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TRANSLATION OF ABSOLUTE EVOLUTION LAWS OF A CONTINUOUS SYSTEM IN ONE COMOVING PHYSICAL FRAME OF REFERENCE. SOME EXACT SOLUTIONS

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Abstract: In this paper I am going to expose some results attained by me and some collaborators of mine on the evolution laws of a continuous system in one comoving physical frame of reference with the purpose of pointing out which advantages we have when we translate the absolute laws expressed in one comoving physical frame of reference, since the physicists work and establish the physical laws in their laboratory, which is a comoving physical frame of reference, by its evolution, as we are going to see soon.

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1. Choice of one Frame of Reference

Let us consider a 4–dimensional normal hyperbolic Riemannian manifold V_4 with the first fundamental form given by the regular evolution of a continuous system $\mathcal{S}$

$$ds^2 = g_{hk}\, dx^h dx^k \ , \tag{1}$$

where x^k are local coordinates or Eulerian coordinates and $+++-$ its signature.

Moreover, let $\mathcal{C}^*$ be the reference configuration with material coordinates or Lagrangian coordinates $y^1, y^2, y^3, y^4 = x^4 = 0$ and $\mathcal{C}$ the variable configuration with equation

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$$x^4 = \text{const} . \tag{2}$$

Every material point $P \in \mathcal{C}$ has the eulerian coordinates

$$P \equiv (x^1, x^2, x^3, x^4 = y^4) , \tag{3}$$

and the material coordinates

$$P \equiv (y^1, y^2, y^3, y^4) , \tag{4}$$

since a homeomorphism exists among the 3–manifolds $\mathcal{C}^*$ and every $\mathcal{C}$; consequently the relation

$$\det\|\frac{\partial x^h}{\partial y^k}\| \neq 0 \tag{5}$$

is locally satisfied.

Let us now consider the stream lines of the material points P of the system $\mathcal{S}$ as the time–like congruence $\{L\}$ that we will employ, from now in the future, as the physical frame of reference, and let the vector $\mathbf{u}$ be tangent to the stream lines, set towards the future, with the norm

$$\|\mathbf{u}\| = -1$$

and the controvariant eulerian components

$$u^\rho(x) = \frac{\partial_4 x^\rho}{\sqrt{-\|\partial_4 P\|}} \quad (\rho = 1, 2, 3), \quad u^4(x) = \frac{1}{\sqrt{-\|\partial_4 P\|}} ,$$

where ∂_4 denotes the partial derivative with respect to y^4 $\left(\partial_4 \equiv \frac{\partial}{\partial y^4}\right)$.

Besides we put

$$\begin{cases} \overset{*}{g}_{hk}(y) = g_{rs}(x) \dfrac{\partial x^r}{\partial y^h} \dfrac{\partial x^s}{\partial y^k}, \\ \overset{*}{u}{}^\rho(y) = u^k(x) \dfrac{\partial y^\rho}{\partial x^k} = \dfrac{\delta_4^\rho}{\sqrt{-\|\partial_4 P\|}} = 0, \\ \overset{*}{u}{}^4(y) = u^k(x) \dfrac{\partial y^4}{\partial x^k} = \dfrac{1}{\sqrt{-\|\partial_4 P\|}}, \end{cases} \tag{6}$$

to indicate the image of the tensor fields $g_{hk}(x)$, $u^k(x)$ on the fixed reference configuration $\mathcal{C}^*$. Moreover we introduce the vector space T_P tangent, at every point P, to V_4 as product of two orthogonal subspaces Σ_P, Θ_P:

$$T = \Theta_P \times \Sigma_P, \tag{7}$$

where Θ_P is 1–dimensional like–time space, tangent to the stream line for P, while Σ_P a 3–dimensional space orthogonal to Θ_P at P [4].

The tensor field

$$\gamma_{hr} \equiv \mathcal{P}_{\Sigma\Sigma}(g_{hr}) = g_{hr} + u_h u_r \quad (\gamma_{4r} \equiv 0) , \tag{8}$$

obtained by means two projections on the 3–space Σ_P, is the metric tensor of Σ_P or the space projector, or the space metric tensor; in addition

$$\overset{*}{\gamma}_{hr}(y) = \overset{*}{g}_{hr}(y) + \overset{*}{u}_h \overset{*}{u}_r \qquad (\overset{*}{\gamma}_{4r} \equiv 0) \tag{9}$$

are its Lagrangian covariant components.

Let us remark that the Lagrangian coordinates y^k satisfy the following conditions:

$$y^\rho = \text{const.}\,, \quad y^4 = \text{variable on every stream line}\,; \tag{10}$$

therefore we will say the Lagrangian coordinates are adapted to the congruence $\{L\}$.

After that we will pose

$$m_{\alpha\beta}(y^1, y^2, y^3) \equiv \overset{*}{\gamma}_{\alpha\beta}(y^1, y^2, y^3, 0) \quad \alpha, \beta = 1, 2, 3\,; \tag{11}$$

namely m_{hr} is the metric tensor in the reference configuration $\mathcal{C}^*$. That put before, we will call *local deformation tensor* $\overset{*}{\varepsilon}_{hr}(y)$ of the system $\mathcal{S}$ the symmetric space tensor [5]

$$\overset{*}{\varepsilon}_{hr} = \frac{1}{2}\left(\overset{*}{\gamma}_{hr} - \delta_h^\alpha \, \delta_r^\beta \, m_{\alpha\beta}\right)\,. \tag{12}$$

Let us point out that the congruence $\{L\}$ is the family of trajectories of a one–parameter transformation group of which the vector field $\mathbf{u}$ is the generator field. According to this idea, by the operator $\mathcal{L}_{\mathbf{u}}$ (Lie derivative) we have:

$$\begin{cases} \mathcal{L}_{\mathbf{u}} g_{hr} = \nabla_h u_r + \nabla_r u_h \equiv K_{hr} \quad \text{(Killing tensor)}, \\ \mathcal{L}_{\mathbf{u}} \gamma_{hr} = \mathcal{P}_{\Sigma\Sigma}(K_{hr}) \equiv \widetilde{K}_{hr} \quad \text{(the rate of deformation tensor)}, \\ \mathcal{L}_{\mathbf{u}} \varepsilon_{hr} = \frac{1}{2}\widetilde{K}_{hr}, \end{cases} \tag{13}$$

and analogously, as image on $\mathcal{C}^*$,

$$\begin{cases} \mathcal{L}_{\mathbf{u}} \overset{*}{g}_{hr} = \overset{*}{\nabla}_h \overset{*}{u}_r + \overset{*}{\nabla}_r \overset{*}{u}_h, \\ \mathcal{L}_{\mathbf{u}} \overset{*}{\gamma}_{hr} = \overset{*}{u}{}^4 \, \partial_4 \overset{*}{\gamma}_{hr}, \\ \mathcal{L}_{\mathbf{u}} \overset{*}{\varepsilon}_{hr} = \overset{*}{u}{}^4 \, \partial_4 \overset{*}{\varepsilon}_{hr} = \frac{1}{2} \mathcal{L}_{\mathbf{u}} \overset{*}{\gamma}_{hr}\,. \end{cases} \tag{14}$$

2. Conservation Law of Pure Matter. Stress Tensor. Reversible Systems and Energy Momentum Tensor

The conservation law of pure matter has the following eulerian form:

$$\nabla_h(\mu\, u^h) = 0, \tag{15}$$

where μ is the pure matter density; while if $\overset{*}{\mu}$ is the pure matter density of $\mathcal{S}$ in the reference configuration $\mathcal{C}^*$, we can write also the lagrangian form

$$\overset{*}{\mu} = \mu\sqrt{\frac{\overset{*}{\gamma}}{m}} \ , \quad m = \det\|m_{\rho\sigma}\| \ , \quad \overset{*}{\gamma} = \det\| \overset{*}{\gamma}_{\rho\sigma} \| \ . \tag{16}$$

Let us remember that we can define at every point $P \in \mathcal{C}$ the Lagrangian components of the stress tensor

$$Y^{rs}(y) \quad r, s = 1, 2, 3, 4, \tag{17}$$

and the Eulerian components

$$X^{rs}(x) = Y^{hk}(y)\frac{\partial x^r}{\partial y^h}\frac{\partial x^s}{\partial y^k} \ ; \tag{18}$$

this tensor is spatial symmetric tensor.

Hereafter we will consider only reversible systems, that is the systems for which it is possible to define a state function s, the specific entropy of the system $\mathcal{S}$, satisfying the condition

$$ds = \frac{\partial q}{T} \ ,$$

where ∂q is the heat absorbed in the algebraical sense by the unity of mass for an infinitesimal transformation of $\mathcal{S}$, and T is the absolute temperature of that mass. Consequently, we can define a proper energy density of pure matter

$$\mu c^2 \ ,$$

and a proper thermodynamic energy density

$$\mu w \ ;$$

moreover, if we add the hypothesis that there is no exchange of heat among contiguous elements of the system $\mathcal{S}$, the energy–momentum tensor of $\mathcal{S}$ has the eulerian controvariant components

$$T^{hk} = c^2 \mu \left(1 + \frac{w}{c^2}\right) u^h u^k + X^{hk} \tag{19}$$

and the Lagrangian components

$$\overset{*}{T}{}^{hk} = c^2 \mu \left(1 + \frac{w}{c^2}\right) \overset{*}{u}{}^h \overset{*}{u}{}^k + Y^{hk} \ . \tag{20}$$

3. The Main Local Equation and the Boundary Conditions

If we make the local projection of the conservation equation

$$\nabla_k \overset{*}{T}{}^{hk} = 0 \tag{21}$$

on the 3–space Σ_P we obtain [2]:

$$\mathcal{P}_\Sigma \left(\nabla_k \overset{*}{T}{}^{hk} \right) \equiv \overset{*}{\gamma}_{rh} \nabla_k \overset{*}{T}{}^{hk} = 0, \tag{22}$$

that is

$$\mu \left(1 + \frac{w}{c^2} \right) \overset{*}{A}_\rho = -\overset{*}{C}_\tau \; Y^\tau_\rho - \widetilde{\nabla}_\tau, Y^\tau_\rho, \tag{23}$$

where the space vector $\mathbf{A}$ is the 4–absolute acceleration of an infinitesimal volume of $\mathcal{S}$ at P, $\mathbf{C}$ is the curvature vector of the stream line for P, and the operator $\widetilde{\nabla}_\tau$ is the covariant derivative with respect the space metric tensor γ_{hk}. Besides for the equation (23) which we call the *main equation* of a continuous system, we can deduce the *boundary conditions*:

$$Y^{\rho\nu} \widetilde{n}_\nu = \overset{*}{f}{}^\rho \; (Q) \quad \forall Q \in \partial\mathcal{C} \tag{24}$$

at the boundary of the configuration $\mathcal{C}$, where we have

$$\mathbf{f}(Q) = \mathcal{P}_\Sigma[\mathbf{F}(Q)] \quad \forall Q \in \partial\mathcal{C} \tag{25}$$

with $\mathbf{F}$ the 4–force density applying at every point Q of $\partial\mathcal{C}$, and the unit vector

$$\mathbf{n} = \widetilde{n}^\rho \widetilde{\mathbf{e}}_\rho \;, \quad \widetilde{\mathbf{e}}_\rho \equiv \widetilde{\partial}_\rho P \equiv \left(\partial_\rho + \overset{*}{u}_\rho \overset{*}{u}{}^4 \; \partial_4 \right) P \tag{26}$$

is orthogonal at Q to the boundary into Σ_Q.

We can join other types of boundary conditions to the equation (22); I am indicating the following:

— Let Γ be the tube of the stream lines of the particles of $\mathcal{S}$ from $\mathcal{C}^*$, Q any point of the boundary of Γ, $\partial\Gamma$, and $\mathbf{n}_Q$ the unit vector orthogonal to $\partial\Gamma$ at Q, set towards the interior of Γ; that stated in advance, we suppose known the equations of the stream lines of the points of $\partial\mathcal{C}^*$

$$x^h(y_{\underset{Q}{*}}, y^4) = \varphi^h(y_{\underset{Q}{*}}, y^4) \; . \tag{27}$$

— We could give only on $\mathcal{C}^*$ three functions

$$\xi^\alpha(y^1, y^2, y^3) = x^\alpha(y_{\underset{Q}{*}}, 0) \tag{28}$$

and besides the three functions

$$\eta^\alpha(y^1, y^2, y^3) = [\partial_4 x^\alpha]_{y^4=0} \qquad \left(\eta^4 \equiv 1 \right) \tag{29}$$

that is position and 4–velocity of every point of $\mathcal{S}$ at $\mathcal{C}^*$.

In this case we say we give the *initial conditions*.

4. Symbolic Equation of Relativistic Mechanics of Continuous Systems

It is possible to demonstrate that the main local equation (23) and the boundary conditions (24) are equivalent to the scalar equation [2]

$$-\int_C \mu\left(1+\frac{w}{c^2}\right)\mathbf{A}\cdot\mathbf{z}\,\mathrm{d}\mathcal{C} + \int_C \Phi^\rho\cdot\widetilde{\partial}_\rho\mathbf{z}\,\mathrm{d}\mathcal{C} - \int_C \overset{*}{C}_\lambda Y^{\lambda\sigma}\,\widetilde{z}_\sigma\,\mathrm{d}\mathcal{C} + \int_{\partial C}\mathbf{f}\cdot\mathbf{z}\mathrm{d}\mathcal{C} = 0, \quad (30)$$

where $\mathbf{z}$ is an arbitrary regular vectorial field defined in the configuration $\mathcal{C}$, C is a field interior to $\mathcal{C}$.

By choosing as vectorial field $\mathbf{z}$ an arbitrary infinitesimal displacement

$$\mathbf{z} = \mathbf{u}\,dy^4 + \widetilde{d}P = \overset{*}{u}{}^4\,\partial_4 P\,dy^4 + dy^\tau\,\widetilde{\mathbf{e}}_\tau \quad (31)$$

the equation (30) becomes

$$-\int_C \mu\left(1+\frac{w}{c^2}\right)\mathbf{A}\cdot\widetilde{d}P\,\mathrm{d}\mathcal{C} + dy^4\int_C Y^{\rho\sigma}\overset{*}{u}{}^4\partial_4\overset{*}{\varepsilon}_{\rho\sigma}\mathrm{d}\mathcal{C} + \int_C Y^\rho_\lambda\Gamma^\lambda_{\rho\tau}dy^\tau\,\mathrm{d}\mathcal{C} - \int_C Y^\rho_\lambda\overset{*}{C}_\rho dy^\lambda\,\mathrm{d}\mathcal{C} + \int_{\partial C}\mathbf{f}\cdot\widetilde{d}P = 0, \quad (32)$$

where every integral has a physical meaning into the frame of reference:

$$\partial L^{(m)} \equiv -\int_C \mu\left(1+\frac{w}{c^2}\right)\mathbf{A}\cdot\widetilde{d}P\mathrm{d}\mathcal{C}$$ infinitesimal work of the forces of inertia;

$$\partial L^{(i)} \equiv dy^4\int_C Y^{\rho\sigma}\overset{*}{u}{}^4\partial_4\overset{*}{\varepsilon}_{\rho\sigma}\mathrm{d}\mathcal{C}$$ infinitesimal work of the interior forces;

$$\partial L^{(c)} \equiv \int_C Y^\rho_\lambda\widetilde{\Gamma}^\lambda_{\rho\tau}dy^\tau\,\mathrm{d}\mathcal{C}$$ infinitesimal complementary work;

$$\partial L^{(r)} \equiv -\int_C Y^\rho_\lambda\overset{*}{C}_\rho dy^\lambda\,\mathrm{d}\mathcal{C}$$ infinitesimal work of the interaction forces;

$$\partial L^{(f)} \equiv \int_{\partial C}\mathbf{f}\cdot\widetilde{d}P\mathrm{d}\mathcal{C}$$ infinitesimal work of the surface forces.

(33)

Consequently, the equation (32) can be written as

$$\partial L^{(m)} + \partial L^{(i)} + \partial L^{(c)} + \partial L^{(r)} + \partial L^{(f)} = 0. \quad (34)$$

5. Reversible Material Systems. Constitutive Equations

As we know, the equation

$$\frac{\partial q}{T} = \mathcal{L}_{\mathbf{u}} s \cdot dy^4 \,, \tag{35}$$

where s is the entropy density, characterizes the locally reversible processes for which it is possible to define the free energy density

$$\mathcal{F} = u - E\, s\, T, \tag{36}$$

where u is the internal energy density and E the mechanical equivalent of heat. The free energy density $\mathcal{F}$ satisfies the equation

$$\mu \overset{*}{u}{}^4 \partial_4 \mathcal{F} = -Y^{\rho\tau} \overset{*}{u}{}^4 \partial_4 \overset{*}{\varepsilon}_{\rho\tau} - E\, \mu\, s\, \overset{*}{u}{}^4 \partial_4 T \tag{37}$$

consequence of the first and second principle of thermodynamics and $(33)_2$. It is possible to deduce from this equation (37) the constitutive equations for a reversible system $\mathcal{S}$:

$$Y^{hk} = -\mu \left(\delta^h_\rho - \delta^h_4 \frac{\overset{*}{u}_\rho}{\overset{*}{u}_4} \right) \left(\delta^k_\tau - \delta^k_4 \frac{\overset{*}{u}_\tau}{\overset{*}{u}_4} \right) \frac{\partial \mathcal{F}}{\partial \overset{*}{\varepsilon}_{\rho\tau}} \qquad \text{(isothermal processes)}, \tag{38}$$

$$Y^{hk} = -\mu \left(\delta^h_\rho - \delta^h_4 \frac{\overset{*}{u}_\rho}{\overset{*}{u}_4} \right) \left(\delta^k_\tau - \delta^k_4 \frac{\overset{*}{u}_\tau}{\overset{*}{u}_4} \right) \frac{\partial u}{\partial \overset{*}{\varepsilon}_{\rho\tau}} \qquad \text{(isoentropic processes)}. \tag{39}$$

Now we could demonstrate that the local projection of the conservation law (21) on the subspace Θ_P (see (7)) becomes an identity if we limit ourselves only to the reversible systems, by making us of (38) and (39).

6. The Cauchy Problem to Determine the Evolution of a Reversible System

To determine the evolution of a reversible material system and consequently the tensor fields

$$\overset{*}{\gamma}_{\rho\tau}(y) \;,\; \left(\overset{*}{\gamma}_{4r} = 0 \right), \overset{*}{u}_h \;\; \left(\overset{*}{u}{}^4 \overset{*}{u}_4 = -1 \right), \mu(y)\,, \tag{40}$$

we must add the Einstein's field equations

$$\overset{*}{G}_{hk} \equiv \overset{*}{R}_{hk} - \frac{1}{2} \overset{*}{R} \overset{*}{g}_{hk} = -\chi \left[c^2 \mu \left(1 + \frac{w}{c^2} \right) \overset{*}{u}_h \overset{*}{u}_k + Y_{hk} \right] \tag{41}$$

to the equations (16), (23), (38) or (39).

The function w, proper thermodynamic energy density, coincides with $\mathcal{F}$ in the isothermal processes, with u in the isoentropic processes. All the projections of equation (41) give:

$$\begin{cases} \overset{*}{s}_{\alpha\rho} = -\chi\left\{\left[\frac{1}{2}c^2\mu\left(1+\frac{w}{c^2}\right) - Y^{\nu}_{\nu}\right]\overset{*}{\gamma}_{\alpha\rho} + Y_{\alpha\rho}\right\}, \\ \overline{S}_{\alpha} = 0, \\ \widetilde{R} + \mathcal{I} = -2\chi c^2\mu\left(1+\frac{w}{c^2}\right), \end{cases} \tag{42}$$

where we have

$$\overset{*}{s}_{\alpha\rho} \equiv \mathcal{P}_{\Sigma\Sigma}\left(\overset{*}{R}_{\alpha\rho}\right) \quad , \quad \left(\overset{*}{s}_{4h} = 0\right) \ ; \tag{43}$$

$$\overline{S}_{\alpha} \equiv \frac{1}{2}\left[\widetilde{\nabla}_{\alpha}\,\widetilde{K}^{\ \nu}_{\nu} - \widetilde{\nabla}_{\beta}\left(\widetilde{K}^{\ \beta}_{\alpha} + \widetilde{\Omega}^{\ \beta}_{\alpha}\right)\right] + \overset{*}{C}^{\beta}\,\widetilde{\Omega}_{\beta\alpha} \ ; \tag{44}$$

$$\widetilde{\Omega}_{hr} \equiv \mathcal{P}_{\Sigma\Sigma}\left(\Omega_{hr}\right) \equiv \mathcal{P}_{\Sigma\Sigma}\left(\nabla_h \overset{*}{u}_r - \nabla_r \overset{*}{u}_h\right) \ , \ \left(\widetilde{\Omega}_{4h} = 0\right) \ ; \tag{45}$$

$$\overset{*}{C}^{r} \equiv \overset{*}{u}^{h}\nabla_h \overset{*}{u}^{r}, \ \widetilde{K}_{hr} \equiv \mathcal{P}_{\Sigma\Sigma}\left(K_{hr}\right) \equiv \mathcal{P}_{\Sigma\Sigma}\left(\nabla_h \overset{*}{u}_r + \nabla_r \overset{*}{u}_h\right), \ \left(\widetilde{K}_{4h} = 0\right); \tag{46}$$

the scalar invariants $\widetilde{R}$ and $\mathcal{I}$ have respectively the expressions

$$\begin{aligned} \widetilde{R} &\equiv \overset{*}{\gamma}^{\alpha\beta}\widetilde{P}_{\alpha\beta} \\ &= \overset{*}{\gamma}^{\alpha\beta}\left[-\widetilde{\partial}_{\beta}\widetilde{\left\{{}^{\ \rho}_{\alpha\ \rho}\right\}} + \widetilde{\partial}_{\rho}\widetilde{\left\{{}^{\ \rho}_{\beta\ \alpha}\right\}} - \widetilde{\left\{{}^{\ \sigma}_{\alpha\ \rho}\right\}}\widetilde{\left\{{}^{\ \rho}_{\beta\ \sigma}\right\}} + \widetilde{\left\{{}^{\ \sigma}_{\beta\ \alpha}\right\}}\widetilde{\left\{{}^{\ \rho}_{\rho\ \sigma}\right\}}\right] \end{aligned} \tag{47}$$

$$\mathcal{I} = \frac{1}{4}\left[\left(\widetilde{K}^{\ \alpha}_{\alpha}\right)^2 - \widetilde{K}^{\ \alpha\rho}\widetilde{K}_{\alpha\rho} + 3\widetilde{\Omega}^{\ \alpha\rho}\widetilde{\Omega}_{\alpha\rho}\right] \ . \tag{48}$$

Besides the coefficients $\widetilde{\left\{{}^{\ \sigma}_{\alpha\ \rho}\right\}}$ are the Christoffel symbols of the second kind builded with the metric space tensor $\overset{*}{\gamma}_{hk}$ and the operator $\widetilde{\partial}_{\rho}$ is the transverse derivation

$$\widetilde{\partial}_{\rho} \equiv \partial_{\rho} + \overset{*}{u}_{\rho}\overset{*}{u}^{4}\partial_4 \ . \tag{49}$$

But E. Laserra and I. Bochicchio have demonstrated [8] that only the six equations $(42)_1$ will be added to the equations (16), (23), (38) or (39) to determine the evolution of a reversible system (the restricted evolution problem).

The four remaining equations that we can deduce from the equations

$$\begin{cases} \mathcal{P}_{\Sigma\Theta}\left(\overset{*}{G}_{hr}\right) = -\chi\,\mathcal{P}_{\Sigma\Theta}\left(\overset{*}{T}_{hr}\right), \\ \mathcal{P}_{\Theta\Sigma}\left(\overset{*}{G}_{hr}\right) = -\chi\mathcal{P}_{\Theta\Sigma}\left(\overset{*}{T}_{hr}\right), \\ \mathcal{P}_{\Theta\Theta}\left(\overset{*}{G}_{hr}\right) = -\chi\mathcal{P}_{\Theta\Theta}\left(\overset{*}{T}_{hr}\right), \end{cases} \tag{50}$$

calculated on $\mathcal{C}^*$, that is the equations

$$\begin{cases} \left(\overline{S}_\alpha\right)_{\mathcal{C}^*} \equiv \left\{\frac{1}{2}\left[\widetilde{\nabla}_\alpha \widetilde{K}^\nu_\nu - \widetilde{\nabla}_\beta \left(\widetilde{K}^\beta_\alpha + \widetilde{\Omega}^\beta_\alpha\right)\right] + \overset{*}{C}{}^\beta \, \widetilde{\Omega}_{\beta\alpha}\right\}_{\mathcal{C}^*} = 0 \ (\alpha, \beta = 1, 2, 3), \\ \left(\widetilde{R} + \mathcal{I}\right)_{\mathcal{C}^*} = -2\chi \overset{*}{\mu} \left(1 + \frac{w}{c^2}\right)_{\mathcal{C}^*}, \end{cases} \tag{51}$$

answer the purpose of solving the *initial conditions problem* to give the Cauchy data on the reference configuration $\mathcal{C}^*$.

The unknown functions present in these four equations are the six components of the tensor field $\left(\widetilde{K}_{hr}\right)_{\mathcal{C}^*}$; therefore the initial conditions problem has infinite solutions.

Definitively, after we have determined one solution of the initial conditions problem, we can study the *restricted evolution problem*, that is we can look for the solution of the equations:

$$\begin{cases} \overset{*}{s}_{\alpha\rho} = -\chi \left\{\left[\frac{c^2}{2}\mu \left(1 + \frac{w}{c^2}\right) - Y^\nu_\nu\right] \overset{*}{\gamma}_{\alpha\rho} + Y_{\alpha\rho}\right\}, \\ \left[\mu \left(c^2 + w\right) \delta^\nu_\rho + Y^\nu_\rho\right] \overset{*}{u}{}^4 \nabla_4 \overset{*}{u}_\nu = -\widetilde{\nabla}_\nu Y^\nu_\rho \qquad (\alpha, \nu, \rho = 1,\ 2,\ 3), \\ Y^{\nu\rho} = -\mu \frac{\partial w}{\partial \overset{*}{\varepsilon}_{\nu\rho}}, \quad \overset{*}{\varepsilon}_{\nu\rho} = \frac{1}{2}\left(\overset{*}{\gamma}_{\nu\rho} - m_{\nu\rho}\right), \\ \overset{*}{u}{}^4 \overset{*}{u}_4 = -1, \quad \mu = \overset{*}{\mu}\sqrt{\frac{m}{\overset{*}{\gamma}}}, \end{cases} \tag{52}$$

since E. Laserra and I. Bochicchio have demonstrated that the restricted evolution problem admits an unique physical solution that satisfies the initial conditions problem on the reference configuration $\mathcal{C}^*$ [9].

7. Exact Solutions of Some Problems

The breaking of the Cauchy problem in two different problems, the initial data problem and the restricted evolution problem affords an undoubted advantage experimented by E. Laserra and E. Mazziotti in some lately issued papers [7], [10].

A) E. Laserra has considered the interior case of spherically symmetric dust in which the metric of the manifold V_4 can be given the form

$$ds^2 = A^2(r,t)dr^2 + B^2(r,t)(d\theta^2 + \sin^2\theta \, d\varphi^2) - dt^2, \tag{53}$$

where t is the proper time of every particle and r, θ, φ are spherical coordinates.

Here we have:

$$\begin{cases} A^2(r,t) = \overset{*}{g}_{11} = \overset{*}{\gamma}_{11}, \\ B^2(r,t) = \overset{*}{g}_{22} = \overset{*}{\gamma}_{22}, \\ B^2(r,t)\sin^2\theta = \overset{*}{g}_{33} = \overset{*}{\gamma}_{33}, \\ \overset{*}{g}_{44} = -\left(\overset{*}{u}_4\right)^2 = -1 \ ; \end{cases} \tag{54}$$

$$\overset{*}{T}_{hk} = \mu\,\overset{*}{u}_h\,\overset{*}{u}_k; \tag{55}$$

$$\begin{cases} \overset{*}{u}{}^{\alpha} = 0 \qquad \overset{*}{u}{}^{4} = 1, \\ \overset{*}{u}_{\alpha} = 0 \qquad \overset{*}{u}_4 = -1. \end{cases} \tag{56}$$

Consequently the tensor fields $\widetilde{\Omega}_{hr}$, $\overset{*}{C}{}^{r}$ are null everywhere in $\{L\}$, the congruence is irrotational and geodesic; Einstein's field equations (41) become in $\{L\}$

$$\begin{cases} \overset{*}{s}_{\alpha\rho} - \frac{1}{2}\overset{*}{R}\,\overset{*}{\gamma}_{\alpha\rho} = 0, \\ \overline{S}_{\alpha} = 0, \\ \widetilde{R} + \mathcal{I} = -2\chi c^2\mu, \end{cases} \tag{57}$$

the main local equation (23) is reduced to an identity, and the conservation law of pure matter (15) can be written

$$\partial_t\mu = -\frac{1}{2}\,\mu\,\widetilde{K}_{\rho}{}^{\rho} \ . \tag{58}$$

Besides, by positions (54) the equations (57) become

$$\begin{cases} -2\frac{A^2\ddot{B}}{B} - \frac{A^2}{B^2}\left(1+\dot{B}^2\right) + \frac{(B')^2}{B^2} = 0 \qquad \left(\ddot{B} \equiv \partial_t\partial_t B \ , \ B' \equiv \partial_r B\right), \\ -\frac{\ddot{A}\,B^2}{A} - \ddot{B}B - \frac{\dot{A}B\dot{B}}{A} + \frac{B^2}{A^2}\left(\frac{B''}{B} - \frac{A'B'}{AB}\right) = 0, \\ B^{-1}\left(\dot{B}' - \frac{\dot{A}\,B'}{A}\right) = 0, \\ 2\frac{\dot{A}\,\dot{B}}{AB} + \frac{1+\dot{B}^2}{B^2} - \frac{1}{A^2}\left[2\frac{B''}{B} + \frac{(B')^2}{B^2} - 2\frac{A'B'}{AB}\right] = c^2\chi\mu, \\ \partial_t\left(\mu AB^2\right) = 0. \end{cases} \tag{59}$$

Since the equation $(59)_3$ admits the integral

$$A(r,t) = f(r)\partial_r B(r,t) \tag{60}$$

being $f(r)$ an arbitrary positive function, we can replace to the differential system (59) the equivalent following system

$$\begin{cases} \partial_t \left[B \left(f^{-2} - 1 - \dot{B}^2 \right) \right] = 0, \\ \partial_r \left[\dot{B}^{-1} \left\{ \partial_t \left[B \left(f^{-2} - 1 - \dot{B}^2 \right) \right] \right\} \right] = 0, \\ A = f(r) B', \\ (B')^{-1} B^{-2} \partial_r \left[B \left(f^{-2} - 1 - \dot{B}^2 \right) \right] = -c^2 \chi \mu, \\ \partial_t \left(\mu B' B^2 \right) = 0. \end{cases} \tag{61}$$

But the equation $(61)_2$ can be deduced from $(61)_1$; therefore we will study the equivalent system

$$\begin{cases} \partial_t \left[B \left(f^{-2} - 1 - \dot{B}^2 \right) \right] = 0, \qquad A(r,t) = f(r) B'(r,t), \\ \partial_r \left[B \left(f^{-2} - 1 - \dot{B}^2 \right) \right] = -\chi c^2 \mu B' B^2 \qquad \partial_t \left(\mu B' B^2 \right) = 0. \end{cases} \tag{62}$$

Consequently the initial conditions problem is expressed by the equations

$$\begin{cases} \left[\partial_t A(r,t) \right]_{t=0} = f(r) \left[\partial_r \partial_t B(r,t) \right]_{t=0}, \\ \partial_r \left[B \left(f^{-2} - 1 - \dot{B}^2 \right) \right]_{t=0} = -c^2 \chi \left(\mu B' B^2 \right)_{t=0}, \end{cases} \tag{63}$$

and the restricted evolution problem by

$$\begin{cases} \partial_t \left[B \left(f^{-2} - 1 - \dot{B}^2 \right) \right] = 0, \\ A(r,t) = f(r) B'(r,t), \\ \partial_t \left(\mu B' B^2 \right) = 0. \end{cases} \tag{64}$$

Since the congruence $\{L\}$ is geodesic, we can assume the intrinsic radius $\overset{*}{r}$ of the O–spheres of $\mathcal{C}^*$ as radial coordinate

$$\overset{*}{r} \; = \; B(r,0) \tag{65}$$

and the metric of the configuration $\mathcal{C}^*$ (itself an O–sphere) can be written

$$d\sigma^2 = a^2(\overset{*}{r}) \left(d\overset{*}{r} \right)^2 + (\overset{*}{r})^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \tag{66}$$

being

$$a(\overset{*}{r}) = A(\overset{*}{r}, 0) = f(\overset{*}{r}) \; . \tag{67}$$

If we put moreover

$$\psi(\overset{*}{r}) = \left[\partial_t A(\overset{*}{r}, t) \right]_{t=0} \quad , \quad \overline{\chi}(\overset{*}{r}) = \left[\partial_t B(\overset{*}{r}, t) \right]_{t=0} \tag{68}$$

to solve the initial conditions problem means to determine three functions

$$a(\overset{*}{r}), \quad \psi(\overset{*}{r}), \quad \overline{\chi}(\overset{*}{r}), \tag{69}$$

which satisfy the equations

$$\psi\left(\overset{*}{r}\right) = a(\overset{*}{r})\overline{\chi}'(\overset{*}{r}), \qquad \left(\overset{*}{r}\right)^{-2}\frac{d}{d\overset{*}{r}}\left[\overset{*}{r}\left(1+\overline{\chi}^2-a^{-2}\right)\right] = \chi c^2 \overset{*}{\mu}(\overset{*}{r}) \tag{70}$$

in the configuration $\mathcal{C}^*$. Therefore the initial conditions problem has an infinity of solutions.

If we give $\overset{*}{\mu}(\overset{*}{r})$ and $\overline{\chi}(\overset{*}{r})$ into $\mathcal{C}^*$, we can deduce $a(\overset{*}{r})$ from (70). The investigation of $(70)_2$ allows to get

$$\overline{\chi}^2 - \frac{2\,m(\overset{*}{r})}{\overset{*}{r}} = \frac{1-a^2(\overset{*}{r})}{a^2(\overset{*}{r})}, \tag{71}$$

where $m(\overset{*}{r})$ has the expression

$$m(\overset{*}{r}) = 4\pi\int_0^{\overset{*}{r}} \overset{*}{\mu}(s)\, s^2 ds \tag{72}$$

and represents the amount of pure matter included in an Euclidean sphere with the (intrinsic) radius $\overset{*}{r}$, that we call "Euclidean mass". If the configuration $\mathcal{C}^*$ is not Euclidean, the amount of pure matter included into the O–sphere with the intrinsic radius $\overset{*}{r}$ is

$$M(\overset{*}{r}) = 4\pi\int_0^{\overset{*}{r}} \overset{*}{\mu}(s)\ a(s)\ s^2 ds \tag{73}$$

and the formula (71) gives a relative difference of the function $a^2(\overset{*}{r})$ with regard to its Euclidean value $\left[a^2(\overset{*}{r}) = 1\right]$.

After we have resolved the initial conditions problem we can determine univocally the three scalar functions

$$m(\overset{*}{r}, t), \quad A(\overset{*}{r}, t), \quad B(\overset{*}{r}, t), \tag{74}$$

that satisfy the equations (see (64), (67))

$$\begin{cases} \partial_t\left[B\left(\dot{B}^2 + 1 - \frac{1}{a^2(\overset{*}{r})}\right)\right] = 0, \\ A(\overset{*}{r}, t) = a(\overset{*}{r})B'(\overset{*}{r}, t), \\ \partial_t\left[\mu(\overset{*}{r}, t)\, B' B^2\right] = 0 \end{cases} \tag{75}$$

in a part of the world–tube described by $\mathcal{S}$, containing $\mathcal{C}^*$, and verify, on $\mathcal{C}^*$, the conditions

$$\mu(\overset{*}{r}, 0) = \overset{*}{\mu}(\overset{*}{r}),\ A(\overset{*}{r}, 0) = a(\overset{*}{r}),\ B(\overset{*}{r}, 0) = \overset{*}{r},\ \dot{B}(\overset{*}{r}, 0) = \overline{\chi}(\overset{*}{r}). \tag{76}$$

As the equation $(75)_1$ can be replaced by

$$\dot{B}^2 = \frac{1}{a^2} - 1 + \frac{2\,m(\overset{*}{r})}{B}\,, \tag{77}$$

and the integration of $(75)_3$ gives

$$\mu(\overset{*}{r},t) \;=\; \frac{1}{A\,B^2}\overset{*}{\mu}(\overset{*}{r})\;a(\overset{*}{r})\left(\overset{*}{r}\right)^2\;, \tag{78}$$

the evolution problem equations are also

$$\begin{cases} A(\overset{*}{r},t) &= a(\overset{*}{r})B'(\overset{*}{r},t), \\ \dot{B}^2 &= \frac{1}{a^2} - 1 + \frac{2\,m(\overset{*}{r})}{B}, \\ \mu(\overset{*}{r},t) &= \frac{\overset{*}{\mu}(\overset{*}{r})\left(\overset{*}{r}\right)^2}{B'B^2}. \end{cases} \tag{79}$$

The equations $(79)_{1,2}$ had been already obtained by Tolman–Bondi [14], [1].

The special case $\mu(\overset{*}{r}) =$ const. reduces the equation $(79)_2$ to the Freedman equation. In the case $\mathcal{C}^*$ 3–Euclidean space, if the pure matter is initially or expanding or contracting, Laserra has found respectively

$$B(\overset{*}{r},t) = \left[3t\sqrt{\frac{m(\overset{*}{r})}{2}} \pm \sqrt{\left(\overset{*}{r}\right)^3}\;\right]^{\frac{2}{3}}\,, \tag{80}$$

where

$$m(\overset{*}{r}) = 4\pi\int_0^{\overset{*}{r}} \overset{*}{\mu}\,(s)\,s^2 ds\ . \tag{81}$$

Note that (81) is the (72).

B) By ending I wish to do a short account of the exact solutions obtained by E. Mazziotti, who has studied the evolution of a perfect fluid which produces a spherically symmetric 4–manifold V_4, by a state equation between pressure p and density

$$p \;=\; p(\mu)\ . \tag{82}$$

For this scheme we consider the energy–momentum tensor has the Lagrangian expression

$$\overset{*}{T}{}^{hk} \;=\; p\,\overset{*}{g}{}^{hk} + \left(\mu c^2 + p\right)\overset{*}{u}{}^{h}\,\overset{*}{u}{}^{k} \tag{83}$$

and the metric of V_4 can be written

$$ds^2 = e^{2\lambda(r,t)}\,dr^2 + Y^2\,(r,t)\,d\Omega^2 - e^{2v(r,t)}\,dt^2, \tag{84}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2$ and

$$\begin{gathered} \overset{*}{g}_{11} = \overset{*}{\gamma}_{11} = e^{2\lambda(r,t)}, \quad \overset{*}{g}_{22} = \overset{*}{\gamma}_{22} = Y^2, \\ \overset{*}{g}_{33} = \overset{*}{\gamma}_{33} = Y^2 \sin^2\theta, \quad \overset{*}{g}_{44} = -e^{2v(r,t)}, \quad \overset{*}{g}_{rs} = 0, \; r \neq s, \end{gathered} \tag{85}$$

$$\overset{*}{u}^{\alpha} = 0, \; \overset{*}{u}^{4} = e^{-v} = \frac{1}{\sqrt{-\overset{*}{g}_{44}}}, \; \overset{*}{u}_4 = -e^v. \tag{86}$$

The Einstein's field equations (42) become

$$\begin{cases} \overset{*}{s}_{\alpha\rho} - \frac{1}{2}\overset{*}{R}\,\overset{*}{\gamma}_{\alpha\rho} = -\chi\, p \overset{*}{\gamma}_{\alpha\rho}, \\ \overline{S}_{\alpha} = 0, \\ \widetilde{R} + \mathcal{I} = -2\chi\,\mu\, c^2, \end{cases} \tag{87}$$

and the conservation equations (21), (15) translated in the frame of reference {Ł} are the following:

$$\begin{cases} \widetilde{\partial}_{\alpha} p + (p+\mu)\,\overset{*}{C}_{\alpha} = 0, \\ \overset{*}{u}^{4}\,\partial_4 \mu + \frac{1}{2}\,(p+\mu)\,\widetilde{K}^{\alpha}_{\alpha} = 0, \end{cases} \tag{88}$$

where $\widetilde{\partial}_{\alpha} \equiv \partial_{\alpha} + \overset{*}{u}^{4}\,\partial_4$.

By equations (85), (86) and the positions

$$\chi = 1, \quad c = 1, \quad \gamma' \equiv \partial_r Y, \quad \dot{Y} \equiv \partial_t Y, \tag{89}$$

the field equations (87) have the explicit form

$$\begin{cases} \frac{1}{e^{2\lambda}}\left[-\left(\frac{Y'}{Y}\right)^2 - 2\frac{Y'}{Y}v'\right] + \frac{1}{Y^2} + \frac{1}{e^{2v}}\left(2\frac{\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y} - 2\frac{\dot{Y}}{Y}\dot{v}\right) = -8\pi p, \\ \frac{1}{e^{2\lambda}}\left[-\frac{Y''}{Y} - v'' - (v')^2 - \frac{Y'}{Y}v' + \lambda'\frac{Y'}{Y} + \lambda' v'\right] \\ \qquad + \frac{1}{e^{2v}}\left[\frac{\ddot{Y}}{Y} + \ddot{\lambda} + \left(\dot{\lambda}\right)^2 + \frac{\dot{Y}}{Y}\dot{\lambda} - \frac{\dot{v}\dot{Y}}{Y} - \dot{\lambda}\dot{v}\right] = -8\pi p, \\ \dot{Y}' - \dot{Y}v' - Y'\dot{\lambda} = 0, \\ \frac{2}{e^{2v}}\dot{\lambda}\frac{\dot{Y}}{Y} + \frac{1}{Y^2}\left(1 + \frac{\dot{Y}^2}{e^{2v}}\right) - \frac{1}{e^{2\lambda}}\left[2\frac{Y''}{Y} + \left(\frac{Y'}{Y}\right)^2 - 2\lambda'\frac{Y'}{Y}\right] = 8\pi\mu\ . \end{cases} \tag{90}$$

Besides, the calculation of $\widetilde{K}^{\alpha}_{\alpha} \equiv \left(\overset{*}{g}^{\alpha\rho} + \overset{*}{u}^{\alpha}\overset{*}{u}^{\rho}\right)\widetilde{K}_{\alpha\rho}$ gives us the possibility of writing the equation (88) in the following way:

$$\begin{cases} \partial_r p + (p+\mu)\, v' = 0, \\ \partial_4 \mu + (p+\mu)\left(\dot{\lambda} + 2\frac{\dot{Y}}{Y}\right) = 0. \end{cases} \tag{91}$$

But we can give the equation $(90)_3$ the two equivalent forms

$$v' = \frac{e^{\lambda}}{\dot{Y}}\partial_t\left(\frac{Y'}{e^{\lambda}}\right), \quad \dot{\lambda} = \frac{e^{v}}{Y'}\partial_r\left(\frac{\dot{Y}}{e^{v}}\right); \tag{92}$$

and if we replace to v', $\dot{\lambda}$ the expressions (92) in the remaining equations (90), we have the equivalent differential system

$$\begin{cases} \frac{1}{\dot{Y}Y^2}\partial_t\left[Y\left(1+\frac{\dot{Y}^2}{e^{2v}}-\frac{(Y')^2}{e^{2\lambda}}\right)\right] = -8\pi p, \\ \frac{1}{2YY'}\partial_r\left\{\frac{1}{\dot{Y}}\partial_t\left[Y\left(1+\frac{\dot{Y}^2}{e^{2v}}-\frac{(Y')^2}{e^{2\lambda}}\right)\right]\right\} \\ \qquad +\frac{1}{2Y'}\left[\frac{1}{\dot{Y}}\partial_t\left(\frac{\dot{Y}^2}{e^{2v}}-\frac{(Y')^2}{e^{2\lambda}}\right)-\frac{1}{Y'}\partial_r\left(\frac{\dot{Y}^2}{e^{2v}}-\frac{(Y')^2}{e^{2\lambda}}\right)\right]v' = -8\pi p, \\ \frac{1}{Y'Y^2}\partial_r\left[Y\left(1+\frac{\dot{Y}^2}{e^{2v}}-\frac{(Y')^2}{e^{2\lambda}}\right)\right] = 8\pi\mu. \end{cases} \tag{93}$$

These last equations, the second not in the explicit form, had been already written by Poduret [12] and Misner–Sharp [11] but for a different and harder proceeding.

When v' or $\dot{\lambda}$ are identically null, two integrals follows from (92), the first of which had been already obtained by E. Laserra (see (60)).

Now we can examine distinctly the initial conditions problem and the restricted evolution problem. We translate the first problem in the differential system

$$\begin{cases} \psi(r) = \frac{e^{\overline{v}}}{\overline{Y}'}\partial_r\left(\frac{\overline{\chi}(r)}{e^{\overline{v}}}\right), \\ \frac{1}{\overline{Y}'\overline{Y}^2}\partial_r\left[\overline{Y}\left(1+\frac{\overline{\chi}^2}{e^{2\overline{v}}}-\frac{\left(\overline{Y}'\right)^2}{e^{2\overline{\lambda}}}\right)\right] = 8\pi, \overset{*}{\mu}(r) \end{cases} \tag{94}$$

with conditions

$$\psi(r) = \dot{\lambda}(r,0), \quad \overline{\chi}(r) = \dot{Y}(r,0), \quad \frac{dp}{d\mu} \neq 0 \text{ in } \mathcal{C}^*, \tag{95}$$

and the second problem in studying the differential system

$$\begin{cases} \dfrac{1}{\dot{Y}\,Y^2}\partial_t\left[Y\left(1+\dfrac{\dot{Y}^2}{e^{2v}}-\dfrac{(Y')^2}{e^{2\lambda}}\right)\right]=-8\pi p,\\ \dfrac{1}{2Y\,Y'}\partial_r\left\{\dfrac{1}{\dot{Y}}\partial_t\left[Y\left(1+\dfrac{\dot{Y}^2}{e^{2v}}-\dfrac{(Y')^2}{e^{2\lambda}}\right)\right]\right\}\\ +\dfrac{1}{2\,Y'}\left[\dfrac{1}{\dot{Y}}\partial_t\left(\dfrac{\dot{Y}^2}{e^{2v}}-\dfrac{(Y')^2}{e^{2\lambda}}\right)-\dfrac{1}{Y'}\partial_r\left(\dfrac{\dot{Y}^2}{e^{2v}}-\dfrac{(Y')^2}{e^{2\lambda}}\right)\right]v'=-8\pi p,\\ \partial_r p+(p+\mu)\,v'=0,\\ \partial_t\mu+(p+\mu)\left(\dot{\lambda}+2\dfrac{\dot{Y}}{Y}\right)=0, \end{cases} \tag{96}$$

in a part of the world tube described by the configuration $\mathcal{C}$ containing $\mathcal{C}^*$. To solve the initial conditions problem we give the functions

$$\overset{*}{\mu}(r)\ ,\ p=p(\overset{*}{\mu}) \tag{97}$$

and we look for the scalar functions

$$\overline{v}(r)\ ,\ \overline{Y}(r)\ ,\ \overline{\lambda}(r)\ ,\ \overline{\chi}(r)\ ,\ \psi(r) \tag{98}$$

that satisfy the equations (94); obviously there exist an infinity of solutions. If we choose the intrinsic radius of every O–sphere of $\mathcal{C}^*$ (itself an O–sphere) as a radial coordinate, we obtain

$$Y(r,0)=\overset{*}{r}, \tag{99}$$

the equation $(94)_2$ becomes

$$\partial_{\overset{*}{r}}\left[\overset{*}{r}\left(1+\frac{\overline{\chi}^2}{e^{2\overline{v}}}-\frac{1}{e^{2\overline{\lambda}}}\right)\right]=\overset{*}{\mu}(\overset{*}{r})\,(\overset{*}{r})^2, \tag{100}$$

the integration of which gives

$$\overset{*}{r}\left(1+\frac{\overline{\chi}^2}{e^{2\overline{v}}}-\frac{1}{e^{2\overline{\lambda}}}\right)=\int_{\overset{*}{r}_0}^{\overset{*}{r}} s^2\overset{*}{\mu}(s)ds=m(\overset{*}{r})\ . \tag{101}$$

When we have chosen a solution of the equation (101), we can determine $\psi(\overset{*}{r})$:

$$\psi(\overset{*}{r})=e^{\overline{v}}\partial_{\overset{*}{r}}\left(\frac{\overline{\chi}}{e^{\overline{v}}}\right)\ . \tag{102}$$

After we are able to determine the functions

$$\lambda(\overset{*}{r},t),\quad Y(\overset{*}{r},t),\quad v(\overset{*}{r},t),\quad \mu(\overset{*}{r},t), \tag{103}$$

which satisfy the equations (96) and, in $\mathcal{C}^*$, the conditions

$$\begin{cases} \lambda(\overset{*}{r},0) = \overline{\lambda}(\overset{*}{r}), & Y(\overset{*}{r},0) = \overline{Y}(\overset{*}{r}), \\ v(\overset{*}{r},0) = \overline{v}(\overset{*}{r}), & \mu(\overset{*}{r},0) = \overline{\mu}(\overset{*}{r}), \\ \dot{Y}(\overset{*}{r},0) = \chi(\overset{*}{r}), & \dot{\lambda}(\overset{*}{r},0) = \psi(\overset{*}{r}). \end{cases} \tag{104}$$

Some other remarks permit us to replace to the differential system (96) the following system

$$\begin{cases} v' = \frac{e^{\lambda}}{\dot{Y}}\partial_t\left(\frac{Y'}{e^{\lambda}}\right) \text{ or } \dot{\lambda} = \frac{e^{v}}{Y'}\partial_r\left(\frac{\dot{Y}}{e^{v}}\right), \\ \frac{1}{\dot{Y}Y^2}\partial_t\left[Y\left(1+\frac{\dot{Y}^2}{e^{2v}}-\frac{(Y')^2}{e^{2\lambda}}\right)\right] = -8\pi p, \\ \partial_r p + (p+\mu)\,v' = 0, \\ \partial_t\mu + (p+\mu)\left(\dot{\lambda}+2\frac{\dot{Y}}{Y}\right) = 0. \end{cases} \tag{105}$$

Mazziotti has translated the equations (105) in four different ways, choosing curvature coordinates, Gaussian coordinates, Gaussian polar coordinates, harmonic coordinates.

If we make use of Gaussian polar coordinates and the auxiliary assumption of radially spatial isotropy, we can put

$$Y(r,t) = x(r)\,y(t) \tag{106}$$

and translate the restricted evolution problem in the two following differential equations

$$\begin{cases} \partial_t\left[y^2\left(1+\frac{\dot{y}^2}{f^2}-\frac{c}{2}y^2\right)\right] = 0, \\ \partial_r\left(x^2\,x'^2\right) = \frac{c}{2}\partial_r x^2, \end{cases} \tag{107}$$

where c is an integration constant and $f(t)$ freely chosen by a convenient scale of t.

By integration, the equations (107) give immediately

$$\begin{cases} y^2\left(1+\frac{\dot{y}^2}{f^2}-\frac{c}{2}y^2\right) = d, \\ x^2\,x'^2 = \frac{c}{2}x^2 + k, \end{cases} \tag{108}$$

where d and k are constants.

If we put in (108) $f = \frac{1}{2}$ we obtain the solution of Gutman [6]

$$y(t) = 1 + Ae^{t} + Be^{-t}, \tag{109}$$

where A and B are constants.

Moreover, if we put $f = \frac{1}{t}$, the unknown function $y(t)$ will be solution of the algebraic equation

$$(y^4 - y^2 + n^2)^{\frac{1}{2}} + y^2 - \frac{1}{2} = t^{\pm 2} \qquad (n\text{– constant}) \tag{110}$$

that is the solution of Wesson [15].

By considering the frames of reference associated to isotropic coordinates and spherical symmetry, Mazziotti has obtained other remarkable results:

1. If the distribution of the fluid is regular everywhere in $\mathcal{C}^*$, shear zero, and the Hubble parameter is constant, all the configurations $\mathcal{C}$ of the fluid are Euclidean hypersurfaces.

2. With the unique assumptions $p = p(\mu)$ and shear zero in $\mathcal{C}^*$, the restricted evolution problem gives dynamic models non different from Einstein–de Sitter or Friedman–Robertson–Walker or Wyman models [16], [13].

The present work has not been published and is not under consideration for publication elsewhere.

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