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SCHUR CONVEXITY AND INEQUALITIES FOR A CLASS OF SYMMETRIC FUNCTIONS

Weifeng Xia[1], Gendi Wang[2], Yuming Chu[3 §]

[1]School of Teachers' Education
Huzhou Teachers College
Huzhou, 313000, P.R. CHINA
e-mail: xwf212@hutc.zj.cn

[2,3]Department of Mathematics
Huzhou Teachers College
Huzhou, 313000, P.R. CHINA
[2]e-mail: wgdi@hutc.zj.cn
[3]e-mail: chuyuming@hutc.zj.cn

Abstract: For $x = (x_1, x_2, \cdots, x_n) \in R^n_+$, the symmetric function $F_n(x, r)$ is defined by

$$F_n(x,r) = F_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} \prod_{j=1}^{r} \frac{x_{i_j}}{1 + x_{i_j}},$$

where $r = 1, 2, \cdots, n$ and $i_1, i_2, \cdots, i_n$ are positive integers.

In this article, the Schur convexity, Schur harmonic convexity and Schur multiplicative convexity of $F_n(x, r)$ are discussed. As applications, some inequalities are established by use of the theory of majorization.

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1. Introduction

In this paper, we shall adopt the notation and terminology as follows: R^n

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§Correspondence author

denotes the $n-$dimensional Euclidean space $(n \geq 2)$, $R_+^n = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$, $R = (-\infty, +\infty)$, $R_+ = (0, +\infty)$ and $N = \{1, 2, \cdots, n, \cdots\}$. For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in R^n$ and $\alpha \in R$, we denote by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n), \\ xy &= (x_1 y_1, x_2 y_2, \cdots, x_n y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \\ e^x &= (e^{x_1}, e^{x_2}, \cdots, e^{x_n}), \\ \alpha + x &= (\alpha + x_1, \alpha + x_2, \cdots, \alpha + x_n) \end{aligned}$$

and

$$\alpha - x = (\alpha - x_1, \alpha - x_2, \cdots, \alpha - x_n).$$

Moreover, we denote by

$$x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha),$$

$$\log x = (\log x_1, \log x_2, \cdots, \log x_n)$$

and

$$\frac{1}{x} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right)$$

for $x \in R_+^n$.

For $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, $r \in N$ and $r \leq n$, the Hamy symmetric function $H_n(x, r)$ is defined by T. Hara, M. Uchiyama and S. Takahasi [11] as follows:

$$H_n(x, r) = H_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \left(\prod_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}},$$

where $i_1, i_2, \cdots, i_n \in N$.

Corresponding to this is the $r-$th order Hamy mean

$$\sigma_n(x, r) = \sigma_n(x_1, x_2, \cdots, x_n; r) = \frac{1}{C_n^r} H_n(x, r),$$

where $C_n^r = \frac{n!}{(n-r)!r!}$. T. Hara, M. Uchiyama and S. Takahasi [11] established the following refinement of the classical arithmetic and geometric means inequalities:

$$G_n(x) = \sigma_n(x, n) \leq \sigma_n(x, n-1) \leq \cdots \leq \sigma_n(x, 2) \leq \sigma_n(x, 1) = A_n(x).$$

Here, $A_n(x) = \frac{1}{n} \sum_{i=1}^n x_i$ and $G_n = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$ denote the classical arithmetic and geometric means of x, respectively.

The paper [15] by H.T. Ku, M.C. Ku and X.M. Zhang contains some interesting inequalities including the fact that $(\sigma_n(x,r))^{\frac{1}{r}}$ is log-concave, the more results can be found in the book [1] by P.S. Bullen.

Recently, the Schur convexity of the Hamy symmetric function $H_n(x,r)$ was discussed and some analytic inequalities were established by K.Z. Guan [10].

In this article, we define the following new symmetric function

$$F_n(x,r) = F_n(x_1, x_2, \cdots, x_n; r) = \sum_{1\le i_1<i_2\cdots<i_r\le n} \prod_{j=1}^{r} \frac{x_{i_j}}{1+x_{i_j}}, \qquad (1.1)$$

for $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, $r \in N$ and $r \le n$. Here, $i_1, i_2, \cdots, i_n \in N$.

The main purpose of this paper is to discuss the Schur convexity, Schur harmonic convexity and Schur multiplicative convexity for the symmetric function $F_n(x,r)$. As applications, some inequalities are established by use of the theory of majorization.

Schur convex and Schur multiplicatively convex functions are defined as follows.

Definition 1.1. Let $E \subseteq R^n$ be a set, a real-valued function F on E is called a Schur convex function if

$$F(x_1, x_1, \cdots, x_n) \le F(y_1, y_2, \cdots, y_n)$$

for each pair of n-tuples $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in E, such that $x \prec y$, i.e.

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the i-th largest component of x. F is called Schur concave if $-F$ is Schur convex .

Definition 1.2. Let $E \subseteq R_+^n$ be a set, a real-valued function F on E is called a Schur multiplicatively convex function if

$$F(x_1, x_2, \cdots, x_n) \le F(y_1, y_2, \cdots, y_n)$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\log x \prec \log y$. F is called Schur multiplicatively concave if $\frac{1}{F}$ is Schur multiplicatively convex.

Next, we introduce the notion of Schur harmonic convexity.

Definition 1.3. Let $E \subseteq R_+^n$ be a set. A real-valued function F on E is called a Schur harmonic convex function if

$$F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n) \tag{1.2}$$

for each pair of $n-$tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in E, such that $\frac{1}{x} \prec \frac{1}{y}$. F is called a Schur harmonic concave function on E if inequality (1.2) is reversed.

The Schur convexity was introduced by I. Schur [16] in 1923, G.H.Hardy, J.E. Little and G. Pólya were also interested in some inequalities that are related to the Schur convexity [12]. It has many important applications in extended mean values [3], theory of statistical experiments [20], graphs and matrices [5], combinatorial optimization [13], reliability [14], gamma functions [17], information-theoretic topics [6], stochastic orderings [19] and other related fields. Recently, the Schur multiplicative convexity was introduced and investigated in paper [4, 8].

Up to now, no one has ever researched the Schur harmonic convexity. To investigate the Schur harmonic convexity for symmetric function $F_n(x, r)$ is one of the main purpose in this article.

2. Lemmas

In this section, we introduce and establish some lemmas, which are used in the proof of our main results.

Lemma 2.1. (see [16]) *Let $f : R_+^n \to R_+$ be a continuous symmetric function. If f is differentiable in R_+^n, then f is Schur convex in R_+^n if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \tag{2.1}$$

for all $i, j = 1, 2, \cdots, n$ and $x = (x_1, \cdots, x_n) \in R_+^n$. And f is Schur concave in R_+^n if and only if inequality (2.1) is reversed. Here f is a symmetric function in R_+^n which means that $f(Px) = f(x)$ for any $x \in R_+^n$ and any $n \times n$ permutation matrix P.

Remark 2.1. Since f is symmetric, the Schur's condition in Lemma 2.1, i.e. (2.1) can be reduced as

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0.$$

Lemma 2.2. (see [4, 8]) *Let $f : R_+^n \to R_+$ be a continuous symmtric function. If f is differentiable in R_+^n, then f is Schur multiplicatively convex in R_+^n if and only if*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.2}$$

for all $x = (x_1, x_2, \cdots, x_n) \in R_+^n$. And f is Schur multiplicatively concave in R_+^n if and only if inequality (2.2) is reversed.

Lemma 2.3. *Let $f : R_+^n \to R_+$ be a continuous symmetric function. If f is differentiable in R_+^n, then f is Schur harmonic convex in R_+^n if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.3}$$

for all $x = (x_1, x_2, \cdots, x_n) \in R_+^n$. And f is Schur harmonic concave in R_+^n if and only if inequality (2.3) is reversed.

Proof. From Definitions 1.1 and 1.3, we clearly see the following fact.

Fact A. *$f : R_+^n \to R_+$ is Schur harmonic convex if and only if $F(x) = \frac{1}{f(\frac{1}{x})} : R_+^n \to R_+$ is Schur concave.*

Therefore, Lemma 2.3 follows from Lemma 2.1, Remark 2.1 and Fact A together with simple computation. □

Lemma 2.4. (see [8-10]) *Let $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq s$, then*

$$\frac{c-x}{\frac{nc}{s}-1} = \left(\frac{c-x_1}{\frac{nc}{s}-1}, \frac{c-x_2}{\frac{nc}{s}-1}, \cdots, \frac{c-x_n}{\frac{nc}{s}-1}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

Lemma 2.5. (see [9]) *Let $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq 0$, then*

$$\frac{c+x}{\frac{nc}{s}+1} = \left(\frac{c+x_1}{\frac{nc}{s}+1}, \frac{c+x_2}{\frac{nc}{s}+1}, \cdots, \frac{c+x_n}{\frac{nc}{s}+1}\right) \prec (x_1, x_2, \cdots, x_n) = x.$$

3. Main Results

Theorem 3.1. *$F_n(x, r)$ is Schur concave in R_+^n.*

Proof. By Lemma 2.1 and Remark 2.1 we only need to prove that

$$(x_1 - x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1} - \frac{\partial F_n(x,r)}{\partial x_2}\right) \leq 0 \tag{3.1}$$

for all $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ and $r = 1, 2, \cdots, n$.

The proof is divided into seven cases.

Case 1. If $r = 1$, then (1.1) leads to

$$F_n(x,1) = F_n(x_1, x_2, \cdots, x_n; 1) = \sum_{i=1}^{n} \frac{x_i}{1+x_i}. \tag{3.2}$$

Elementary computation and (3.2) yield that

$$(x_1 - x_2)\left(\frac{\partial F_n(x,1)}{\partial x_1} - \frac{\partial F_n(x,1)}{\partial x_2}\right) = -\frac{(x_1-x_2)^2(2+x_1+x_2)}{(1+x_1)^2(1+x_2)^2} \le 0.$$

Case 2. If $n = 2$ and $r = 2$, then from (1.1) we clearly see that

$$F_2(x,2) = F_2(x_1, x_2; 2) = \frac{x_1 x_2}{(1+x_1)(1+x_2)}. \tag{3.3}$$

Equation (3.3) leads to

$$(x_1 - x_2)\left(\frac{\partial F_2(x,2)}{\partial x_1} - \frac{\partial F_2(x,2)}{\partial x_2}\right) = -\frac{(x_1-x_2)^2(1+x_1+x_2)}{(1+x_1)^2(1+x_2)^2} \le 0.$$

Case 3. If $n = 3$ and $r = 2$, then by (1.1) we have that

$$\begin{aligned} F_3(x,2) &= F_3(x_1, x_2, x_3; 2) \\ &= \frac{x_1}{1+x_1}\left(\frac{x_2}{1+x_2} + \frac{x_3}{1+x_3}\right) + \frac{x_2}{1+x_2}\frac{x_3}{1+x_3}. \end{aligned} \tag{3.4}$$

Equation (3.4) gives

$$\begin{aligned} &(x_1 - x_2)\left(\frac{\partial F_3(x,2)}{\partial x_1} - \frac{\partial F_3(x,2)}{\partial x_2}\right) \\ &= -\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}\left[1 + x_1 + x_2 + \frac{x_3(2+x_1+x_2)}{1+x_3}\right] \le 0. \end{aligned}$$

Case 4. If $n \ge 4$ and $r = 2$, then (1.1) leads to

$$\begin{aligned} F_n(x,2) &= F_n(x_1, x_2, \cdots, x_n; 2) \\ &= \frac{x_1}{1+x_1}\frac{x_2}{1+x_2} + \left(\frac{x_1}{1+x_1} + \frac{x_2}{1+x_2}\right)\sum_{i=3}^{n}\frac{x_i}{1+x_i} \\ &\quad + \sum_{3 \le i < j \le n} \frac{x_i x_j}{(1+x_i)(1+x_j)}. \end{aligned} \tag{3.5}$$

Elementary computation and (3.5) yield that

$$(x_1 - x_2)\left(\frac{\partial F_n(x,2)}{\partial x_1} - \frac{\partial F_n(x,2)}{\partial x_2}\right)$$

$$= -\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}\left[1+x_1+x_2+(2+x_1+x_2)\sum_{i=3}^{n}\frac{x_i}{1+x_i}\right] \le 0.$$

Case 5. If $n \ge 3$ and $r = n$, then

$$F_n(x,n) = F_n(x_1,x_2,\cdots,x_n;n) = \prod_{i=1}^{n}\frac{x_i}{1+x_i} \tag{3.6}$$

and

$$\begin{aligned}
&(x_1-x_2)\left(\frac{\partial F_n(x,n)}{\partial x_1}-\frac{\partial F_n(x,n)}{\partial x_2}\right)\\
= \quad & -\frac{(x_1-x_2)^2(1+x_1+x_2)}{x_1x_2(1+x_1)(1+x_2)}F_n(x,n) \le 0.
\end{aligned}$$

Case 6. If $n \ge 4$ and $r = n-1$, then

$$\begin{aligned}
F_n(x,n-1) &= F_n(x_1,x_2,\cdots,x_n;n-1)\\
&= \frac{x_1}{1+x_1}\frac{x_2}{1+x_2}F_{n-2}(x_3,x_4,\cdots,x_n;n-3)\\
&\quad +\left(\frac{x_1}{1+x_1}+\frac{x_2}{1+x_2}\right)\prod_{i=3}^{n}\frac{x_i}{1+x_i}
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&(x_1-x_2)\left(\frac{\partial F_n(x,n-1)}{\partial x_1}-\frac{\partial F_n(x,n-1)}{\partial x_2}\right)\\
= \quad & -\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}\Bigg[(1+x_1+x_2)F_{n-2}(x_3,x_4,\cdots,x_n;n-3)\\
&+(2+x_1+x_2)\prod_{i=3}^{n}\frac{x_i}{1+x_i}\Bigg] \le 0.
\end{aligned}$$

Case 7. If $n \ge 5$ and $3 \le r \le n-2$, then

$$\begin{aligned}
F_n(x,r) &= F_n(x_1,x_2,\cdots,x_n;r)\\
&= \frac{x_1}{1+x_1}\frac{x_2}{1+x_2}F_{n-2}(x_3,x_4,\cdots,x_n;r-2)\\
&\quad +(\frac{x_1}{1+x_1}+\frac{x_2}{1+x_2})F_{n-2}(x_3,x_4,\cdots,x_n;r-1)\\
&\quad +F_{n-2}(x_3,x_4,\cdots,x_n;r)
\end{aligned} \tag{3.8}$$

and

$$(x_1-x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\right)$$

$$= -\frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}\Big[(1+x_1+x_2)F_{n-2}(x_3,x_4,\cdots,x_n;r-2) \\ +(2+x_1+x_2)F_{n-2}(x_3,x_4,\cdots,x_n;r-1)\Big] \le 0.$$

Therefore, (3.1) follows from Cases 1-7 and the proof of Theorem 3.1 is completed. □

Theorem 3.2. *$F_n(x,r)$ is Schur harmonic convex in R^n_+.*

Proof. According to Lemma 2.3 we only need to prove that

$$(x_1-x_2)\left(x_1^2\frac{\partial F_n(x,r)}{\partial x_1} - x_2^2\frac{\partial F_n(x,r)}{\partial x_2}\right) \ge 0 \tag{3.9}$$

for all $x=(x_1,x_2,\cdots,x_n)\in R^n_+$ and $r=1,2,\cdots,n$.

The proof is divided into seven cases.

Case I. If $r=1$, then (3.2) leads to

$$(x_1-x_2)\left(x_1^2\frac{\partial F_n(x,1)}{\partial x_1} - x_2^2\frac{\partial F_n(x,1)}{\partial x_2}\right) \\ = \frac{(x_1-x_2)^2(x_1+x_2+2x_1x_2)}{(1+x_1)^2(1+x_2)^2} \ge 0.$$

Case II. If $n=2$ and $r=2$, then (3.3) yields that

$$(x_1-x_2)\left(x_1^2\frac{\partial F_2(x,2)}{\partial x_1} - x_2^2\frac{\partial F_2(x,2)}{\partial x_2}\right) \\ = \frac{(x_1-x_2)^2x_1x_2}{(1+x_1)^2(1+x_2)^2} \ge 0.$$

Case III. If $n=3$ and $r=2$, then from (3.4) we have

$$(x_1-x_2)\left(x_1^2\frac{\partial F_3(x,2)}{\partial x_1} - x_2^2\frac{\partial F_3(x,2)}{\partial x_2}\right) \\ = \frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}\left[x_1x_2+(x_1+x_2+2x_1x_2)\frac{x_3}{1+x_3}\right] \ge 0.$$

Case IV. If $n\ge 4$ and $r=2$, then (3.5) implies that

$$(x_1-x_2)\left(x_1^2\frac{\partial F_n(x,2)}{\partial x_1} - x_2^2\frac{\partial F_n(x,2)}{\partial x_2}\right) \\ = \frac{(x_1-x_2)^2}{(1+x_1)^2(1+x_2)^2}\left[x_1x_2+(x_1+x_2+2x_1x_2)\sum_{i=3}^{n}\frac{x_i}{1+x_i}\right] \ge 0.$$

Case V. If $n \geq 3$ and $r = n$, then (3.6) reveals

$$\begin{aligned}&(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x,n)}{\partial x_1} - x_2^2 \frac{\partial F_n(x,n)}{\partial x_2}\right)\\ = \ & \frac{(x_1 - x_2)^2 F_n(x,n)}{(1+x_1)(1+x_2)} \geq 0.\end{aligned}$$

Case VI. If $n \geq 4$ and $r = n-1$, then (3.7) shows that

$$\begin{aligned}&(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x,n-1)}{\partial x_1} - x_2^2 \frac{\partial F_n(x,n-1)}{\partial x_2}\right)\\ = \ & \frac{(x_1 - x_2)^2}{(1+x_1)^2(1+x_2)^2}\Bigg[x_1 x_2 F_{n-2}(x_3, x_4, \cdots, x_n; n-3)\\ & +(x_1 + x_2 + 2x_1x_2)\prod_{i=3}^{n} \frac{x_i}{1+x_i}\Bigg] \geq 0.\end{aligned}$$

Case VII. If $n \geq 5$ and $3 \leq r \leq n-2$, then (3.8) deduces that

$$\begin{aligned}&(x_1 - x_2)\left(x_1^2 \frac{\partial F_n(x,r)}{\partial x_1} - x_2^2 \frac{\partial F_n(x,r)}{\partial x_2}\right)\\ = \ & \frac{(x_1 - x_2)^2}{(1+x_1)^2(1+x_2)^2}\Big[x_1 x_2 F_{n-2}(x_3, x_4, \cdots, x_n; r-2)\\ & +(x_1 + x_2 + 2x_1x_2) F_{n-2}(x_3, x_4, \cdots, x_n; r-1)\Big] \geq 0.\end{aligned}$$

Therefore, (3.9) follows from Cases I-VII and the proof of Theorem 3.2 is completed. □

Next, we denote by $\Omega_n = \{(x_1, x_2, \cdots, x_n) \in R_+^n : x_i x_j \geq 1, i \neq j, i, j = 1, 2, \cdots, n\}$ and $B_n = \{(x_1, x_2, \cdots, x_n) \in R_+^n : x_i x_j \leq 1, i \neq j, i, j = 1, 2, \cdots, n\}$.

Theorem 3.3. *For the Schur multiplicative convexity of symmetric function $F_n(x,r)$ in R_+^n, we have*

(1) $F_n(x,n) = F_n(x_1, x_2, \cdots, x_n; n)$ is Schur multiplicatively concave in R_+^n;

(2) $F_n(x,1) = F_n(x_1, x_2, \cdots, x_n; 1)$ is Schur multiplicatively convex in B_n;

(3) If $r < n$, then $F_n(x,r) = F_n(x_1, x_2, \cdots, x_n; r)$ is Schur multiplicatively concave in Ω_n.

Proof. (1) For any $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, from (3.3) and (3.6) we get

$$\begin{aligned}&(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x,n)}{\partial x_1} - x_2 \frac{\partial F_n(x,n)}{\partial x_2}\right)\\ = \ & -\frac{(x_1 - x_2)(\log x_1 - \log x_2)}{(1+x_1)(1+x_2)} F_n(x,n) \leq 0. \qquad (3.10)\end{aligned}$$

Therefore, Theorem 3.3(1) follows from Lemma 2.2 and (3.10).

(2) For any $x = (x_1, x_2, \cdots, x_n) \in B_n$, by (3.2) we have

$$\begin{aligned}&(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x,1)}{\partial x_1} - x_2 \frac{\partial F_n(x,1)}{\partial x_2}\right)\\ =& \frac{(x_1 - x_2)(\log x_1 - \log x_2)}{(1+x_1)^2(1+x_2)^2}(1 - x_1 x_2) \geq 0. \qquad (3.11)\end{aligned}$$

Therefore, Theorem 3.3(2) follows from Lemma 2.2 and (3.11).

(3) The proof is divided into five cases.

Case A. If $r = 1$, then (3.11) yields that

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x,1)}{\partial x_1} - x_2 \frac{\partial F_n(x,1)}{\partial x_2}\right) \leq 0.$$

Case B. If $n = 3$ and $r = 2$, then (3.4) gives

$$\begin{aligned}&(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_3(x,2)}{\partial x_1} - x_2 \frac{\partial F_3(x,2)}{\partial x_2}\right)\\ &= \frac{(x_1 - x_2)(\log x_1 - \log x_2)}{(1+x_1)^2(1+x_2)^2}\left[\frac{x_3}{1+x_3}(1 - x_1 x_2) - x_1 x_2\right] \leq 0.\end{aligned}$$

Case C. If $n \geq 4$ and $r = 2$, then (3.5) leads to

$$\begin{aligned}&(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x,2)}{\partial x_1} - x_2 \frac{\partial F_n(x,2)}{\partial x_2}\right)\\ &= \frac{(x_1 - x_2)(\log x_1 - \log x_2)}{(1+x_1)^2(1+x_2)^2}\left[(1 - x_1 x_2)\sum_{i=3}^{n} \frac{x_i}{1+x_i} - x_1 x_2\right] \leq 0.\end{aligned}$$

Case D. If $n \geq 4$ and $r = n - 1$, then (3.7) implies

$$\begin{aligned}&(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x,n-1)}{\partial x_1} - x_2 \frac{\partial F_n(x,n-1)}{\partial x_2}\right)\\ &= \frac{(x_1 - x_2)(\log x_1 - \log x_2)}{(1+x_1)^2(1+x_2)^2}\left[(1 - x_1 x_2)\prod_{i=3}^{n} \frac{x_i}{1+x_i}\right.\\ &\left. - x_1 x_2 F_{n-2}(x_3, x_4, \cdots, x_n; n-3)\right] \leq 0.\end{aligned}$$

Case E. If $n \geq 5$ and $3 \leq r \leq n - 2$, then (3.8) reveals

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial F_n(x,r)}{\partial x_1} - x_2 \frac{\partial F_n(x,r)}{\partial x_2}\right)$$

$$= \frac{(x_1 - x_2)(\log x_1 - \log x_2)}{(1+x_1)^2(1+x_2)^2}\Big[(1 - x_1x_2)F_{n-2}(x_3, x_4, \cdots, x_n; r-1)$$
$$- x_1x_2F_{n-2}(x_3, x_4, \cdots, x_n; r-2)\Big] \le 0.$$

Therefore, Theorem 3.3(3) follows from Cases A-E and Lemma 2.2. □

4. Applications

In this section, we establish some inequalities by use of Theorems 3.1-3.3 and the theory of majorization.

Theorem 4.1. *Suppose that* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ *and* $\sum_{i=1}^n x_i = s$. *If* $c \ge s$ *and* $1 \le r \le n$, *then*

$$(i) \quad F_n(x, r) \le F_n\left(\frac{c-x}{\frac{nc}{s}-1}, r\right);$$

$$(ii) \quad F_n\left(\frac{1}{x}, r\right) \ge F_n\left(\frac{\frac{nc}{s}-1}{c-x}, r\right).$$

Proof. Theorem 4.1 (i) follows from Lemma 2.4 and Theorem 3.1, and Theorem 4.1 (ii) follows from Lemma 2.4 and Theorem 3.2. □

Theorem 4.2. *Suppose that* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ *and* $\sum_{i=1}^n x_i = s$. *If* $c \ge 0$ *and* $1 \le r \le n$, *then*

$$(i) \quad F_n(x, r) \le F_n\left(\frac{c+x}{\frac{nc}{s}+1}, r\right);$$

$$(ii) \quad F_n(\frac{1}{x}, r) \ge F_n\left(\frac{\frac{nc}{s}+1}{c+x}, r\right).$$

Proof. Theorem 4.2 (i) follows from Lemma 2.5 and Theorem 3.1, and Theorem 4.2 (ii) follows from Lemma 2.5 and Theorem 3.2. □

Theorem 4.3. *Suppose that* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ *and* $A_n(x) = \frac{1}{n}\sum_{i=1}^n x_i$. *If* $1 \le r \le n$, *then*

$$(i) \quad \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} \prod_{j=1}^r \frac{x_{i_j}}{1+x_{i_j}} \le C_n^r\left[\frac{A_n(x)}{1+A_n(x)}\right]^r = C_n^r\left[\frac{A_n(x)}{A_n(1+x)}\right]^r;$$

$$(ii) \quad \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} \prod_{j=1}^r \frac{1}{1+x_{i_j}} \ge \frac{C_n^r}{[1+A_n(x)]^r} = \frac{C_n^r}{[A_n(1+x)]^r}.$$

Proof. We clearly see that

$$(A_n(x), A_n(x), \cdots, A_n(x)) \prec (x_1, x_2, \cdots, x_n). \tag{4.1}$$

Therefore, Theorem 4.3 (i) follows from (4.1) and Theorem 3.1 together with (1.1), and Theorem 4.3 (ii) follows from (4.1) and Theorem 3.2 together with (1.1). □

If we take $r = 1$ and $r = n$ in Theorem 4.3, respectively, then we get the following Corollaries 4.1 and 4.2.

Corollary 4.1. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, *then*

$$(i) \quad \sum_{i=1}^{n} \frac{x_i}{1+x_i} \le \frac{nA_n(x)}{A_n(1+x)};$$

$$(ii) \quad \sum_{i=1}^{n} \frac{1}{1+x_i} \ge \frac{n}{A_n(1+x)}.$$

Corollary 4.2. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ *and* $G_n(x) = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$, *then*

$$(i) \quad \frac{G_n(x)}{G_n(1+x)} \le \frac{A_n(x)}{A_n(1+x)};$$

$$(ii) \quad A_n(1+x) \ge G_n(1+x).$$

Remark 4.1. Inequality in Corollary 4.2 (ii) is the well-known unweighted arithmetic-geometric means inequality, and the inequality in Corollary 4.2 (i) was proved by V. Govedarica and M.V. Jovanović in paper [7], here we give a new proof of this inequality which is briefer and shorter than the proof in [7].

Remark 4.2. If $\sum_{i=1}^n x_i = 1$ in Corollary 4.2 (i), then we get the Weierstrass inequality [2, p. 260]

$$\prod_{i=1}^{n} (x_i^{-1} + 1) \ge (n+1)^n.$$

Theorem 4.4. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ *and* $1 \le r \le n$, *then*

$$(i) \quad \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} \prod_{j=1}^{r} \frac{x_{i_j}}{1+x_{i_j}} \ge C_n^r \left(\frac{n}{n + \sum_{i=1}^n \frac{1}{x_i}} \right)^r;$$

$$(ii) \quad \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} \prod_{j=1}^{r} \frac{1}{1+x_{i_j}} \le C_n^r \left(\frac{\sum_{i=1}^n \frac{1}{x_i}}{n + \sum_{i=1}^n \frac{1}{x_i}} \right)^r.$$

Proof. It is easy to see that

$$\left(\frac{\sum_{i=1}^n \frac{1}{x_i}}{n}, \frac{\sum_{i=1}^n \frac{1}{x_i}}{n}, \cdots, \frac{\sum_{i=1}^n \frac{1}{x_i}}{n}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right). \tag{4.2}$$

Then, Theorem 4.4 (i) follows from (4.2), Theorem 3.2 and (1.1), and Theorem 4.4 (ii) follows from (4.2), Theorem 3.1 and (1.1). □

If we take $r = 1$ and $r = n$ in Theorem 4.4, respectively, then we get the following Corollaries 4.3 and 4.4.

Corollary 4.3. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$ *and* $H_n(x) = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$, *then*

(i) $\quad A_n\left(\frac{x}{1+x}\right) \geq \frac{H_n(x)}{1+H_n(x)}$;

(ii) $\quad A_n\left(\frac{1}{1+x}\right) \leq \frac{1}{H_n(x)+1}$.

Corollary 4.4. *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, *then*

(i) $\quad \frac{G_n(x)}{G_n(1+x)} \geq \frac{H_n(x)}{1+H_n(x)}$;

(ii) $\quad G_n(1+x) \geq 1 + H_n(x)$.

Theorem 4.5. *The following statements are true*

(i) *If* $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, *then*

$$G_n(1+x) \geq 1 + G_n(x);$$

(ii) *If* $0 < x_i \leq 1, i = 1, 2, \cdots, n$, *then*

$$A_n\left(\frac{x}{1+x}\right) \geq \frac{G_n(x)}{1+G_n(x)};$$

(iii) *If* $x_i \geq 1, i = 1, 2, \cdots, n$, *then*

$$A_n\left(\frac{x}{1+x}\right) \leq \frac{G_n(x)}{1+G_n(x)};$$

(iv) *If* $x_i \geq 1, i = 1, 2, \cdots, n$ *and* $2 \leq r \leq n$, *then*

$$\sum_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \prod_{j=1}^r \frac{x_{i_j}}{1+x_{i_j}} \leq C_n^r \left[\frac{G_n(x)}{1+G_n(x)}\right]^r.$$

Proof. (i) For $x = (x_1, x_2, \cdots, x_n) \in R_+^n$, we clearly see that

$$\log(G_n(x), G_n(x), \cdots, G_n(x)) \prec \log(x_1, x_2, \cdots, x_n). \tag{4.3}$$

Therefore, Theorem 4.5 (*i*) follows from (4.3) and Theorem 3.3 (1) together with (1.1).

(*ii*) Let $x = (x_1, x_2, \cdots, x_n)$. If $0 < x_i \leq 1, i = 1, 2, \cdots, n$, then

$$x \in B_n. \tag{4.4}$$

Theorem 4.5 (*ii*) follows from (4.3), (4.4) and Theorem 3.3 (2) together with (1.1).

(*iii*) Let $x = (x_1, x_2, \cdots, x_n)$. If $x_i \geq 1, i = 1, 2, \cdots, n$, then

$$x \in \Omega_n. \tag{4.5}$$

Then (4.3) and (4.5) together with Theorem 3.3 (3) yields that

$$F_n(x_1, x_2, \cdots, x_n; 1) \leq F_n(G_n(x), G_n(x), \cdots, G_n(x); 1). \tag{4.6}$$

Therefore, Theorem 4.5 (*iii*) follows from (4.6) and (1.1).

(*iv*) Theorem 4.5 (*iv*) follows from (4.3), (4.5), Theorem 3.3 (3) and (1.1). □

Theorem 4.6. *If $x = (x_1, x_2, \cdots, x_n) \in R^n_+$, then*

(*i*) $$\prod_{i=1}^{n} \frac{1 + x_i}{2 + x_i} \geq \left(\frac{1}{2}\right)^{n-1} \frac{1 + \sum_{i=1}^{n} x_i}{2 + \sum_{i=1}^{n} x_i};$$

(*ii*) $$\sum_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{2 + x_{i_j}} \geq \left(\frac{1}{2}\right)^{r-1} \frac{1 + \sum_{i=1}^{n} x_i}{2 + \sum_{i=1}^{n} x_i} C_{n-1}^{r-1} + \left(\frac{1}{2}\right)^{r} C_{n-1}^{r} \quad \textit{for} \quad 1 \leq r \leq n-1;$$

(*iii*) $$\prod_{i=1}^{n} \frac{1}{2 + x_i} \leq \left(\frac{1}{2}\right)^{n-1} \frac{1}{2 + \sum_{i=1}^{n} x_i};$$

(*iv*) $$\sum_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \prod_{j=1}^{r} \frac{1}{2 + x_{i_j}} \leq \left(\frac{1}{2}\right)^{r-1} \frac{C_{n-1}^{r-1}}{2 + \sum_{i=1}^{n} x_i} + \left(\frac{1}{2}\right)^{r} C_{n-1}^{r} \quad \textit{for} \quad 1 \leq r \leq n-1.$$

Proof. Theorem 4.6 follows from Theorem 3.1, Theorem 3.2 and (1.1) together with the fact that

$$(1 + x_1, 1 + x_2, \cdots, 1 + x_n) \prec \left(1 + \sum_{i=1}^{n} x_i, 1, 1, \cdots, 1\right). \qquad \square$$

Theorem 4.7. *Let $\mathcal{A} = A_1 A_2 \cdots A_{n+1}$ be a $n-$dimensional simplex in*

R^n and P be an arbitrary point in the interior of $\mathcal{A}$. If B_i is the intersection point of straight line A_iP and hyperplane $\sum_i = A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}, i = 1,2,\cdots,n+1$. Then for $r \in \{1,2,\cdots,n+1\}$ we have

$$(i)\qquad \sum_{1\le i_1<i_2\cdots<i_r\le n+1}\prod_{j=1}^{r}\frac{PB_{i_j}}{A_{i_j}B_{i_j}+PB_{i_j}} \le \frac{C_{n+1}^r}{(n+2)^r};$$

$$(ii)\qquad \sum_{1\le i_1<i_2\cdots<i_r\le n+1}\prod_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{A_{i_j}B_{i_j}+PB_{i_j}} \ge C_{n+1}^r\left(\frac{n+1}{n+2}\right)^r;$$

$$(iii)\qquad \sum_{1\le i_1<i_2\cdots<i_r\le n+1}\prod_{j=1}^{r}\frac{PA_{i_j}}{A_{i_j}B_{i_j}+PA_{i_j}} \le C_{n+1}^r\left(\frac{n}{2n+1}\right)^r;$$

$$(iv)\qquad \sum_{1\le i_1<i_2\cdots<i_r\le n+1}\prod_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{A_{i_j}B_{i_j}+PA_{i_j}} \ge C_{n+1}^r\left(\frac{n+1}{2n+1}\right)^r.$$

Proof. It is easy to see that $\sum_{i=1}^{n+1}\frac{PB_i}{A_iB_i} = 1$ and $\sum_{i=1}^{n+1}\frac{PA_i}{A_iB_i} = n$, these imply that

$$\left(\frac{1}{n+1},\frac{1}{n+1},\cdots,\frac{1}{n+1}\right) \prec \left(\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2},\cdots,\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\right) \tag{4.7}$$

and

$$\left(\frac{n}{n+1},\frac{n}{n+1},\cdots,\frac{n}{n+1}\right) \prec \left(\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2},\cdots,\frac{PA_{n+1}}{A_{n+1}B_{n+1}}\right). \tag{4.8}$$

Therefore, Theorem 4.7 follows from (4.7), (4.8), Theorem 3.1, Theorem 3.2 and (1.1).

Remark 4.3. D.S. Mitrinović, J.E. Pečarić and V. Volenec [18, p. 473-479] established a series of inequalities for $\frac{PA_i}{A_iB_i}$ and $\frac{PB_i}{A_iB_i}$, $i = 1,2,\cdots,n+1$. Obviously, our inequalities in Theorem 4.7 are different from theirs.

Theorem 4.8. *Suppose that $A \in M_n(C)(n \ge 2)$ is a complex matrix, $\lambda_1,\lambda_2,\cdots,\lambda_n$ and $\sigma_1,\sigma_2,\cdots,\sigma_n$ are the eigenvalues and the singular values of A, respectively. If A is a positive Hermitian matrix, then*

$$(i)\qquad \sum_{1\le i_1<i_2\cdots<i_r\le n}\prod_{j=1}^{r}\frac{\lambda_{i_j}}{1+\lambda_{i_j}} \le C_n^r\left(\frac{\operatorname{tr}A}{n+\operatorname{tr}A}\right)^r \quad \text{for} \quad 1\le r\le n;$$

$$(ii)\qquad \sum_{1\le i_1<i_2\cdots<i_r\le n}\prod_{j=1}^{r}\frac{1}{1+\lambda_{i_j}} \ge C_n^r\left(\frac{n}{n+\operatorname{tr}A}\right)^r \quad \text{for} \quad 1\le r\le n;$$

$$(iii)\qquad \sum_{1\le i_1<i_2\cdots<i_r\le n}\prod_{j=1}^{r}\frac{1+\lambda_{i_j}}{2+\lambda_{i_j}} \le C_n^r\left(\frac{\sqrt[n]{\det(I+A)}}{1+\sqrt[n]{\det(I+A)}}\right)^r$$

$$for \quad 1 \le r \le n;$$

$$(iv) \quad \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \operatorname{tr} A} \ge \frac{n\sqrt[n]{\det A}}{\operatorname{tr} A + \sqrt[n]{\det A}};$$

$$(v) \quad \prod_{i=1}^{n}(1+\lambda_i) \le \prod_{i=1}^{n}(1+\sigma_i);$$

$$(vi) \quad \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n}\sigma_i} \le \sum_{i=1}^{n} \frac{\sigma_i}{\sigma_i + \sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n}\sigma_i}.$$

Proof. $(i)-(ii)$ We clearly see that $\lambda_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \lambda_i = \operatorname{tr} A$, which leads to

$$\left(\frac{\operatorname{tr} A}{n}, \frac{\operatorname{tr} A}{n}, \cdots, \frac{\operatorname{tr} A}{n}\right) \prec (\lambda_1, \lambda_2, \cdots, \lambda_n). \tag{4.9}$$

Therefore, Theorem 4.8 (i) and (ii) follow from (4.9), Theorem 3.1, Theorem 3.2 and (1.1).

(iii) It is easy to see that $1+\lambda_1, 1+\lambda_2, \cdots, 1+\lambda_n$ are the eigenvalues of matrix $I + A$ and $\prod_{i=1}^{n}(1+\lambda_i) = \det(I+A)$, these yield that

$$\begin{aligned}&\log\left(\sqrt[n]{\det(I+A)}, \sqrt[n]{\det(I+A)}, \cdots, \sqrt[n]{\det(I+A)}\right) \\ &\prec \log(1+\lambda_1, 1+\lambda_2, \cdots, 1+\lambda_n).\end{aligned} \tag{4.10}$$

Therefore, Theorem 4.8 (iii) follows from (4.10), Theorem 3.3(1) and (3) together with (1.1).

(iv) Theorem 4.8(iv) follows from (1.1), Theorem 3.3(2) and the fact that

$$\frac{\lambda_i}{\operatorname{tr} A} < 1, i = 1, 2, \cdots, n$$

and

$$\log\left(\frac{\sqrt[n]{\det A}}{\operatorname{tr} A}, \frac{\sqrt[n]{\det A}}{\operatorname{tr} A}, \cdots, \frac{\sqrt[n]{\det A}}{\operatorname{tr} A}\right) \prec \log\left(\frac{\lambda_1}{\operatorname{tr} A}, \frac{\lambda_2}{\operatorname{tr} A}, \cdots, \frac{\lambda_n}{\operatorname{tr} A}\right).$$

(v) A result due to H. Weyl [21] (see also [16, p. 231]) gives

$$\log(\lambda_1, \lambda_2, \cdots, \lambda_n) \prec \log(\sigma_1, \sigma_2, \cdots, \sigma_n). \tag{4.11}$$

Therefore, Theorem 4.8 (v) follows from (4.11), Theorem 3.3(1) and (1.1).

(vi) Theorem 4.8(vi) follows from Theorem 3.3(2) and (1.1) together with the fact that

$$\log\left(\frac{\lambda_1}{\sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n}\sigma_i}, \frac{\lambda_2}{\sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n}\sigma_i}, \cdots, \frac{\lambda_n}{\sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n}\sigma_i}\right)$$

$$\prec \log\left(\frac{\sigma_1}{\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \sigma_i}, \frac{\sigma_2}{\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \sigma_i}, \cdots, \frac{\sigma_n}{\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \sigma_i}\right). \quad \square$$

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