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OPTIMAL CONTROL AND HAMILTON-POISSON FORMALISM

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Abstract: In this paper we are concerned with the extremal curves of the Hamilton-Poisson dynamical systems associated with a certain class of invariant optimal control problems on matrix Lie groups. Explicit computations are done in a special case.

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1. Introduction

A wide range of dynamical systems from fields as diverse as mechanics, electrical networks, molecular chemistry, and computer science can be modelled by invariant systems on matrix Lie groups (see, e.g. [2], [5], [6]). Invariant control systems on Lie groups were first considered by Brockett [1] and by Jurdjevic and Sussmann [3].

We identify a class of left-invariant optimal control problems $(\mathcal{P})$ on some matrix Lie group G and then define – via the Pontryagin's maximum principle – the appropriate Hamiltonian H on the dual $\mathfrak{g}^*$ of the Lie algebra of G. Elements of Hamilton-Poisson formalism are then used to describe normal extremal curves. Explicit computations, in terms of elliptic functions, are done in the special case of the rotation group $\mathsf{SO}\,(3)$.

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2. Control Systems on Matrix Lie Groups

A (real) *matrix Lie group* is a closed subgroup G of some general linear group $\mathsf{GL}(n,\mathbb{R})$. It is then known that G is a smooth embedded submanifold of the matrix space $\mathbb{R}^{n\times n}$ of all $n\times n$ matrices over $\mathbb{R}$. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Both the tangent and the cotangent bundle of G can be trivialized by the left translations. $T\mathsf{G}$ will be identified with $\mathsf{G}\times\mathfrak{g}$, whereas $T^*\mathsf{G}$ will be identified with $\mathsf{G}\times\mathfrak{g}^*$. More precisely, $gA\in T_g\mathsf{G}$ is identified with $(g,A)\in\mathsf{G}\times\mathfrak{g}$ and $\xi\in T_g^*\mathsf{G}$ is identified with $(g,p)\in\mathsf{G}\times\mathfrak{g}^*$ by $p=dL_g^*(\xi)\,:\,p(A)=\xi(gA)$ for $A\in\mathfrak{g}$. Each $A\in\mathfrak{g}$ defines a smooth function H_A on $T^*\mathsf{G}$ by $H_A(\xi)=\xi(gA)=p(A)$. H_A is left-invariant on $\mathsf{G}\times\mathfrak{g}^*$ and so is (identified with) a linear function on $\mathfrak{g}^*$. The canonical symplectic form ω on $T^*\mathsf{G}$ sets up a correspondence between functions H on $T^*\mathsf{G}$ and vector fields $\vec{H}$ on $T^*\mathsf{G}$ given by $\omega_{(g,p)}\left(\vec{H}(g,p),V\right)=dH(g,p)\cdot V$ for all tangent vectors at (g,p). In the left-invariant realization of $T^*\mathsf{G}$, the Hamiltonian vector field $\vec{H}$ of a left-invariant H is given by (X,Y^*) with $X(p)=dH(p)$ and $Y^*(p)=\mathrm{ad}^*_{dH(p)}p$ (ad^* denotes the coadjoint representation of $\mathfrak{g}$).

An arbitrary left-invariant control system Σ, evolving on some matrix Lie group G, is written

$$\dot{g}=\Xi\,(g,u),\quad g\in\mathsf{G},\ u\in U, \tag{1}$$

with the dynamics $\Xi:\mathsf{G}\times U\to T\mathsf{G}$ such that $\Xi\,(gh,u)=g\,\Xi\,(g,u)$ for all $g,h\in\mathsf{G}$ and $u\in U$. Typically, the control set U is a subset of some Euclidean space $\mathbb{R}^\ell$, $1\le\ell\le\dim\mathsf{G}$. We shall assume that $U=\mathbb{R}^\ell$. We have $\Xi\,(g,u)=g\,\Xi\,(1,u)\in T_g\mathsf{G}$, where the parametrization $\Xi\,(1,\cdot):\mathbb{R}^\ell\to\mathfrak{g}$ is smooth. The dynamics Ξ is completely determined by the parametrized set $\Gamma=\mathrm{im}\,\Xi\,(1,\cdot)$. So a *left-invariant control system* Σ is (defined by) a pair (G,Γ), where $\mathsf{G}\le\mathsf{GL}(n,\mathbb{R})$ is a matrix Lie group and Γ is a parametrized subset of the Lie algebra $\mathfrak{g}$.

An *admissible control* is a map $u(\cdot):[0,T(u)]\to\mathbb{R}^\ell$ that is bounded and Lebesgue measurable on $[0,T(u)]$. The class of admissible controls, denoted by $\mathcal{U}$, is tacitly assumed to be fixed, once for all. If $u(\cdot)\in\mathcal{U}$, a *trajectory* of Σ corresponding to $u(\cdot)$ is an absolutely continuous curve $g(\cdot):[0,T(u)]\to\mathsf{G}$ such that $\dot{g}(t)=g(t)\,\Xi\,(g(t),u(t))$ for a.e. $t\in[0,T(u)]$. A *controlled trajectory* of Σ is a pair $(g(\cdot),u(\cdot))$, where $u(\cdot)\in\mathcal{U}$ and $g(\cdot)$ is a trajectory of Σ corresponding to $u(\cdot)$. In many cases, a trajectory cannot arise from more than one admissible control, so it is not necessary to distinguish between "trajectories" and "controlled trajectories". Carathéodory's Existence and Uniqueness

Theorem guarantees that, for any $u(\cdot) \in \mathcal{U}$ and any $g_0 \in \mathsf{G}$, there exists a unique trajectory $g(\cdot)$, defined on a maximal interval $I \subseteq [0, T(u)]$, such that $g(0) = g_0$. We make the assumption that the solution curve actually exists on the whole interval $[0, T(u)]$.

3. Elements of Hamilton-Poisson Formalism

A *Hamilton-Poisson dynamical system* is a triplet $(M, \{\cdot,\cdot\}, H)$, where $(M, \{\cdot, \cdot\})$ is a Poisson manifold and H is a smooth function on M. The *Hamiltonian vector field* $\vec{H}$ is defined by $\vec{H}[F] = \{F, H\}$ for every $F \in C^\infty(M)$. The bivector field $\Lambda \in \Gamma(\wedge^2 TM)$ such that $\{F, G\} = \wedge(dF, dG)$ for every $F, G \in C^\infty(M)$ induces a vector-bundle map $\wedge^\sharp : T^* \to TM$. A nondegenerate Poisson structure (i.e., for which the rank of $\wedge^\sharp$ is everywhere equal to the dimension of M) is equivalent to that of a symplectic structure. The subset $\wedge^\sharp(T^*M)$ of TM is a completely integrable generalized distribution whose leaves are symplectic.

If (φ^t) is the flow of $\vec{H}$, then $\frac{d}{dt}(F \circ \varphi^t) = \{F, H\} \circ \varphi^t = \{F \circ \varphi^t, H\}$. For short, $\dot{F} = \{F, H\}$ for any $F \in C^\infty(M)$.

$\mathfrak{g}^*$ has a natural Poisson structure, called the *"minus Lie-Poisson structure"* and given by $\{F, G\}_-(p) = -p\left([dF(p), dG(p)]\right)$ for all $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$ ($dF(p)$ is a linear function on $\mathfrak{g}^*$ and hence is an element of $\mathfrak{g}$). The (minus) Lie-Poisson bracket on $\mathfrak{g}^*$ can be derived from the canonical Poisson structure on the cotangent bundle $T^*\mathsf{G}$ by the Poisson reduction. The Poisson manifold $(\mathfrak{g}^*, \{\cdot,\cdot\}_-)$ will be denoted by $\mathfrak{g}^*_-$. If $(E_k)_{1\leq k\leq m}$ is a basis for $\mathfrak{g}$, the structure constants C^k_{ij} are obtained from $[E_i, E_j] = \sum_{k=1}^m C^k_{ij} E_k$. Any $p \in \mathfrak{g}^*$ can be expressed uniquely as $p = \sum_{k=1}^m p_k E^\flat_k$, where $\left(E^\flat_k\right)_{1\leq k\leq m}$ is the basis of $\mathfrak{g}^*$ dual to $(E_k)_{1\leq k\leq m}$. Then the Lie-Poisson bracket becomes $\{F, G\}_-(p) = -\sum_{i,j,k=1}^m C^k_{ij} p_k \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}$. The symplectic leaves of (the Poisson manifold) $\mathfrak{g}^*_-$ are exactly the coadjoint orbits of G. Each left-invariant Hamiltonian $\mathcal{H} : T^*\mathsf{G} \to \mathbb{R}$ is identified with its reduction $H : \mathfrak{g}^*_- \to \mathbb{R}$. The integral curves $(g(\cdot), p(\cdot))$ of any such Hamiltonian vector field satisfy the Hamilton's equations $\dot{g} = g\, dH(p)$ and $\dot{p} = \mathrm{ad}^*_{dH(p)} p$. In coordinates, the *reduced Hamilton's equations* become

$$\dot{p}_i = \{p_i, H\}_- = -\sum_{j,k=1}^m C^k_{ij} p_k \frac{\partial H}{\partial p_j}, \quad i = 1, \ldots, m. \tag{2}$$

On semisimple matrix Lie groups, the Killing form sets up a correspondence between $\mathfrak{g}$ and $\mathfrak{g}^*$. If we denote by P the element in $\mathfrak{g}$ which corresponds to p, then equations (2) take the dual form, known as the *Lax-pair formulation*: $\dot{P} = [P, dH(p)]$.

4. A Class of Optimal Control Problems

We consider an *optimal control problem* $\mathcal{P} = (\Sigma, L, \delta)$ given by the specification of (i) a left-invariant control system $\Sigma = (\mathsf{G}, \Gamma)$, (ii) a smooth function $L : \mathbb{R}^\ell \to \mathbb{R}$, called the Lagrangian, and (iii) the boundary data $\delta = (g_0, g_1, T) \in \mathsf{G} \times \mathsf{G} \times]0, \infty[$, consisting of an initial state g_0, a target state g_1, and the final time T. More explicitly, we want to minimize a cost functional $\mathcal{J} = \int_0^T L(u(t))\, dt$ over the controlled trajectories of a left-invariant control system

$$\dot{g} = g\, \Xi\, (1, u), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{3}$$

subject to the boundary conditions $g(0) = g_0$ and $g(T) = g_1$. The final time $T > 0$ is fixed in advance. Pontryagin's Maximum Principle is a necessary condition of optimality expressed most naturally in the language of the geometry of the cotangent bundle of G. There is a Hamiltonian function $\mathcal{H}^\lambda$ on $T^*\mathsf{G}$ given by $\mathcal{H}^\lambda(\xi, u(t)) = \lambda L(u(t)) + \xi\, (g\, \Xi\, (1, u(t))$, where λ is a parameter that can be either equal to 0 or -1.

The Maximum Principle. (see [2]) *Suppose the controlled trajectory* $(\bar{g}(\cdot), \bar{u}(\cdot))$ *defined on the interval* $[0, T]$ *is optimal. Then* $(\bar{g}(\cdot), \bar{u}(\cdot))$ *is the projection of an integral curve* $(\bar{\xi}(\cdot), \bar{u}(\cdot))$ *of the Hamiltonian vector field* $\vec{\mathcal{H}}^\lambda(\xi, u)$ *on the interval* $[0, T]$ *such that:*

(MP1) If $\lambda = 0$, *then* $\bar{\xi}(t) \neq 0$ *for any* $t \in [0, T]$.

(MP2) The time-varying Hamiltonian $\mathcal{H}^\lambda(\bar{\xi}(t), \bar{u}(t))$ *satisfies the following maximality condition*

$$\mathcal{H}^\lambda(\bar{\xi}(t), \bar{u}(t)) \leq \mathcal{H}^\lambda(\bar{\xi}(t), u)$$

for any $u \in \mathbb{R}^\ell$ *and a.e. in* $[0, T]$.

Integral curves $(\xi(\cdot), u(\cdot))$ that satisfy conditions (MP1) and (MP2) are called *extremals*. An extremal curve is called *normal* if $\lambda = -1$ and *abnormal* if $\lambda = 0$. We shall be concerned only with normal extremals.

We limit ourselves to a certain class $(\mathcal{P})$ of left-invariant optimal control problems by making two additional assumptions:

Assumption 1. The parametrized set $\Gamma = \operatorname{im} \Xi\, (1, \cdot) \subseteq \mathfrak{g}$ is an affine subspace of dimension ℓ. So

$$\Gamma = \left\{ A_0 + u_1 A_1 + \cdots + u_\ell A_\ell \,|\, u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell \right\}. \tag{4}$$

Assumption 2. The Lagrangian L has the form

$$L(u) = \frac{1}{2}\left(c_1 u_1^2 + \cdots + c_\ell u_\ell^2\right); \quad c_1, \ldots, c_\ell > 0. \tag{5}$$

To any such invariant optimal control problem there corresponds a Hamilton-Poisson system $(\mathfrak{g}_-^*, H)$, whose integral curves are closely related to the solution curves. The following result holds (see [4]).

Theorem 1. *For an optimal control problem $\mathcal{P} = (\Sigma, L, \delta)$ every normal extremal $(\bar{\xi}(\cdot) = (\bar{g}(\cdot), \bar{p}(\cdot)), \bar{u}(\cdot))$ is such that*

$$\bar{u}_i(t) = \frac{1}{c_i} p(t)(A_i), \quad i = 1, \ldots, \ell,$$

where the components of $\bar{p}(\cdot) = \sum_{k=1}^m \bar{p}_k(\cdot) E_k^\flat : [0,T] \to \mathfrak{g}_-^*$ are solutions of the equations (2) corresponding to the reduced Hamiltonian H given by $H(p) = p(A_0) + \frac{1}{2}\sum_{i=1}^\ell \frac{1}{c_i} p(A_i)^2$ for $p \in \mathfrak{g}_-^*$.

Note that the differential equation $\dot{p} = \mathrm{ad}_{dH(p)}^* p$ implies $p(t) = \mathrm{Ad}_{g(t)}^* p(0)$ for some $p(0) \in \mathfrak{g}_-^*$ with Ad^* equal to the coadjoint action of G on $\mathfrak{g}_-^*$. Hence $\bar{p}(\cdot)$ is contained in the coadjoint orbit of G through $p(0)$.

5. Some Explicit Computations

Consider the rotation group $\mathsf{SO}(3) \leq \mathsf{GL}(3, \mathbb{R})$ with associated Lie algebra $\mathfrak{so}(3)$. We will identify $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ via the pairing

$$\left\langle \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Then each extremal curve $p(\cdot)$ is identified with a curve $P(\cdot)$ in $\mathfrak{so}(3)$ via the formula $\langle P(t), A \rangle = p(t)(A)$ for all $A \in \mathfrak{so}(3)$. Thus

$$P(t) = \begin{bmatrix} 0 & -P_3(t) & P_2(t) \\ P_3(t) & 0 & -P_1(t) \\ -P_2(t) & P_1(t) & 0 \end{bmatrix},$$

where $P_i(t) = \langle P(t), E_i \rangle = p(t)(E_i)$, $i = 1, 2, 3$ (E_1, E_2, E_3 form the standard basis).

Consider the problem of minimizing $\frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + c_2 u_2^2(t)\right) dt$ subject to

$\dot{g} = g\,(u_1 E_1 + u_2 E_2)\,,\ g \in \mathsf{SO}\,(3)$ and the boundary conditions $g(0) = g_0$ and $g(T) = g_1$. Here $c_1, c_2 > 0,\ E_1, E_2$ are members of the standard basis of $\mathfrak{so}\,(3),\ u = (u_1, u_2) \in \mathbb{R}^2$, and $T > 0$ is fixed. The reduced Hamiltonian is $H = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2$. So the extremal controls are given by $\bar{u}_1 = \frac{1}{c_1}P_1$ and $\bar{u}_2 = \frac{1}{c_2}P_2$, where the functions $P_i : [0, T] \to \mathbb{R}$ are solutions of the system

$$\dot{P}_1 = -\frac{1}{c_2}P_2 P_3, \tag{6}$$

$$\dot{P}_2 = \frac{1}{c_1}P_1 P_3, \tag{7}$$

$$\dot{P}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1}\right) P_1 P_2. \tag{8}$$

The extremal trajectories are the intersections of the elliptic cylinders $\frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2 = 2H$ and the spheres $P_1^2 + P_2^2 + P_3^2 = 2K$.

The reduced Hamilton's equations (6)-(8) may be explicitly integrated by *elliptic functions*. For instance, if $0 < c_1 < c_2$ and $(c_2 - c_1)P_1^2(0) < c_1 P_3^2(0)$, then

$$P_3(t) = \sqrt{2(K - Hc_2)} \cdot \mathrm{nd}\left(\sqrt{\frac{2(K - Hc_1)}{c_1 c_2}}\, t,\ \sqrt{\frac{H(c_2 - c_1)}{K - Hc_1}}\right),$$

where nd denotes the reciprocal of the Jacobi elliptic function dn.

6. Final Remark

An extensive and systematic study of the class of optimal control problems identified in this paper, particularly on specific matrix Lie groups of low dimension, will be carried out in future works.

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