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FIXED POINT THEOREMS FOR TWO CLASSES OF MULTI-VALUED MAPPINGS

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Abstract: In this paper two fixed point theorems for two classes of multi-valued contractive type mappings are established. The results presented in this paper improve and generalize some results in literatures.

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1. Introduction and Preliminaries

Let (X, d) be a metric space and put

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$$CL(X) = \{A : A \text{ is a nonempty colsed subset of } X\};$$
$$C(X) = \{A : A \text{ is a nonempty compact subset of } X\};$$
$$CB(X) = \{A : A \text{ is a nonempty bounded colsed subset of } X\}.$$

For $c \in X, A, B \in CL(X)$ and $T : X \to CL(X)$, define

$$d(c, A) = \inf_{a \in A} d(c, a), \quad H_-(A, B) = \sup_{x \in A} d(x, B),$$
$$H_+(A, B) = H_-(B, A), \quad H(A, B) = \max\{H_-(A, B), H_-(B, A)\},$$

where H is the Hausdorff metric in $CB(X)$.

In 1969, Nadler [11] first discussed the existence and stability of fixed points for the multi-valued contraction mapping $f : X \to CB(X)$ satisfying the following condition

$$H(fx, fy) \le qd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where q is a constant in $[0, 1)$. Later, many researchers gave correlative generalizations of Nadler's results, for example, see [1-12] and the references cited therein. Lim [3] improved Nadler's Stability Theorem from $CB(X)$ to $CL(X)$ and Wang [12] acquired some extensions of Nadler and Lim's results from the class of multi-valued contraction mappings (1.1) to the larger class of multi-valued mappings (1.2) below:

$$H_-(Tx, Ty) \le \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X, \tag{1.2}$$

where α and β are constants $\in [0, 1)$ with $\alpha + 2\beta < 1$. In 2006, Liu, Chen, Kang and Kim [7] generalized the results of Nadler [11] and Wang [12] and established a few fixed point theorems for the multi-valued mapping $T : X \to CL(X)$ satisfying

$$H_-(Tx, Ty) \le \alpha \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)], \frac{[1 + d(y, Ty)]d(y, Tx)}{1 + d(x, Tx)}\right\}, \quad \forall x, y \in X, \tag{1.3}$$

where α is a constant in $[0, 1)$.

Motivated and inspired by the works [1-5, 7, 11, 12], in this paper, we introduce and study two new classes of multi-valued contractive type mappings as follows:

$$H_-(Tx, Ty) \le \alpha \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)], \right.$$
$$\frac{d(x, y)d(y, Tx)}{1 + H_-(Tx, Ty)}, \frac{d(x, Tx)d(y, Tx)}{1 + H_-(Tx, Ty)}, \frac{d(y, Ty)d(y, Tx)}{1 + H_-(Tx, Ty)},$$

$$\frac{d(x,Ty)d(y,Tx)}{1+H_-(Tx,Ty)}, \frac{d^2(y,Tx)}{1+H_-(Tx,Ty)}, \frac{d^2(x,Tx)}{1+d(x,y)},$$
$$\frac{d(y,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d^2(y,Tx)}{1+d(x,y)},$$
$$\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,y)d(y,Tx)}{1+d(x,Tx)}, \frac{[1+d(y,Ty)]d(y,Tx)}{1+d(x,Tx)},$$
$$\frac{d(x,Ty)d(y,Tx)}{1+d(x,Tx)}, \frac{d^2(y,Tx)}{1+d(x,Tx)}, \frac{d(x,y)d(y,Tx)}{1+d(y,Ty)},$$
$$\frac{[1+d(x,Tx)]d(y,Tx)}{1+d(y,Ty)}, \frac{d(x,Ty)d(y,Tx)}{1+d(y,Ty)}, \frac{d^2(y,Tx)}{1+d(y,Ty)},$$
$$\left.\frac{d(x,y)d(y,Tx)}{1+d(x,Ty)}, \frac{d(x,Tx)d(y,Tx)}{1+d(x,Ty)}, \frac{d(y,Ty)d(y,Tx)}{1+d(x,Ty)}, \frac{d^2(y,Tx)}{1+d(x,Ty)}\right\},$$
$$\forall x,y \in X, \quad (1.4)$$

where α is a constant in $[0,1)$ and

$$H_-(Tx,Ty) < \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)],\right.$$
$$\frac{d(x,y)d(y,Tx)}{1+H_-(Tx,Ty)}, \frac{d(x,Tx)d(y,Tx)}{1+H_-(Tx,Ty)}, \frac{d(y,Ty)d(y,Tx)}{1+H_-(Tx,Ty)},$$
$$\frac{d(x,Ty)d(y,Tx)}{1+H_-(Tx,Ty)}, \frac{d^2(y,Tx)}{1+H_-(Tx,Ty)}, \frac{d^2(x,Tx)}{1+d(x,y)},$$
$$\frac{d(y,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d(x,Ty)d(y,Tx)}{1+d(x,y)}, \frac{d^2(y,Tx)}{1+d(x,y)},$$
$$\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,y)d(y,Tx)}{1+d(x,Tx)}, \frac{[1+d(y,Ty)]d(y,Tx)}{1+d(x,Tx)},$$
$$\frac{d(x,Ty)d(y,Tx)}{1+d(x,Tx)}, \frac{d^2(y,Tx)}{1+d(x,Tx)}, \frac{d(x,y)d(y,Tx)}{1+d(y,Ty)},$$
$$\frac{[1+d(x,Tx)]d(y,Tx)}{1+d(y,Ty)}, \frac{d(x,Ty)d(y,Tx)}{1+d(y,Ty)}, \frac{d^2(y,Tx)}{1+d(y,Ty)},$$
$$\frac{d(x,y)d(y,Tx)}{1+d(x,Ty)}, \frac{d(x,Tx)d(y,Tx)}{1+d(x,Ty)},$$
$$\left.\frac{d(y,Ty)d(y,Tx)}{1+d(x,Ty)}, \frac{d^2(y,Tx)}{1+d(x,Ty)}\right\}, \quad \forall x,y \in X \text{ with } x \neq y. \quad (1.5)$$

Under certain conditions we give sufficient conditions which insure the existence of fixed points for the multi-valued contractive type mappings T satisfying (1.4) and (1.5), respectively. The results presented in this paper extend and improve some known results in [1, 3, 7, 11, 12].

The below lemmas play important and key roles in this paper.

Lemma 1.1. (see [12]) *Let* (X,d) *be a metric space and* $A, B \in CL(X)$. *Then:*

(a) $H_-(A,B) \geq 0$;

(b) $H_-(A,B) = 0$ *if and only if* $A \subseteq B$;

(c) $H_-(A,B) \leq H_-(A,C) + H_-(C,B)$ *for any* $A, B, C \in CL(X)$.

Lemma 1.2. (see [7]) *Let* (X,d) *be a metric space and* $A, B \in CL(X)$. *Then:*

(a) *for any* $\varepsilon > 0$ *and* $a \in A$, *there exists* $b \in B$ *such that*

$$d(a,b) < H_-(A,B) + \varepsilon;$$

(b) *for any* $\varepsilon > 0$, $r > 1$ *and* $a \in A$, *there exists* $b \in B$ *such that* $d(a,b) \leq rH_-(A,B)$.

Lemma 1.3. (see [7]) *Let* (X,d) *be a metric space. Then*

(a) $|d(x,A) - d(x,B)| \leq H(A,B)$, $\forall x \in X, A, B \in CL(X)$;

(b) $|d(x,A) - d(y,A)| \leq d(x,y)$, $\forall x, y \in X, A \in CL(X)$.

Lemma 1.4. *Let* (X,d) *be a metric space,* $A, B \in CL(X)$ *and* $x \in X$. *Then*

$$d(x,A) \leq d(x,B) + H_-(B,A).$$

Proof. Let $A, B \in CL(X), x \in X$ and $b \in B$. It follows that

$$d(x,A) \leq d(x,b) + d(b,A) \leq d(x,b) + H_-(B,A),$$

which implies that

$$d(x,A) \leq d(x,B) + H_-(B,A).$$

This completes the proof. □

2. Fixed Point Theorems

In this section, we prove the existence of fixed points for the multi-valued contractive type mappings (1.4) and (1.5), respectively.

Theorem 2.1. *Let* (X,d) *be a complete metric space and* $T : X \to CL(X)$ *be a multi-valued mapping satisfying* (1.4). *Then* T *has a fixed point in* X.

Proof. Let $\varepsilon > 0$ be arbitrary. Now we assert that there exists a sequence $\{x_n\}_{n\geq 0}$ satisfying $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1) + n\alpha^n \varepsilon, \quad \forall n \geq 0. \tag{2.1}$$

Take an arbitrary point $x_0 \in X$. Choose a point $x_1 \in Tx_0$. Obviously, (2.1) holds for $n = 0$. Suppose that (2.1) holds for some $n \geq 0$. It follows from Lemma 1.2 that there exists some point $x_{n+2} \in Tx_{n+1}$ such that

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &\leq H_-(Tx_n, Tx_{n+1}) + \alpha^{n+1}(1-\alpha)\varepsilon \\
&\leq \alpha \max\Big\{ d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\
&\quad \frac{1}{2}[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)], \frac{d(x_n, x_{n+1})d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \\
&\quad \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \frac{d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \\
&\quad \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \frac{d^2(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \\
&\quad \frac{d^2(x_n, Tx_n)}{1 + d(x_n, x_{n+1})}, \frac{d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, x_{n+1})}, \\
&\quad \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, x_{n+1})}, \frac{d^2(x_{n+1}, Tx_n)}{1 + d(x_n, x_{n+1})}, \\
&\quad \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, x_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \\
&\quad \frac{[1 + d(x_{n+1}, Tx_{n+1})]d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \\
&\quad \frac{d^2(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \frac{d(x_n, x_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \\
&\quad \frac{[1 + d(x_n, Tx_n)]d(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \\
&\quad \frac{d^2(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \frac{d(x_n, x_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_{n+1})}, \\
&\quad \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_{n+1})}, \frac{d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_{n+1})}, \\
&\quad \frac{d^2(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_{n+1})} \Big\} + \alpha^{n+1}(1-\alpha)\varepsilon \\
&\leq \alpha \max\Big\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}d(x_n, x_{n+2}), 0 \Big\} + \alpha^{n+1}(1-\alpha)\varepsilon
\end{aligned}$$

$$= \alpha \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\} + \alpha^{n+1}(1-\alpha)\varepsilon. \quad (2.2)$$

Suppose that $d(x_n, x_{n+1}) \le d(x_{n+1}, x_{n+2})$. In view of (2.2), we infer that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\le \alpha d(x_{n+1}, x_{n+2}) + \alpha^{n+1}(1-\alpha)\varepsilon, \\ &\le \alpha d(x_{n+1}, x_{n+2}) + \alpha(1-\alpha)d(x_n, x_{n+1}) + \alpha^{n+1}(1-\alpha)\varepsilon, \end{aligned}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \le \alpha d(x_n, x_{n+1}) + \alpha^{n+1}\varepsilon. \quad (2.3)$$

Suppose that $d(x_n, x_{n+1}) > d(x_{n+1}, x_{n+2})$. (2.2) means that

$$d(x_{n+1}, x_{n+2}) \le \alpha d(x_n, x_{n+1}) + \alpha^{n+1}(1-\alpha)\varepsilon \le \alpha d(x_n, x_{n+1}) + \alpha^{n+1}\varepsilon,$$

that is, (2.3) holds. Taking into account (2.3), we obtain that

$$d(x_{n+1}, x_{n+2}) \le \alpha[\alpha^n d(x_0, x_1) + n\alpha^n\varepsilon] + \alpha^{n+1}\varepsilon = \alpha^{n+1}d(x_0, x_1) + (n+1)\alpha^{n+1}\varepsilon.$$

Hence (2.1) holds for $n+1$. Consequently, (2.1) holds for every $n \ge 0$. For any $m \ge 1$ and $p \ge 1$, by (2.1) we conclude that

$$\begin{aligned} d(x_m, x_{m+p}) \le \sum_{n=m}^{m+p-1} d(x_n, x_{n+1}) &\le \sum_{n=m}^{m+p-1} [\alpha^n d(x_0, x_1) + n\alpha^n\varepsilon] \\ &\le \frac{\alpha^m}{1-\alpha} d(x_0, x_1) + \varepsilon \sum_{n=m}^{m+p-1} n\alpha^n. \end{aligned}$$

Notice that $\alpha \in [0, 1)$ and

$$\lim_{m\to+\infty} \left[\frac{\alpha^m}{1-\alpha} d(x_0, x_1) + \varepsilon \sum_{n=m}^{+\infty} n\alpha^n\right] = 0.$$

Hence $\{x_n\}_{n\ge 0}$ is a Cauchy sequence and there exists some $z \in X$ with $\lim_{n\to+\infty} x_n = z$ by completeness of X.

Next we investigate that z is the fixed point of T. In fact, by (1.4), Lemmas 1.1, 1.3 and 1.4, we derive that

$$\begin{aligned} d(z, Tz) &\le d(z, Tx_n) + H_-(Tx_n, Tz) \\ &\le d(z, x_n) + d(x_n, Tx_n) + H_-(Tx_n, Tz) \le d(z, x_n) + d(x_n, Tx_n) \\ &+ \alpha \max\Bigg\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2}[d(x_n, Tz) + d(z, Tx_n)], \\ &\frac{d(x_n, z)d(z, Tx_n)}{1 + H_-(Tx_n, Tz)}, \frac{d(x_n, Tx_n)d(z, Tx_n)}{1 + H_-(Tx_n, Tz)}, \frac{d(z, Tz)d(z, Tx_n)}{1 + H_-(Tx_n, Tz)}, \\ &\frac{d(x_n, Tz)d(z, Tx_n)}{1 + H_-(Tx_n, Tz)}, \frac{d^2(z, Tx_n)}{1 + H_-(Tx_n, Tz)}, \frac{d^2(x_n, Tx_n)}{1 + d(x_n, z)}, \end{aligned}$$

$$\frac{d(z,Tz)d(z,Tx_n)}{1+d(x_n,z)}, \frac{d(x_n,Tz)d(z,Tx_n)}{1+d(x_n,z)}, \frac{d^2(z,Tx_n)}{1+d(x_n,z)},$$
$$\frac{d(x_n,Tx_n)d(z,Tz)}{1+d(x_n,z)}, \frac{d(x_n,z)d(z,Tx_n)}{1+d(x_n,Tx_n)}, \frac{[1+d(z,Tz)]d(z,Tx_n)}{1+d(x_n,Tx_n)},$$
$$\frac{d(x_n,Tz)d(z,Tx_n)}{1+d(x_n,Tx_n)}, \frac{d^2(z,Tx_n)}{1+d(x_n,Tx_n)}, \frac{d(x_n,z)d(z,Tx_n)}{1+d(z,Tz)},$$
$$\frac{[1+d(x_n,Tx_n)]d(z,Tx_n)}{1+d(z,Tz)}, \frac{d(x_n,Tz)d(z,Tx_n)}{1+d(z,Tz)}, \frac{d^2(z,Tx_n)}{1+d(z,Tz)},$$
$$\frac{d(x_n,z)d(z,Tx_n)}{1+d(x_n,Tz)}, \frac{d(x_n,Tx_n)d(z,Tx_n)}{1+d(x_n,Tz)},$$
$$\left.\frac{d(z,Tz)d(z,Tx_n)}{1+d(x_n,Tz)}, \frac{d^2(z,Tx_n)}{1+d(x_n,Tz)}\right\} \le d(z,x_n)+d(x_n,x_{n+1})$$
$$+\alpha\max\Big\{d(x_n,z), d(x_n,x_{n+1}), d(z,Tz), \frac{1}{2}[d(x_n,Tz)+d(z,x_{n+1})],$$
$$d(x_n,z)d(z,x_{n+1}), d(x_n,x_{n+1})d(z,x_{n+1}), d(z,Tz)d(z,x_{n+1}),$$
$$d(x_n,Tz)d(z,x_{n+1}), d^2(z,x_{n+1}), d^2(x_n,x_{n+1}), d(x_n,x_{n+1})d(z,Tz),$$
$$[1+d(z,Tz)]d(z,x_{n+1}), [1+d(x_n,x_{n+1})]d(z,x_{n+1})\Big\}, \quad \forall n \ge 0. \quad (2.1)$$

Letting $n \to +\infty$ in the above inequalities, by Lemma 1.3 we deduce that

$$d(z,Tz) \le \alpha\max\Big\{0, d(z,Tz), \frac{1}{2}d(z,Tz)\Big\} = \alpha d(z,Tz),$$

which yields that $z \in Tz$. This completes the proof. □

Remark 2.1. Theorem 2.1 extends and improves Theorem 3.1 of Liu, Chen, Kang and Kim [7], Theorem 5 of Nadler [11] and Theorem 2.4 of Wang [12]. The following example shows that Theorem 2.1 generalizes properly the corresponding results in [7, 11, 12].

Example 2.1. Let $X = \{0,1,6\}$ with the usual metric. Define a multi-valued mapping $T : X \to CL(X)$ by $T0 = \{6\}, T1 = \{0\}$ and $T6 = \{1,6\}$. It is easy to verify that the multi-valued mapping T satisfies (1.4) with $\alpha = \frac{1}{5}$. Thus Theorem 2.1 guarantees that T has a fixed point in the complete metric space X. But we could not use Theorem 3.1 of Liu, Chen, Kang and Kim [7], Theorem 5 of Nadler [11] and Theorem 2.4 of Wang [12] to show that T has a fixed point in X since for any constant $\alpha \in [0,1)$

$$H_-(T0,T1) = 6$$
$$< \alpha\max\left\{d(0,1), d(0,6), d(1,0), \frac{1}{2}[d(0,0)+d(1,6)], \frac{[1+d(1,0)]d(1,6)}{1+d(0,6)}\right\} = 6\alpha$$

does not hold.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to C(X)$ be a continuous multi-valued mapping satisfying* (1.5). *Assume that there exists a sequence $\{x_n\}_{n\geq 0} \subset X$ with*

$$d(x_n, x_{n+1}) = d(x_n, Tx_n),\ x_{n+1} \in Tx_n, \quad \forall n \geq 0, \tag{2.4}$$

and the sequence has a cluster point $z \in X$. Then z is a fixed point of T in X.

Proof. We assert that $x_n \neq x_{n+1}$ for each $n \geq 0$. Otherwise there exists some $k \geq 0$ such that $x_k = x_{k+1}$. Consequently, $x_k = x_{k+1} \in Tx_k$ and $d(x_{k+1}, x_{k+2}) = d(x_{k+1}, Tx_{k+1}) = 0$, which gives that $x_k = x_{k+1} = x_{k+2}$. It is easy to see that $x_n = x_k$ for any $n > k$. Thus $\{x_n\}_{n\geq 0}$ does not have a cluster point, which is impossible.

We now show that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \quad \forall n \geq 0. \tag{2.5}$$

Let n be a nonnegative integer. It follows from (1.5) and (2.4) that

$$\begin{aligned} H_-(Tx_n, Tx_{n+1}) < \max\Big\{ & d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ & \frac{1}{2}[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)], \frac{d(x_n, x_{n+1})d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \\ & \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \frac{d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \\ & \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \frac{d^2(x_{n+1}, Tx_n)}{1 + H_-(Tx_n, Tx_{n+1})}, \frac{d^2(x_n, Tx_n)}{1 + d(x_n, x_{n+1})}, \\ & \frac{d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, x_{n+1})}, \\ & \frac{d^2(x_{n+1}, Tx_n)}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \\ & \frac{d(x_n, x_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \frac{[1 + d(x_{n+1}, Tx_{n+1})]d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \\ & \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \frac{d^2(x_{n+1}, Tx_n)}{1 + d(x_n, Tx_n)}, \\ & \frac{d(x_n, x_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \frac{[1 + d(x_n, Tx_n)]d(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \\ & \frac{d(x_n, Tx_{n+1})d(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \frac{d^2(x_{n+1}, Tx_n)}{1 + d(x_{n+1}, Tx_{n+1})}, \end{aligned}$$

$$\frac{d(x_n,x_{n+1})d(x_{n+1},Tx_n)}{1+d(x_n,Tx_{n+1})}, \frac{d(x_n,Tx_n)d(x_{n+1},Tx_n)}{1+d(x_n,Tx_{n+1})},$$
$$\left.\frac{d(x_{n+1},Tx_{n+1})d(x_{n+1},Tx_n)}{1+d(x_n,Tx_{n+1})}, \frac{d^2(x_{n+1},Tx_n)}{1+d(x_n,Tx_{n+1})}\right\}$$
$$\leq \max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{1}{2}d(x_n,x_{n+2}), 0\right\}$$
$$= \max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\right\},$$

which gives that

$$\begin{aligned} d(x_{n+1},x_{n+2}) &= d(x_{n+1},Tx_{n+1}) \leq H_-(Tx_n,Tx_{n+1}) \\ &< \max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\} = d(x_n,x_{n+1}), \end{aligned}$$

hence (2.5) holds for $n \geq 0$. It is clear that $\{d(x_n,x_{n+1})\}_{n\geq 0}$ is a strictly decreasing sequence and it converges to some $r \geq 0$. Since $\{x_n\}_{n\geq 0}$ has a cluster point z, it follows that there exists a subsequence $\{x_{n_i}\}_{i\geq 1}$ of $\{x_n\}_{n\geq 0}$ such that $\lim_{i\to+\infty} x_{n_i} = z$. In view of Lemma 2.3 we deduce that

$$\begin{aligned} &|d(x_{n_i},Tx_{n_i}) - d(x_{n_i},Tz)| \leq H(Tx_{n_i},Tz) \to 0 \quad \text{as } i \to +\infty, \\ &|d(x_{n_i},Tz) - d(z,Tz)| \leq d(x_{n_i},z) \to 0 \quad \text{as } i \to +\infty, \end{aligned}$$

which mean that

$$d(x_{n_i},x_{n_i+1}) = d(x_{n_i},Tx_{n_i}) \to d(z,Tz) = r \quad \text{as } i \to +\infty. \tag{2.6}$$

Notice that the continuity of T guarantees that

$$d(x_{n_i+1},Tz) \leq H_-(Tx_{n_i},Tz) \leq H(Tx_{n_i},Tz) \to 0 \quad \text{as } i \to +\infty. \tag{2.7}$$

From $Tz \in C(X)$, we can select a sequence $\{z_{n_i}\}_{i\geq 1} \subset Tz$ such that $d(x_{n_i+1},z_{n_i}) = d(x_{n_i+1},Tz)$ for each $i \geq 1$. The compactness of Tz ensures that there exists a subsequence $\{z_{n_{i_k}}\}_{k\geq 1}$ of $\{z_{n_i}\}_{i\geq 1}$ with $\lim_{k\to+\infty} z_{n_{i_k}} = u \in Tz$. It follows that

$$d(x_{n_{i_k}+1},u) \leq d(x_{n_{i_k}+1},z_{n_{i_k}}) + d(z_{n_{i_k}},u) \to 0 \quad \text{as } k \to +\infty. \tag{2.8}$$

By virtue of Lemma 2.3, we conclude that

$$\begin{aligned} &|d(x_{n_{i_k}+1},Tx_{n_{i_k}+1}) - d(x_{n_{i_k}+1},Tu)| \leq H(Tx_{n_{i_k}+1},Tu) \to 0 \quad \text{as } k \to +\infty, \\ &|d(x_{n_{i_k}+1},Tu) - d(u,Tu)| \leq d(x_{n_{i_k}+1},u) \to 0 \quad \text{as } k \to +\infty, \end{aligned}$$

which imply that

$$d(x_{n_{i_k}+1},x_{n_{i_k}+2}) = d(x_{n_{i_k}+1},Tx_{n_{i_k}+1}) \to d(u,Tu) = r \quad \text{as } k \to +\infty. \tag{2.9}$$

In light of (2.6) and (2.8) we derive that

$$d(x_{n_{i_k}},u) \leq d(x_{n_{i_k}},x_{n_{i_k}+1}) + d(x_{n_{i_k}+1},u) \to r \quad \text{as } k \to +\infty,$$

that is, $d(z,u) = r$. In view of (2.6)-(2.9), we deduce that

$$r = d(z,u) = d(u,Tu) = d(z,Tz). \tag{2.10}$$

Suppose that $z \neq u$. By virtue of (1.5) and (2.10), we get that

$$\begin{aligned}
r = d(u,Tu) &\leq H_-(Tz,Tu) \\
&< \max\Big\{ d(z,u), d(z,Tz), d(u,Tu), \frac{1}{2}[d(z,Tu)+d(u,Tz)], \\
&\frac{d(z,u)d(u,Tz)}{1+H_-(Tz,Tu)}, \frac{d(z,Tz)d(u,Tz)}{1+H_-(Tz,Tu)}, \frac{d(u,Tu)d(u,Tz)}{1+H_-(Tz,Tu)}, \\
&\frac{d(z,Tu)d(u,Tz)}{1+H_-(Tz,Tu)}, \frac{d^2(u,Tz)}{1+H_-(Tz,Tu)}, \frac{d^2(z,Tz)}{1+d(z,u)}, \\
&\frac{d(u,Tu)d(u,Tz)}{1+d(z,u)}, \frac{d(z,Tu)d(u,Tz)}{1+d(z,u)}, \frac{d^2(u,Tz)}{1+d(z,u)}, \\
&\frac{d(z,Tz)d(u,Tu)}{1+d(z,u)}, \frac{d(z,u)d(u,Tz)}{1+d(z,Tz)}, \frac{[1+d(u,Tu)]d(u,Tz)}{1+d(z,Tz)}, \\
&\frac{d(z,Tu)d(u,Tz)}{1+d(z,Tz)}, \frac{d^2(u,Tz)}{1+d(z,Tz)}, \frac{d(z,u)d(u,Tz)}{1+d(u,Tu)}, \\
&\frac{[1+d(z,Tz)]d(u,Tz)}{1+d(u,Tu)}, \frac{d(z,Tu)d(u,Tz)}{1+d(u,Tu)}, \frac{d^2(u,Tz)}{1+d(u,Tu)}, \\
&\frac{d(z,u)d(u,Tz)}{1+d(z,Tu)}, \frac{d(z,Tz)d(u,Tz)}{1+d(z,Tu)}, \frac{d(u,Tu)d(u,Tz)}{1+d(z,Tu)}, \frac{d^2(u,Tz)}{1+d(z,Tu)} \Big\} \\
&\leq \max\left\{ r, \frac{r^2}{1+r}, 0 \right\} = r,
\end{aligned}$$

which is impossible. Hence $z = u$ and $z \in Tz$ by (2.10). This completes the proof. □

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