves the statement (c).

Finally, we shall demonstrate (3.11). Let us consider two real numbers r_1 and s_1 such that $r_1 = ks_1$ and

$$\begin{aligned}& 1 < \frac{s}{p} < r < r_1 \leq \infty, \quad 1 < \frac{r}{q} < s < s_1 \leq \infty, \\ & N\left(\frac{p}{s}-\frac{1}{r_1}\right) < 1, \quad N\left(\frac{q}{r}-\frac{1}{s_1}\right) < 1.\end{aligned} \tag{3.34}$$

Note that such choice is possible owing to Lemma 2.3. Let us write the equation satisfied by u as follows

$$u(t) = e^{(t/2)\Delta}u(t/2) + \int_{t/2}^{t} e^{(t-\sigma)\Delta} f(\,|\nabla v(\sigma)|)\,\mathrm{d}\sigma.$$

Then, by using the smoothing properties of the heat semigroup (2.2), as well as (3.3), (3.34), Lemma 2.3 and the estimate (3.10), we obtain

$$\begin{aligned}
t^{\frac{\alpha}{2}-\frac{N}{2r_1}}||u(t)||_{r_1} \le\ & C\mathcal{H} \sup_{t>0}[t^{\alpha_1}||u(t)||_r] + t^{\frac{\alpha}{2}-\frac{N}{2r_1}} \\
& \times \int_{t/2}^{t} ||\, e^{(t-\sigma)\Delta} f\,(\,|\nabla v(\sigma)|\,)\,||_{r_1}\,\mathrm{d}\sigma \\
\le\ & C\mathcal{H} \sup_{t>0}[t^{\alpha_1}||u(t)||_r] + \mathcal{C}\mathcal{H}\, t^{\frac{\alpha}{2}-\frac{N}{2r_1}} \\
& \times \int_{t/2}^{t} (t-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r_1}\right)} ||\nabla v(\sigma)||_s^p\,\mathrm{d}\sigma \\
\le\ & C\,M + C\,M^p t^{\frac{\alpha}{2}-\frac{N}{2r_1}} \int_{t/2}^{t} (t-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r_1}\right)} \sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma \\
\le\ & C\,M + C\,M^p \int_{1/2}^{1} (1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r_1}\right)} \sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma.
\end{aligned}$$

The latter inequality leads to

$$\sup_{t>0}\left[t^{\frac{\alpha}{2}-\frac{N}{2r_1}}||u(t)||_{r_1}\right] \le C(M) < \infty. \tag{3.35}$$

Let us also write

$$\nabla u(t) = e^{(t/2)\Delta}\nabla u(t/2) + \int_{t/2}^{t} \nabla e^{(t-\sigma)\Delta} f(\,|\nabla v(\sigma)|)\,\mathrm{d}\sigma.$$

Then, we have

$$\begin{aligned}
t^{\frac{\alpha}{2}-\frac{N}{2r_1}+\frac{1}{2}}||\nabla u(t)||_{r_1} \le\ & t^{\frac{\alpha}{2}-\frac{N}{2r_1}+\frac{1}{2}}||e^{(t/2)\Delta}\nabla u(t/2)||_{r_1} \\
& + t^{\frac{\alpha}{2}-\frac{N}{2r_1}+\frac{1}{2}} \int_{t/2}^{t} ||\nabla e^{(t-\sigma)\Delta} f\,(\,|\nabla v(\sigma)|\,)\,||_{r_1}\,\mathrm{d}\sigma \\
\le\ & C\mathcal{H} \sup_{t>0}\left[t^{\alpha_1+\frac{1}{2}}||\nabla u(t)||_r\right] + \mathcal{C}\mathcal{H}\, t^{\frac{\alpha}{2}-\frac{N}{2r_1}+\frac{1}{2}} \\
& \times \int_{t/2}^{t} (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r_1}\right)} ||\nabla v(\sigma)||_s^p\,\mathrm{d}\sigma \\
\le\ & C\,M + C\,M^p \int_{1/2}^{1} (1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r_1}\right)} \sigma^{-\left(\beta_1+\frac{1}{2}\right)p}\,\mathrm{d}\sigma.
\end{aligned}$$

So we get

$$\sup_{t>0}\left[t^{\frac{\alpha}{2}-\frac{N}{2r_1}+\frac{1}{2}}||\nabla u(t)||_{r_1}\right] < C(M) < \infty. \tag{3.36}$$

Analogously, we find the following estimates on the second component v:

$$\sup_{t>0}\left[t^{\frac{\beta}{2}-\frac{N}{2s_1}}||v(t)||_{s_1}\right] \leq C(M) < \infty \tag{3.37}$$

and

$$\sup_{t>0}\left[t^{\frac{\beta}{2}-\frac{N}{2s_1}+\frac{1}{2}}||\nabla v(t)||_{s_1}\right] < C(M) < \infty. \tag{3.38}$$

We iterate this procedure. For the next step, we replace in (3.34), r by r_1, s by s_1 and we consider two real numbers r_2 and s_2 such that $r_2 = ks_2$ and

$$1 < \frac{s_1}{p} < r_1 < r_2 \leq \infty, \quad 1 < \frac{r_1}{q} < s_1 < s_2 \leq \infty,$$

$$N\left(\frac{p}{s_1} - \frac{1}{r_2}\right) < 1, \quad N\left(\frac{q}{r_1} - \frac{1}{s_2}\right) < 1.$$

Using similar estimates as the previous step, we obtain

$$\sup_{t>0}\Bigg[t^{\frac{\alpha}{2}-\frac{N}{2r_2}}||u(t)||_{r_2},\, t^{\frac{\alpha}{2}-\frac{N}{2r_2}+\frac{1}{2}}||\nabla u(t)||_{r_2},\, t^{\frac{\beta}{2}-\frac{N}{2s_2}}||v(t)||_{s_2},$$
$$t^{\frac{\beta}{2}-\frac{N}{2s_2}+\frac{1}{2}}||\nabla v(t)||_{s_2}\Bigg] < \infty.$$

Let us consider the sequences $(r_i)_i$ and $(s_i)_i$ such that $r_0 = r$, $s_0 = s$ and $r_i = ks_i$, $\forall\, i = 1, 2, \ldots$ and such that

$$1 < \frac{s_i}{p} < r_i < r_{i+1} \leq \infty, \quad 1 < \frac{r_i}{q} < s_i < s_{i+1} \leq \infty,$$

$$N\left(\frac{p}{s_i} - \frac{1}{r_{i+1}}\right) < 1, \quad sN\left(\frac{q}{r_i} - \frac{1}{s_{i+1}}\right) < 1.$$

Now, by Lemma 2.3, one can choose the sequences $(r_i)_i$ and $(s_i)_i$ such that they reach ∞ for some finite i. We finally obtain

$$\sup_{t>0}\left[t^{\frac{\alpha}{2}}||u(t)||_{\infty},\, t^{\frac{\alpha+1}{2}}||\nabla u(t)||_{\infty},\, t^{\frac{\beta}{2}}||v(t)||_{\infty},\, t^{\frac{\beta+1}{2}}||\nabla v(t)||_{\infty}\right] \leq C(M)$$
$$< \infty, \tag{3.39}$$

with $C(M)$ converges to zero as M converges to zero. This finishes the proof of Theorem 3.1. □

Theorem 3.4. (Continuous Dependence) *Assume the hypotheses of Theorem* 3.1 *above. If* $\Phi = (\varphi_1, \varphi_2)$ *and* $\Psi = (\psi_1, \psi_2)$ *satisfy* (3.9), *and if* $U_1 = (u_1, v_1)$ *and* $U_2 = (u_2, v_2)$ *respectively are the solutions of the system*

(3.5)–(3.6) *with initial values* Φ *and* Ψ *then we have*

$$\sup_{t>0}\Big[t^{\alpha_1}||u_1(t)-u_2(t)||_r,\, t^{\alpha_1+\frac{1}{2}}||\nabla u_1(t)-\nabla u_2(t)||_r,\, t^{\beta_1}||v_1(t)-v_2(t)||_s,$$
$$t^{\beta_1+\frac{1}{2}}||\nabla v_1(t)-\nabla v_2(t)||_s\Big] \le (1-d)^{-1}\mathcal{N}(\Phi-\Psi), \qquad (3.40)$$

where d *is given in Theorem* 3.1*. Moreover, we have*

$$\sup_{t>0}\Big[t^{\alpha/2}||u_1(t)-u_2(t)||_\infty,\, t^{(\alpha+1)/2}||\nabla u_1(t)-\nabla u_2(t)||_\infty,\, t^{\beta/2}||v_1(t)-v_2(t)||_\infty,$$
$$t^{(\beta+1)/2}||\nabla v_1(t)-\nabla v_2(t)||_\infty\Big] \le C(1-d')^{-1}(1-d)^{-1}\mathcal{N}(\Phi-\Psi), \quad (3.41)$$

where d' *is given by (3.52) below. The positive constant* M *is chosen small enough so that* $0<d'<1$.

Furthermore, if the initial values Φ *and* Ψ *are such that*

$$\mathcal{N}_\delta(\Phi-\Psi) := \sup_{t>0}\Big[t^{\alpha_1+\delta}\,||e^{t\Delta}(\varphi_1-\psi_1)||_r,\, t^{\alpha_1+\delta+\frac{1}{2}}\,||\nabla e^{t\Delta}(\varphi_1-\psi_1)||_r,$$
$$t^{\beta_1+\delta}\,||e^{t\Delta}(\varphi_2-\psi_2)||_s,\, t^{\beta_1+\delta+\frac{1}{2}}\,||\nabla e^{t\Delta}(\varphi_2-\psi_2)||_s\Big] < \infty, \quad (3.42)$$

for some $0<\delta<\delta_0$*, where*

$$\delta_0 = 1-\left(\alpha_1+\frac{1}{2}\right)q, \qquad (3.43)$$

then we have

$$\sup_{t>0}\Big[t^{\alpha_1+\delta}||u_1(t)-u_2(t)||_r,\, t^{\alpha_1+\delta+\frac{1}{2}}||\nabla u_1(t)-\nabla u_2(t)||_r,\, t^{\beta_1+\delta}||v_1(t)-v_2(t)||_s,$$
$$t^{\beta_1+\delta+\frac{1}{2}}||\nabla v_1(t)-\nabla v_2(t)||_s\Big] \le (1-d_\delta)^{-1}\mathcal{N}_\delta(\Phi-\Psi), \qquad (3.44)$$

where d_δ *is given by (3.54) below. The positive constant* M *is chosen small enough so that* $0<d_\delta<1$.

Proof. The inequality (3.40) results by considering $\mathcal{F}_\Phi(U_1)=U_1$ and $\mathcal{F}_\Psi(U_2)=U_2$ in the estimate (3.26) in the proof of Theorem 3.1. We now prove (3.41). Let us denote $||U||_{X_\infty}$ by

$$||U||_{X_\infty} := \sup_{t>0}\Big[t^{\frac{\alpha}{2}}||u(t)||_\infty, t^{\frac{\alpha+1}{2}}||\nabla u(t)||_\infty, t^{\frac{\beta}{2}}||v(t)||_\infty, t^{\frac{\beta+1}{2}}||\nabla v(t)||_\infty\Big], \quad (3.45)$$

where $U=(u,v)$ is a solution of (3.5)–(3.6). From (3.11), this norm is finite. Now, let $U_1(t)=(\,u_1(t),v_1(t)\,)$, $U_2(t)=(\,u_2(t),v_2(t)\,)$, $\Phi=(\varphi_1,\varphi_2)$ and $\Psi=$

(ψ_1, ψ_2). We write for $U_i = (u_i, v_i)$; $i = 1, 2$,

$$u_i(t) = e^{(t/2)\Delta} u_i(t/2) + \int_{t/2}^{t} e^{(t-\sigma)\Delta} f(\,|\nabla v_i(\sigma)|)\,\mathrm{d}\sigma, \tag{3.46}$$

$$v_i(t) = e^{(t/2)\Delta} v_i(t/2) + \int_{t/2}^{t} e^{(t-\sigma)\Delta} g(\,|\nabla u_i(\sigma)|)\,\mathrm{d}\sigma. \tag{3.47}$$

By the smoothing effect of the heat semigroup (2.2), as well as (3.3), (3.39) and (3.40) we have, for all $t > 0$,

$$\begin{aligned} t^{\alpha/2}||u_1(t) - u_2(t)||_\infty &\le t^{\alpha/2}||e^{(t/2)\Delta}(u_1(t/2) - u_2(t/2))||_\infty \\ &\quad + t^{\alpha/2} \int_{t/2}^{t} ||e^{(t-\sigma)\Delta}[f(|\nabla v_1(\sigma)|) - f(|\nabla v_2(\sigma)|)]||_\infty\,\mathrm{d}\sigma \\ &\le Ct^{\alpha_1}||u_1(t/2) - u_2(t/2)||_r + \mathcal{C}t^{\alpha/2} \\ &\quad \times \int_{t/2}^{t} ||[\,|\nabla v_1(\sigma)|^{p-1} + |\nabla v_2(\sigma)|^{p-1}]|\nabla v_1 - \nabla v_2|||_\infty\,\mathrm{d}\sigma \\ &\le C(1-d)^{-1}\mathcal{N}(\Phi - \Psi) + C(M)\,t^{\alpha/2} \\ &\quad \times \left(\int_{t/2}^{t} \sigma^{-\frac{1}{2}(\beta+1)p}\,\mathrm{d}\sigma\right)||U_1 - U_2||_{X_\infty}. \end{aligned}$$

Then by Lemma 2.3 (vii), we have

$$\begin{aligned} t^{\alpha/2}||u_1(t) - u_2(t)||_\infty &\le C(1-d)^{-1}\mathcal{N}(\Phi - \Psi) + C(M) \\ &\quad \times \left(\int_{1/2}^{1} \sigma^{-\frac{1}{2}(\beta+1)p}\,\mathrm{d}\sigma\right)\;||U_1 - U_2||_{X_\infty}, \end{aligned}$$

hence

$$t^{\alpha/2}||u_1(t) - u_2(t)||_\infty \le C(1-d)^{-1}\mathcal{N}(\Phi - \Psi) + d_1'||U_1 - U_2||_{X_\infty}, \tag{3.48}$$

where

$$d_1' = C(M)\int_{1/2}^{1} \sigma^{-\frac{1}{2}(\beta+1)p}\,\mathrm{d}\sigma,$$

which is a finite positive constant. By the same way we get

$$\begin{aligned} t^{(\alpha+1)/2}&||\nabla u_1(t) - \nabla u_2(t)||_\infty \\ &\le C(1-d)^{-1}\mathcal{N}(\Phi - \Psi) + d_2'||U_1 - U_2||_{X_\infty}, \end{aligned} \tag{3.49}$$

$$t^{\beta/2}||v_1(t) - v_2(t)||_\infty \le C(1-d)^{-1}\mathcal{N}(\Phi - \Psi) + d_3'||U_1 - U_2||_{X_\infty}, \tag{3.50}$$

$$\begin{aligned} t^{(\beta+1)/2}&||\nabla v_1(t) - \nabla v_2(t)||_\infty \\ &\le C(1-d)^{-1}\mathcal{N}(\Phi - \Psi) + d_4'||U_1 - U_2||_{X_\infty}, \end{aligned} \tag{3.51}$$

where

$$d_2' = C(M)\int_{1/2}^{1}(1-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}(\beta+1)p}\,\mathrm{d}\sigma,$$

$$d_3' = C(M)\int_{1/2}^{1}\sigma^{-\frac{1}{2}(\alpha+1)q}\,\mathrm{d}\sigma,$$

$$d_4' = C(M)\int_{1/2}^{1}(1-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}(\alpha+1)q}\,\mathrm{d}\sigma,$$

which are finite positive constants. We have from (3.48)–(3.51)

$$||U_1-U_2||_{X_\infty} \le C(1-d)^{-1}\mathcal{N}(\Phi-\Psi) + d'||U_1-U_2||_{X_\infty},$$

where

$$d' = \max(d_1', d_2', d_3', d_4'). \tag{3.52}$$

Since, for M perhaps smaller, $0 < d' < 1$, we get

$$||U_1-U_2||_{X_\infty} \le C(1-d')^{-1}(1-d)^{-1}\mathcal{N}(\Phi-\Psi),$$

which leads to the estimate (3.41).

We have the following stronger decay property which will be needed for obtaining the asymptotic behavior of some global solutions. If in addition Φ and Ψ satisfy (3.42), then following the same steps as in Theorem 3.1 but with the norm

$$||U||_{X,\delta} := \sup_{t>0}\left[t^{\alpha_1+\delta}||u(t)||_r,\, t^{\alpha_1+\delta+\frac{1}{2}}||\nabla u(t)||_r,\, t^{\beta_1+\delta}||v(t)||_s,\, t^{\beta_1+\delta+\frac{1}{2}}||\nabla v(t)||_s\right] < \infty,$$

where $U=(u,v)$. We have

$$\begin{aligned} t^{\alpha_1+\delta}||F_\Phi(U_1)(t)-F_\Psi(U_2)(t)||_r \le\ & t^{\alpha_1+\delta}||e^{t\Delta}(\varphi_1-\psi_1)||_r + 2C\,M^{p-1} \\ &\times t^{\alpha_1-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)-\left(\beta_1+\frac{1}{2}\right)p+1}\left(\int_0^1(1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p-\delta}\,\mathrm{d}\sigma\right)||U_1-U_2||_{X,\delta} \\ \le\ & t^{\alpha_1+\delta}||e^{t\Delta}(\varphi_1-\psi_1)||_r + 2C\,M^{p-1} \\ &\times\left(\int_0^1(1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p-\delta}\,\mathrm{d}\sigma\right)||U_1-U_2||_{X,\delta}. \end{aligned}$$

Also,

$$\begin{aligned} & t^{\alpha_1+\delta+\frac{1}{2}}||\nabla F_\Phi(U_1)(t)-\nabla F_\Psi(U_2)(t)||_r \\ &\qquad \le t^{\alpha_1+\delta+\frac{1}{2}}||\nabla e^{t\Delta}(\varphi_1-\psi_1)||_r + 2C\,M^{p-1} \end{aligned}$$

$$\times\, t^{\alpha_1-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)-\left(\beta_1+\frac{1}{2}\right)p+1}\left(\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p-\delta}\,\mathrm{d}\sigma\right)$$
$$\times\, ||U_1-U_2||_{X,\delta}$$
$$\leq t^{\alpha_1+\delta+\frac{1}{2}}||\nabla e^{t\Delta}(\varphi_1-\psi_1)||_r+2C\,M^{p-1}$$
$$\times\left(\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p-\delta}\,\mathrm{d}\sigma\right)||U_1-U_2||_{X,\delta}.$$

Similarly, we obtain

$$t^{\beta_1+\delta}||G_\Phi(U_1)(t)-G_\Psi(U_2)(t)||_s\leq\ t^{\beta_1+\delta}||e^{t\Delta}(\varphi_2-\psi_2)||_s+2C\,M^{q-1}$$
$$\times t^{\beta_1-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)-\left(\alpha_1+\frac{1}{2}\right)q+1}\left(\int_0^1(1-\sigma)^{-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q-\delta}\,\mathrm{d}\sigma\right)||U_1-U_2||_{X,\delta}$$
$$\leq t^{\beta_1+\delta}||e^{t\Delta}(\varphi_2-\psi_2)||_s+2C\,M^{q-1}$$
$$\times\left(\int_0^1(1-\sigma)^{-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q-\delta}\,\mathrm{d}\sigma\right)||U_1-U_2||_{X,\delta}.$$

Also,

$$t^{\beta_1+\delta+\frac{1}{2}}||G_\Phi(U_1)(t)-G_\Psi(U_2)(t)||_s\leq\ t^{\beta_1+\delta+\frac{1}{2}}||e^{t\Delta}(\varphi_2-\psi_2)||_s+2C\,M^{q-1}\times$$
$$t^{\beta_1-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)-\left(\alpha_1+\frac{1}{2}\right)q+1}\left(\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q-\delta}\,\mathrm{d}\sigma\right)||U_1-U_2||_{X,\delta}$$
$$\leq\ t^{\beta_1+\delta+\frac{1}{2}}||e^{t\Delta}(\varphi_2-\psi_2)||_s+2C\,M^{q-1}$$
$$\times\left(\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q-\delta}\,\mathrm{d}\sigma\right)||U_1-U_2||_{X,\delta}.$$

Then

$$||\mathcal{F}_\Phi(U_1)(t)-\mathcal{F}_\Psi(U_2)(t)||_{X,\delta}\ \leq\ \mathcal{N}_\delta(\Phi-\Psi)+d_\delta\,||U_1-U_2||_{X,\delta},\tag{3.53}$$

where

$$d_\delta=\ \max\left(d_{\delta,1}\,M^{p-1},\,d_{\delta,2}\,M^{p-1},\,d_{\delta,3}\,M^{q-1},\,d_{\delta,4}\,M^{q-1}\right),\tag{3.54}$$

with

$$d_{\delta,1}\ =2C\int_0^1(1-\sigma)^{-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p-\delta}\,\mathrm{d}\sigma,\tag{3.55}$$
$$d_{\delta,2}\ =2C\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{p}{s}-\frac{1}{r}\right)}\sigma^{-\left(\beta_1+\frac{1}{2}\right)p-\delta}\,\mathrm{d}\sigma,\tag{3.56}$$
$$d_{\delta,3}\ =2C\int_0^1(1-\sigma)^{-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q-\delta}\,\mathrm{d}\sigma,\tag{3.57}$$
$$d_{\delta,4}\ =2C\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{N}{2}\left(\frac{q}{r}-\frac{1}{s}\right)}\sigma^{-\left(\alpha_1+\frac{1}{2}\right)q-\delta}\,\mathrm{d}\sigma.\tag{3.58}$$

One can easily see that $d_{\delta,1}$, $d_{\delta,2}$, $d_{\delta,3}$ and $d_{\delta,4}$ are finite positive constants, since δ satisfies

$$\delta + \left(\alpha_1 + \frac{1}{2}\right) q < 1 \qquad \text{and} \qquad \delta + \left(\beta_1 + \frac{1}{2}\right) p < 1,$$

which follow for all $0 < \delta < \delta_0$ with δ_0 given by (3.43).

Since $\mathcal{F}_\Phi(U_1) = U_1$ and $\mathcal{F}_\Psi(U_2) = U_2$, then (3.53) becomes

$$\sup_{t>0} \left[t^{\alpha_1+\delta} ||u_1(t) - u_2(t)||_r,\ t^{\alpha_1+\delta+\frac{1}{2}} ||\nabla u_1(t) - \nabla u_2(t)||_r,\ t^{\beta_1+\delta} ||v_1(t) - v_2(t)||_s, \right.$$
$$\left. t^{\beta_1+\delta+\frac{1}{2}} ||\nabla v_1(t) - \nabla v_2(t)||_s \right] \le (1 - d_\delta)^{-1} \mathcal{N}_\delta(\Phi - \Psi).$$

Therefore (3.44) holds for M small enough, so that $0 < d_\delta < 1$, where d_δ is given by (3.54). This finishes the proof of the theorem. □

Corollary 3.5. *Suppose that the hypotheses of Theorem* 3.1 *through formula* (3.8) *are satisfied. Then we have the following:*

(i) If $\varphi_1 \in L^{N/\alpha}(\mathbb{R}^N)$ *and* $\varphi_2 \in L^{N/\beta}(\mathbb{R}^N)$ *such that* $||\varphi_1||_{N/\alpha}$ *and* $||\varphi_2||_{N/\beta}$ *are sufficiently small, then* (φ_1, φ_2) *satisfies* (3.9).

(ii) If $\varphi_1 \in L^{N/\alpha}(\mathbb{R}^N)$ *and* $\varphi_2 \in L^{N/\beta}(\mathbb{R}^N)$ *then there exists* $T > 0$, *such that* (φ_1, φ_2) *satisfies* (3.9), *but only on* $(0, T)$.

(iii) In the above two cases, if (u, v) *is the resulting solution of the system* (1.7)–(1.8), *then* $(u, v) \in C\left([0, \infty), L^{N/\alpha}(\mathbb{R}^N) \times L^{N/\beta}(\mathbb{R}^N)\right)$ *respectively* $(u, v) \in C\left([0, T], L^{N/\alpha}(\mathbb{R}^N) \times L^{N/\beta}(\mathbb{R}^N)\right)$.

The proof of the Corollary 3.5 can be done as in [20, Corollary 2.6], that is why we omit it.

4. Asymptotic Behavior

In this section, we return to the system (1.7)–(1.8). We first give the theorem about the existence of self-similar solutions.

Theorem 4.1. (Self-Similar Solutions) *Let the positive integer* N *and the real numbers* p *and* q *be such that* $1 < p \le q < 2$ *and*

$$N \frac{pq - 1}{2 - q(p - 1)} > 1.$$

Let α, β, α_1 *and* β_1 *be given by* (1.5), (1.6), (1.10) *and* (1.11) *respectively. Also, let* r *and* s *satisfy the conditions specified in Lemma* 2.3. *Suppose* φ_1, φ_2 *be two tempered distributions which respectively are also homogeneous of* *degree $-\alpha$ and $-\beta$, and such that*

$$\varphi_1(x) = \omega_1(x)|x|^{-\alpha} \qquad \text{and} \qquad \varphi_2(x) = \omega_2(x)|x|^{-\beta}, \tag{4.1}$$

where $\omega_1 \in L^r(S^{N-1})$ and $\omega_2 \in L^s(S^{N-1})$ are homogenous of degree 0. Denote by $\Phi = (\varphi_1, \varphi_2)$. Then $\mathcal{N}(\Phi) < \infty$ and the solution $U = (u, v)$ of (1.7)–(1.8) with initial value $c\Phi$, where $|c|$ is a small constant, is a global self–similar solution.

Moreover, for any L^∞ cut–off function η $\big(\eta$ identically equal to 1 near the origin and with compact support $\big)$, we have

(a) $\mathcal{N}\left((1-\eta)\Phi \right) < \infty$;

(b) $\mathcal{N}_\delta\left(\eta\Phi\right) < \infty$ *for $0 < \delta < \delta_0$, where δ_0 is given by (3.43).*

Proof. Since, by (1.3)-(1.4) and Lemma 2.3 (v) we have $N/\alpha \geq N/\beta > 1$, $r > N/\alpha$ and $s > N/\beta$, then $\mathcal{N}(\Phi) < \infty$ results from [5, 21]. Choose now a constant c such that

$$\mathcal{N}(c\Phi) \leq R, \tag{4.2}$$

where R satisfies the condition (3.8). Let U be the solution of the integral system (1.7)–(1.8) with initial data $c\Phi$ constructed by Theorem 3.1. That is U belongs to X_M. For $\lambda > 0$, let us define U_λ by

$$U_\lambda(t,x) = \Big(\lambda^\alpha u(\lambda^2 t, \lambda x), \lambda^\beta v(\lambda^2 t, \lambda x)\Big), \qquad t > 0, \qquad x \in \mathbb{R}^N.$$

To show that U is a self–similar solution, we have to show that

$$U_\lambda \equiv U, \qquad \forall \lambda > 0,$$

on $(0,\infty) \times \mathbb{R}^N$. Since the system (1.7)–(1.8) is invariant under the transformation $\lambda \to U_\lambda$, then U_λ is also a solution of (1.7)–(1.8) with initial data $c\Phi_\lambda := c\left(\lambda^\alpha \varphi_1(\lambda x), \lambda^\beta \varphi_2(\lambda x)\right)$. Since φ_1 and φ_2 are homogeneous of degree respectively $-\alpha$ and $-\beta$, then $\Phi_\lambda = \Phi$ and hence U and U_λ have the same initial value. Now since

$$||U_\lambda||_X = ||U||_X, \qquad \forall \lambda > 0,$$

then U_λ belongs to X_M and by uniqueness, $U_\lambda = U$, $\forall \lambda > 0$, hence U is self–similar. We denote by $U_s = (u_s, v_s)$ a self–similar solution.

The proof of the statements (a) and (b) is similar to the one of [5, Lemma 4.6], see also [20]. So we omit it. □

We now give the asymptotic behavior result.

Theorem 4.2. (Asymptotic Behavior) *Let the positive integer N and the* *real numbers p and q be such that $1 < p \leq q < 2$ and*

$$N\frac{pq-1}{2-q(p-1)} > 1.$$

Let α, β, α_1 and β_1 be given by (1.5), (1.6), (1.10) and (1.11) respectively. Also, let r and s be two numbers satisfying the conditions specified in Lemma 2.3. Suppose φ_1, φ_2 be two tempered distributions which respectively are also homogeneous of degree $-\alpha$ and $-\beta$, and such that

$$\varphi_1(x) = \omega_1(x)|x|^{-\alpha} \qquad \text{and} \qquad \varphi_2(x) = \omega_2(x)|x|^{-\beta},$$

where $\omega_1 \in L^r(S^{N-1})$ and $\omega_2 \in L^s(S^{N-1})$ are homogenous of degree 0. Denote by $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (1-\eta)\Phi + \Psi_0$ where η is a L^∞ cut–off function (η identically equal to 1 near the origin and with compact support) and $\Psi_0 = (\psi_{0,1}, \psi_{0,2})$ is such that $\mathcal{N}(\Psi_0)$ is small and $\mathcal{N}_\delta(\Psi_0) < \infty$ for some $0 < \delta < \delta_0$, where δ_0 is given by (3.43). If necessary we multiply Ψ by some constant such that both quantities $\mathcal{N}(\Phi)$ and $\mathcal{N}(\Psi_0)$ are smaller. Let

$$U_{\rm s}(t,x) = (t^{-\alpha/2}u_{\rm s}(1, \frac{x}{\sqrt{t}}), t^{-\beta/2}v_{\rm s}(1, \frac{x}{\sqrt{t}}))$$

be the self–similar solution of (1.7)–(1.8) with initial value Φ constructed by Theorem 4.1. Let $U = (u, v)$ be the global solution of (1.7)–(1.8) with initial value Ψ, constructed by Theorem 3.1. Then, for $0 < \delta < \delta_0$, there exists a positive constant C_δ such that

$$||u(t) - u_{\rm s}(t)||_r \leq C_\delta t^{-\alpha_1-\delta}, \qquad \forall t > 0, \tag{4.3}$$

$$||\nabla u(t) - \nabla u_{\rm s}(t)||_r \leq C_\delta t^{-\left(\alpha_1+\frac{1}{2}\right)-\delta}, \qquad \forall t > 0, \tag{4.4}$$

$$||v(t) - v_{\rm s}(t)||_s \leq C_\delta t^{-\beta_1-\delta}, \qquad \forall t > 0, \tag{4.5}$$

$$||\nabla v(t) - \nabla v_{\rm s}(t)||_s \leq C_\delta t^{-\left(\beta_1+\frac{1}{2}\right)-\delta}, \qquad \forall t > 0. \tag{4.6}$$

Also,

$$||t^{\frac{2-p(q-1)}{2(pq-1)}}u(t, .\sqrt{t}) - u_{\rm s}(1, .)||_r \leq C_\delta t^{-\delta}, \qquad \forall t > 0, \tag{4.7}$$

$$||t^{\frac{p+1}{2(pq-1)}}\nabla u(t, .\sqrt{t}) - \nabla u_{\rm s}(1, .)||_r \leq C_\delta t^{-\delta}, \qquad \forall t > 0, \tag{4.8}$$

$$||t^{\frac{2-q(p-1)}{2(pq-1)}}v(t, .\sqrt{t}) - v_{\rm s}(1, .)||_s \leq C_\delta t^{-\delta}, \qquad \forall t > 0, \tag{4.9}$$

$$||t^{\frac{q+1}{2(pq-1)}}\nabla v(t, .\sqrt{t}) - \nabla v_{\rm s}(1, .)||_s \leq C_\delta t^{-\delta}, \qquad \forall t > 0. \tag{4.10}$$

Moreover, for M perhaps smaller and for all $0 < \delta < \delta_0$ there exists a positive constant C_δ such that

$$\sup_{t>0}\left[t^{\frac{\alpha}{2}+\delta}||u(t) - u_{\rm s}(t)||_\infty, t^{\frac{\alpha+1}{2}+\delta}||\nabla u(t) - \nabla u_{\rm s}(t)||_\infty\right] \leq C_\delta, \tag{4.11}$$

$$\sup_{t>0}\left[t^{\frac{\beta}{2}+\delta}||v(t)-v_{\mathrm{s}}(t)||_\infty, t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_{\mathrm{s}}(t)||_\infty\right]\leq C_\delta. \tag{4.12}$$

In particular, there exist positive constants d_1, d_2, d_1' and d_2' such that for t large we have

$$d_1 t^{-\frac{\alpha}{2}}\leq ||u(t)||_\infty\leq d_2 t^{-\frac{\alpha}{2}},\quad d_1 t^{-\frac{\alpha+1}{2}}\leq ||\nabla u(t)||_\infty\leq d_2 t^{-\frac{\alpha+1}{2}} \tag{4.13}$$

and

$$d_1' t^{-\frac{\beta}{2}}\leq ||v(t)||_\infty\leq d_2' t^{-\frac{\beta}{2}},\quad d_1' t^{-\frac{\beta+1}{2}}\leq ||\nabla v(t)||_\infty\leq d_2' t^{-\frac{\beta+1}{2}}. \tag{4.14}$$

Remark 4.3. Let r and s be two real numbers which are given by Lemma 2.3. Using interpolation argument, (4.3)–(4.4) and (4.11) we have

$$\sup_{t>0}\left[t^{\alpha_1(r')+\delta}||u(t)-u_s(t)||_{r'},\, t^{\alpha_1(r')+\frac{1}{2}+\delta}||\nabla u(t)-\nabla u_s(t)||_{r'}\right]\leq C_\delta$$

for $r'\in[r,\infty]$, where

$$\alpha_1(r')=\frac{\alpha}{2}-\frac{N}{2r'}.$$

Also, using (4.5)–(4.6) and (4.12), we have

$$\sup_{t>0}\left[t^{\beta_1(s')+\delta}||v(t)-v_s(t)||_{s'},\, t^{\beta_1(s')+\frac{1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_{s'}\right]\leq C_\delta,$$

for $s'\in[s,\infty]$, where

$$\alpha_1(s')=\frac{\beta}{2}-\frac{N}{2s'}.$$

Remark 4.4. Theorem 4.2 shows that the asymptotic behavior is stable under small perturbation by Ψ_0. In particular, one may take $\Psi_{0,1}$ and $\Psi_{0,2}$ as Gaussian functions.

Proof of Theorem 4.2. The proof of (4.3)–(4.6) is similar to the one of [20, Theorem 2.8]. By the continuous dependence of the solutions on the initial value with respect to the norm (3.42), as show in Theorem 3.4, we have

$$\begin{aligned}\sup_{t>0}\Big[\, & t^{\alpha_1+\delta}||u(t)-u_s(t)||_r,\, t^{\alpha_1+\delta+\frac{1}{2}}||\nabla u(t)-\nabla u_s(t)||_r,\\ & t^{\beta_1+\delta}||v(t)-v_s(t)||_s, t^{\beta_1+\delta+\frac{1}{2}}||\nabla v(t)-\nabla v_s(t)||_s\Big]\\ & \leq C\mathcal{N}_\delta(\Psi-\Phi)=C\mathcal{N}_\delta(\eta\Phi+\Psi_0),\end{aligned}$$

where C is a positive constant. Since for $0<\delta<\delta_0$ we have that $\mathcal{N}_\delta(\Psi_0)<\infty$ and by Theorem 4.1 (b) we have $\mathcal{N}_\delta(\eta\Phi)<\infty$, then we get

$$\sup_{t>0}\Big[\, t^{\alpha_1+\delta}||u(t)-u_s(t)||_r,\, t^{\alpha_1+\delta+\frac{1}{2}}||\nabla u(t)-\nabla u_s(t)||_r,$$

$$t^{\beta_1+\delta}||v(t)-v_s(t)||_s,\, t^{\beta_1+\delta+\frac{1}{2}}||\nabla v(t)-\nabla v_s(t)||_s\Big] \le C. \tag{4.15}$$

This gives (4.3)–(4.6) directly. Inequalities (4.7)–(4.10) results by a simple dilation argument.

Now, we will prove the asymptotically self-similar result in the $W^{1,\infty}$–norm. We argue as in [21]. Let us write

$$u(t) = e^{(t/2)\Delta}u(t/2) + a\int_{t/2}^{t} e^{(t-\sigma)\Delta}(|\nabla v(\sigma)|^p)\,\mathrm{d}\sigma,$$

$$u_s(t) = e^{(t/2)\Delta}u_s(t/2) + a\int_{t/2}^{t} e^{(t-\sigma)\Delta}(|\nabla v_s(\sigma)|^p)\,\mathrm{d}\sigma.$$

By the smoothing effect of the heat semigroup (2.2), for $0<\delta<\delta_0$ and $0<t\le T$, we obtain

$$t^{\frac{\alpha}{2}+\delta}||u(t)-u_s(t)||_\infty \le C\mathcal{H}t^{\alpha_1+\delta}||u(t)-u_s(t)||_r + |a|t^{\frac{\alpha}{2}+\delta}\int_{t/2}^{t}||e^{(t-\sigma)\Delta}\left[|\nabla v(\sigma)|^p-|\nabla v_s(\sigma)|^p\right]||_\infty d\sigma.$$

This implies, for all $0<t\le T$, using (4.15), Lemma 2.3 (vii) and the estimate (3.39), that

$$\begin{aligned} t^{\frac{\alpha}{2}+\delta}||u(t)-u_s(t)||_\infty &\le C + C(M)t^{\frac{\alpha}{2}+\delta} \\ &\quad\times\left(\int_{t/2}^{t}\sigma^{-\frac{1}{2}(\beta+1)p-\delta}\,\mathrm{d}\sigma\right)\sup_{(0,T]}\left[t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty\right] \\ &\le C + C(M)\left(\int_{1/2}^{1}\sigma^{-\frac{1}{2}(\beta+1)p-\delta}\,\mathrm{d}\sigma\right)\sup_{(0,T]}\left[t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty\right], \end{aligned}$$

which leads to

$$t^{\frac{\alpha}{2}+\delta}||u(t)-u_s(t)||_\infty \le C + C(M)\sup_{(0,T]}\left[t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty\right]. \tag{4.16}$$

Also, let us write

$$\begin{aligned} &\nabla u(t)-\nabla u_s(t) \\ &= e^{(\frac{t}{2})\Delta}\left(\nabla u(\frac{t}{2})-\nabla u_s(\frac{t}{2})\right) + a\int_{\frac{t}{2}}^{t}\nabla e^{(t-\sigma)\Delta}\left(|\nabla v(\sigma)|^p-|\nabla v_s(\sigma)|^p\right)\,\mathrm{d}\sigma. \end{aligned}$$

Then, by using (2.3), (3.39), Lemma 2.3 (vii) and (4.15) we have

$$t^{\frac{\alpha+1}{2}+\delta}||\nabla u(t)-\nabla u_s(t)||_\infty \le C\mathcal{H}t^{\alpha_1+\frac{1}{2}+\delta}||\nabla u(t)-\nabla u_s(t)||_r + |a|t^{\frac{\alpha+1}{2}+\delta}\int_{t/2}^{t}||\nabla e^{(t-\sigma)\Delta}\left[|\nabla v(\sigma)|^p-|\nabla v_s(\sigma)|^p\right]||_\infty\,\mathrm{d}\sigma$$

$$\leq C + C(M)t^{\frac{\alpha+1}{2}+\delta}\left(\int_{t/2}^{t}(t-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}(\beta+1)p-\delta}\,\mathrm{d}\sigma\right)$$
$$\times \sup_{(0,T]}\left[t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty\right]$$
$$\leq C+C(M)\left(\int_{1/2}^{1}(1-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}(\beta+1)p-\delta}\,\mathrm{d}\sigma\right)\sup_{(0,T]}\left[t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty\right].$$

Hence

$$t^{\frac{\alpha+1}{2}+\delta}||\nabla u(t)-\nabla u_s(t)||_\infty \leq C + C(M)\sup_{(0,T]}\left[t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty\right]. \quad (4.17)$$

Similarly, we get

$$t^{\frac{\beta}{2}+\delta}||v(t)-v_s(t)||_\infty \leq C + C(M)\sup_{(0,T]}\left[t^{\frac{\alpha+1}{2}+\delta}||\nabla u(t)-\nabla u_s(t)||_\infty\right] \quad (4.18)$$

and

$$t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty \leq C + C(M)\sup_{(0,T]}\left[t^{\frac{\alpha+1}{2}+\delta}||\nabla u(t)-\nabla u_s(t)||_\infty\right]. \quad (4.19)$$

Since $C(M)\searrow 0$ as $M\searrow 0$, using (4.16)–(4.19), we deduce that

$$\sup_{(0,T]}\Big[t^{\frac{\alpha}{2}+\delta}||u(t)-u_s(t)||_\infty,\ t^{\frac{\alpha+1}{2}+\delta}||\nabla u(t)-\nabla u_s(t)||_\infty,\ t^{\frac{\beta}{2}+\delta}||v(t)-v_s(t)||_\infty,$$
$$t^{\frac{\beta+1}{2}+\delta}||\nabla v(t)-\nabla v_s(t)||_\infty\Big] \leq C_\delta(M). \quad (4.20)$$

Finally, since the constant $C_\delta(M)$ appearing in (4.20) does not depend on the arbitrary real number $T>0$, we deduce the estimates (4.11)–(4.12).

Since $U_s=(u_s,v_s)$ is self-similar, then $t^{\frac{\alpha}{2}}||u_s(t)||_\infty = ||u_s(1)||_\infty > 0$. Let us denote by $C_s=||u_s(1)||_\infty$. From (3.11), $C_s<\infty$. Now, from (4.11) we have

$$t^{\alpha/2}||u(t)-u_s(t)||_\infty \leq C_\delta t^{-\delta}.$$

Since

$$t^{\alpha/2}||u(t)||_\infty = t^{\alpha/2}||u(t)-u_s(t)+u_s(t)||_\infty \leq t^{\alpha/2}||u(t)-u_s(t)||_\infty + t^{\alpha/2}||u_s(t)||_\infty,$$
$$t^{\alpha/2}||u(t)||_\infty \leq C_\delta t^{-\delta}+C_s.$$

Also, since

$$t^{\alpha/2}||u_s(t)||_\infty - t^{\alpha/2}||u(t)-u_s(t)||_\infty \leq t^{\alpha/2}||u(t)||_\infty,$$

$$C_s - C_\delta t^{-\delta} \leq t^{\alpha/2}||u(t)||_\infty.$$

Because $\delta > 0$, for large t there exist two positive constants $d_1,\ d_2$ such that

$$d_1 t^{-\alpha/2} \leq ||u(t)||_\infty \leq d_2 t^{-\alpha/2}.$$

Also, if we put $C_s' = t^{\frac{\alpha+1}{2}}||\nabla u_s(t)||_\infty = ||\nabla u_s(1)||_\infty > 0$. From (3.11), $C_s' < \infty$. Now, from (4.11) we have

$$C_s' - C_\delta t^{-\delta} \leq t^{(\alpha+1)/2}||\nabla u(t)||_\infty \leq C_\delta t^{-\delta} + C_s'.$$

Then, because $\delta > 0$, for large t we my take the same positive constants $d_1,\ d_2$ such that

$$d_1 t^{-(\alpha+1)/2} \leq ||\nabla u(t)||_\infty \leq d_2 t^{-(\alpha+1)/2}.$$

Similarly, we obtain (4.14). This ends the proof Theorem 4.2. □

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