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BEST SIMULTANEOUS APPROXIMATION VIA KKM-MAPPING METHOD FOR MULTIVALUED MAPPINGS

H.K. Pathak[1], Satyaj Tiwari[2] §

[1]School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur (C.G.), 492010, INDIA
e-mail: hkpathak05@gmail.com

[2]Department of Mathematics
Shri Shankaracharya Institute of Professional Management and Technology
P.O. Sejabahar, Mujgahan, Raipur (C.G.), 490021, INDIA
e-mail: tsatyaj@yahoo.co.in

Abstract: In this paper, the method of KKM-mapping and the new notion of the measure of quasi-convexity is used to prove result on the best simultaneous approximation for multi-maps. We also give some results on best simultaneous approximation on non convex set. The results of this paper generalized some known results in the literature.

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1. Introduction and Preliminaries

Approximation theory for single valued maps is very rich, but for the multi-valued map is not. Note that the multi-valued mappings play a major role in many areas as in studying disjunctive logic programs, game theory, etc. In this paper using the method of KKM-mapping and the notion of the measure of quasi-convexity, we prove results on best simultaneous approximation for multi-maps. For the detail survey of the subject we refer the reader to Singh, Watson and Shrivastav [9] and Yuan [10].

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§Correspondence author

We first review needed definition. Let $(X, \|.\|)$ be a normed linear space. We denote by $\rho(X)$, the family of all nonempty subsets of X, by $CB(X)$, $K(X)$ and $K_{co}(X)$, the families of all closed bounded, compact and compact convex members of $\rho(X)$, respectively. Let K be a compact convex set of normed space X, $n \in \mathbb{N}$ and $G_i : K \times K \to K_{co}(X)$ maps for all $i \in [n]$, where $[n] = 1, 2, 3..., n$. A best simultaneous approximation problem is to find $y_0 \in K$ such that it satisfies the following equality

$$\sum_{i=1}^{n} \|G_i(y_0, y_0)\| = \inf_{x \in C} \sum_{i=1}^{n} \|G_i(x, y_0)\|.$$

for all $x \in K$. In the next co is the convex hull operation. For the given multi-valued mapping $F : X \to \rho(X)$ and sets $A \subset X$ and $B \subset X$ we define sets:

$$F(A) = \cup_{x \in A} F(x), \qquad F^-(B) = \{x \in X : F(x) \cap B \neq \phi\},$$

$$F^+(B) = \{x \in X : F(x) \subset B\}.$$

Let X be a normed space with norm $\|.\|$. For any nonnegative real number r and any subset A of X, we define the $r-$parallel set of A as

$$A + r = \cup\{B[a, r] : a \in A\}.$$

If A is a nonempty subset of X we define

$$\|A\| = \inf\{\|a\| : a \in A\}.$$

A mapping $F : X \to 2^X$ is upper (lower) semi-continuous on X if and only if for every open set $V \subset X$ the set $F^+(V)(F^-(V))$ is open. A mapping $F : X \to \rho(X)$ is continuous if and only if it is upper and lower semi-continuous.

Mappings $F_1, F_2 : X \to \rho(X)(F_1, F_2 : X \to K(X))$ are lower (upper) semi-continuous and the function $f : X \to \mathbb{R}$ is continuous, then the mapping $F_1 + F_2$ defined by $(F_1 + F_2)(x) = F_1(x) + F_2(x)$, $x \in X$ and the mapping $f \cdot F_1$ defined by $(f \cdot F_1)(x) = f(x) \cdot F_1(x)$, $x \in X$ are lower (upper) semi-continuous.

For $A, B \in CB(X)$, the Hausdorff distance denoted by $H(A, B)$ is defined by

$$H(A, B) = \max\{D(A, B), D(B, A)\},$$

where $D(A, B) = \sup_{y \in A} \inf_{x \in B} \|x - y\|$.

A mapping $F : X \to K(X)$ is continuous if and only if F is continuous in the Hausdorff sense.

Definition. Let C be a subset of X, a map $F : C \to 2^Y$ is called quasi-convex (see for example K. Nikodem [7]) if and only if it satisfies the condition

$$F(x_i) \cap S \neq \phi, i = 1, 2 \Rightarrow F(\lambda x_1 + (1 - \lambda)x_2) \cap S \neq \phi$$

for all convex sets $S \subset Y$, x_1, $x_2 \in C$ and $\lambda \in [0, 1]$.

Remark 1. A map $F : C \to 2^X$ is quasi-convex if and only if the set $F^-(S)$ is convex for each convex set $S \subseteq X$.

Definition. Let X and Y be normed spaces and $F : X \to 2^Y$. The real number $mq(F)$, defined by

$$mq(F) = \inf\{r > 0 : co(F^-(S)) \subseteq F^-(S + r) \text{ for all convex } S \subseteq Y\}$$

is called a measure of convexity for map F.

Remark 2. (i) If F is quasi-convex map then $mq(F) = 0$.

(ii) If α is a real number then $mq(\alpha F) = |\alpha| mq(F)$.

Remark 3. Let E be a topological vector space and X be a nonempty subset of E. A mapping $T : X \to \rho(E)$ is called KKM-mapping if

$$co\{x_1, x_2, \cdots, x_n\} \subset \cup_{i=1}^n T(x_i)$$

for each finite subset $\{x_1, x_2, \cdots, x_n\}$ of X.

In 1961, Ky Fan [3] proved the following generalized form of the Knaster-Kuratowski-Mazurkicwicz [5] theorem holds.

Theorem A. (see [3]) *Let E be the topological vector space, X be a non empty subset of E and $T : X \to \rho(E)$ be a KKM-mapping with closed values. If $T(x)$ is compact for at least one $x \in D$, then $\cap_{x\in X} T(x) \neq \phi$.*

2. Main Results

Lemma 2.1. (see [1]) *Let $(X, \|\cdot\|)$ be a normed linear space and $A, B \subset CB(X)$. Then*

$$|\|A\| - \|B\|| \leq H(A, B).$$

By using Lemma 2.1 above, we can obtain the following lemma.

Lemma 2.2. (see [11]) *Let $(X, \|\cdot\|)$ be a normed linear space and $F : X \to CB(X)$ be continuous in Hausdorff sense. Then the function $f : X \to \mathbb{R}$ defined by $f(x) = \|F(x)\| \, \forall \, x \in X$ is continuous.*

Using Ky Fan's result we shall prove the main result of this section as follows.

Theorem 2.3. *Let X be an normed space, $C \in K_{co}(X)$, and $G_i : C \times C \to K_{co}(X)$, $i = 1, 2, 3, \cdots, n$ be continuous maps with convex compact values, then there exist $y_0 \in C$ such that*

$$\sum_{i=1}^{n} \|G_i(y_0, y_0)\| \leq \inf_{x \in C} \sum_{i=1}^{n} (\|G_i(x, y_0)\| + mq(G_i)).$$

Proof. Let for $x \in C$, $T : C \to 2^C$ be defined by

$$T(x) = \{y \in C : \sum_{i=1}^{n} \|G_i(y, y)\| \leq \sum_{i=1}^{n} (\|G_i(x, y)\| + mq(G_i))\}.$$

Since the mapping G_i are continuous, they are also continuous in the Hausdorff sense. From Lemma 2.2, it follows that $T(x)$ is closed. Since $C \in K_{co}(X)$, $T(x)$ is compact for each $x \in C$. We prove that T is a KKM-mapping; i.e., for every $\{x_1, x_2, \cdots, x_n\} \subset C$ we have

$$co\{x_1, x_2, \cdots, x_n\} \subset \cup_{i=1}^{n} T(x_i).$$

If the above does not hold, there exist $y = \sum_{j=1}^{m} \lambda_j x_j$, where $\lambda_j \geq 0, \sum \lambda_j = 1$ such that $y \notin \cup_{j=1}^{m} T(x_j)$. Then, we have

$$\sum_{i=1}^{n} \|G_i(y, y)\| > \sum_{i=1}^{n} (\|G_i(x_j, y)\| + mq(G_i))$$

for $j = 1, 2, 3, \cdots, m$.

Since $G_i : C \times C \to K_{co}(X)$, there exist $v_{ij} \in G_i(x_j, y)$, such that $\|v_{ij}\| = \|G_i(x_j, y)\|$, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, m$. Let $S = co\{v_{ij}, i = 1, 2..n, j = 1, 2, ..m\}$. Then we have

$$(G_i(x_j, y) \cap S) \neq \phi \ \text{ and } \ (x_j, y) \in G_i^{-}(S),$$

for every $i = 1, 2, 3.....n$. Since the set $G_i(y, y) + S$ is convex and $mq(G_i)$ is a measure of quasi-convexity, we have

$$(y, y) \in G_i^{-}(S + mq(G_i) + \epsilon_i)$$

for each $\epsilon_i > 0$, therefore

$$G_i(y, y) \cap (S + mq(G_i) + \epsilon_i) \neq \phi, \quad i = 1, 2, ..., n.$$

We obtain that there exist

$$v_i \in G_i(y, y) \cap (S + mq(G_i) + \epsilon_i).$$

Hence there exist $s \in S$ and $b_i \in X$ such that $\|b_i\| \leq mq(G_i) + \epsilon_i$ and $v_i = s + b_i$. Since $s \in S$ there exist $\mu_j \geq 0, j = 1, 2, 3, \cdots, m$ and $\sum_{i=1}^{n} \mu_j = 1$ such that $s = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_j v_{ij}$.

We have

$$\sum_{i=1}^{n}\|G_i(y,y)\| \le \sum_{i=1}^{n}\|v_i\| \le \sum_{i=1}^{n}(\|s\|+\|b_i\|) = \sum_{i=1}^{n}\|\sum_{j=1}^{m}\mu_j v_{ij}\| + \|b_i\|$$

$$\le \sum_{j=1}^{m}\mu_j\sum_{i=1}^{n}(\|v_{ij}\| + mq(G_i)+\epsilon_i) \le \max_{1\le j\le m}\sum_{i=1}^{n}(\|G_i(x_j,y)\| + mq(G_i)+\epsilon_i).$$

This contradicts $\sum_{i=1}^{n}\|G_i(y,y)\| > \sum_{i=1}^{n}(\|G_i(x_j,y)\| + mq(G_i))$ for every $j = 1,2,3,\cdot,m$, and so T is a KKM-mapping. Applying Theorem A we obtain that there exist $y_0 \in C$ such that $y_0 \in \cap_{x\in C}T(x)$; i.e.,

$$\sum_{i=1}^{n}\|G_i(y_0,y_0)\| \le \inf_{x\in C}\sum_{i=1}^{n}(\|G_i(x,y_0)\| + mq(G_i)).$$

This proves the theorem. □

Corollary 2.4. *Let X be a normed linear space, $C \in K_{co}(X)$ and $G_i : C\times C \to K_{co}(C), i = 1,2\cdots,n$, be continuous mappings. If $x \to G_i(x,.)$ are quasi-convex, then there exist $y_0 \in C$ such that*

$$\sum_{i=1}^{n}\|G_i(y_0,y_0)\| = \inf_{x\in C}\sum_{i=1}^{n}\|G_i(x,y_0)\|.$$

3. Best Simultaneous Approximation on Non Convex Set

Definition 3.1. A function $\Phi : [0,+\infty) \to [0,+\infty)$ is called a semi modules function if it satisfies the following conditions:

1. $\Phi(x+y) \le \Phi(x) + \Phi(y)$.
2. Φ is a continues increasing function.

For $a \in X$, we denote $\Phi(\|a\|)$ by $\|a\|_\Phi$.

Definition 3.2. We say that $x \in C$ is a Φ-best simultaneous approximation from C to the pair of elements $x_1, x_2 \in X$ if

$$\Phi(\|x_1 - x\|) + \Phi(\|x_2 - x\|) \le \inf_{z\in C}\{\Phi(\|x_1 - z\|) + \Phi(\|x_2 - z\|)\}$$

for every $z \in C$.

Now we quote some properties of tangent cones to a convex set C at a point x not necessarily in C. All the result are either well known or elementary, so that their proof are merely sketched.

In what follows, D denote a convex set in a normed linear space X, C a convex subset of D, and x a point in D. The radial cone, the tangent cone, and the normal cone to C at x are, respectively, given by

$$R_C(x) = \{v \in X : v = t(z - x) \text{ for some } t \geq 0 \text{ and } z \in C\};$$

$$T_C(x) = \overline{R_C(x)}; \qquad N_C(x) = \{x' \in X' : \sup_{z \in C} < x', z - x > \ \leq 0\}.$$

Clearly, $R_C(x) = X$ whenever x lies in the interior of C. In the lemma below, the first assertion follows from the convexity of C, the second one from Hahn-Banach Separation Theorem.

In this section C_x stand for the convex hull of $C \cup \{x\}$.

Lemma 3.3. (see [6]) *(i)* $R_C(x) = R_{C_x}(x)$; *(ii) In case X is Hausdorff l.c.s.,* $R_C(x) = \{v \in X : \ < x', \ v > \ \leq 0$ *for all* $x' \in N_C(x)\}$.

We now turn to discussing the following set: $RF_C(x) = \{v \in X : v = 0$ or $v = \lambda(z - x)$ for some $z \in C$ and λ with $Re(\lambda) > 0\}$. Here, $Re(\lambda)$ denote the real part of the complex number λ. Contrary the previous cones, $RF_C(x)$ is not convex in general. Obviously, if X is real $RF_C(x) = R_C(x)$.

Lemma 3.4. (see [6]) *(i)* $RF_C(x) = RF_{C_x}(x)$; *(ii)* $x + RF_C(x) = \{v \in X : tx + (1 - t)v \in C_x\}$ *for some complex number t with* $|t| < 1$.

We consider the following boundary condition:

$$\text{For each } x \in D, F(x) - x \subset T_x(x) \cup RF_D(x).$$

Next, the study of Φ-best simultaneous approximation problem justifies the introduction of the following set:

$$n_C(x; \|.\|_\Phi) = \{v_1, v_2 \in X : \|v_1 - x\|_\Phi + \|v_2 - x\|_\Phi \leq \|v_1 - z\|_\Phi + \|v_2 - z\|_\Phi$$
for all $z \in C_x\}$.

Lemma 3.5. *Let X be a normed space. Then* $(x + T_C(x) \cup RF_C(x)) \cap n_C(x; \|.\|_\Phi) \subset \{v_1, v_2 \in X : \|v_1 - x\|_\Phi + \|v_2 - x\|_\Phi = 0\}$.

Proof. Let $v_i \in (x + RF_C(x)) \cap \ n_C(x, \|.\|_\Phi)$, for $i = 1, 2.$.

By Lemma 3.4 we have $tx + (1 - t)v \in C_x$ for some $|t| < 1$. We also have

$$\|v_1 - x\|_\Phi + \|v_2 - x\|_\Phi \leq \|v_1 - z\|_\Phi + \|v_2 - z\|_\Phi$$

for all $z \in C_x$. Thus in particular

$$\begin{aligned} \|v_1 - x\|_\Phi + \|v_2 - x\|_\Phi &\leq \|v_1 - (tx + (1 - t)v_1)\|_\Phi + \|v_2 - (tx + (1 - t)v_2)\|_\Phi \\ &= |t| \ (\|v_1 - x\|_\Phi + \|v_2 - x\|_\Phi). \end{aligned}$$

Since $|t| < 1$, we necessarily have $\|v_1 - x\|_\Phi + \|v_2 - x\|_\Phi = 0$.

Next, let $v_i \in (x + T_C(x)) \cap n_C(x, \|.\|_\Phi)$. We argue by contradiction, so assume that

$$\||v_1 - x\|_\Phi + \|v_2 - x\|_\Phi = \alpha > 0.$$

Consider the open nbd. of v_i given by

$$U_{v_i} = \{u_i \in X : \|u_1 - x\|_\Phi + \|u_2 - x\|_\Phi < \alpha/4\}, \quad i = 1, 2.$$

We claim that

$$U_{v_i} \cap (x + R_C(x)) = \phi.$$

for $u_i \in U_{v_i}$ and $0 \leq t < 1$ we have

$$\|v_i - tx - (1-t)u_i\|_\Phi \leq t\|v_i - x\|_\Phi + (1-t)\|v_i - u_i\|_\Phi < t\alpha/2 + (1-t)\alpha/4 < \alpha.$$

This shows that $tx + (1-t)u_i$ does not belong to C_x.

Since any point z of C_x must satisfies $\|v_i - z\|_\Phi \geq \alpha$. By assumption $v_i \in n_c(x; \|.\|_\Phi)$, we have $u_i \notin x + R_C(x) = \{u_i \in X : tx + (1-t)u_i \in C_x$ for some $0 \leq t < 1\}$. We conclude by observing that $U_{v_i} \cap (x + R_C(x)) = \phi$, which contradict the fact that $v_i \in x + \overline{R_C(x)}$.

Theorem 3.6. *Let X be a nonempty convex set in a normed linear space E, K a nonempty compact convex subset of X, and $F_i : X \to K_{co}(E)$ a continuous multi-function. Then, for each continuous function $\Phi : [0, \infty) \to [0, \infty)$, at least one of the following properties hold:*

(i) There exist an external B.S.A. $y \in X \setminus K$ such that $\sum_{i=}^{n} \|y - F_i(y)\|_\Phi = \inf_{x \in K_y} \|x - F_i(y)\|_\Phi$.

(ii) There exist an internal B.S.A. $y_0 \in K$ such that $\sum_{i=}^{n} \|y_0 - F_i(y_0)\|_\Phi = \inf_{x \in X} \|x - F_i(y_0)\|_\Phi$.

As usual, we have let $\|z - C\|_\Phi = \inf\{\Phi(z - v) : v \in C\}$ for $z \in X$ and $C \subset E$.

Proof. Consider the multi-function $G : X \to 2^X$ given by

$$G(x) = \{y \in X : \sum_{i=1}^{n} \|y - F_i(y)\|_\Phi \leq \sum_{i=1}^{n} \|x - F_i(y)\|_\Phi\}$$

for $x \in X$. We claim that G is a KKM; i.e., for every $\{x_1, x_2, \cdots, x_m\} \subset X$ we have

$$co\{x_1, x_2, \cdots, x_m\} \subset \cup_{i=1}^{n} G(x_i).$$

If the above does not hold, there exist $y_0 \in co\{x_1, x_2, \cdots, x_m\}$ such that $y_0 =$

$\Sigma\lambda_j x_j \notin \cup_{j=1}^m G(x_j)$ with $\lambda_j \geq 0$ and $\Sigma_{j=1}^m \lambda_j = 1$ and

$$\sum_{i=1}^n \|y_0 - F_i(y_0)\|_\Phi > \sum_{i=1}^n \|x_j - F_i(y_0)\|_\Phi$$

for each $j = 1, 2\cdots, m$. Since each $F_i(y)$ is compact hence for each $j = 1, 2\cdots, m$ there exist $u_{ij} \in F_i(y)$ such that

$$\sum_{i=1}^n \|x_j - u_{ij}\|_\Phi = \sum_{i=1}^n \|x_j - F_i(y)\|_\Phi$$

for each $j = 1, 2\cdots, m$. Let $U_i = \Sigma_{j=1}^m \lambda_j u_{ij}$, then $U_i \in F_i(y)$. Since $F_i(y)$ is also convex. It follows that

$$\begin{aligned}\sum_{i=1}^n \|y_0 - F_i(y_0)\|_\Phi &\leq \sum_{i=1}^n \|y_0 - U_i\|_\Phi \\ &= \sum_{i=1}^n \|\sum_{j=1}^m \lambda_j(x_j - u_{ij})\|_\Phi \leq \sum_{i=1}^n \sum_{j=1}^m \lambda_j \|x_j - u_{ij}\|_\Phi \\ &\leq \sum_{j=1}^m \lambda_j \sum_{i=1}^n \|x_j - u_{ij}\|_\Phi = \sum_{j=1}^m \lambda_j \sum_{i=1}^n \|x_j - F_i(y_0)\|_\Phi \\ &\leq \max_{1\leq j\leq m} \lambda_j \sum_{i=1}^n \|x_j - F_i(y_0)\|_\Phi = \sum_{i=1}^n \|x_j - F_i(y_0)\|_\Phi,\end{aligned}$$

which is a contradiction.

Now we claim that each $G(x)$ is closed in X. Let $\theta_i : X \to R$ such that $\theta_i(y) = \|y - F_i(y)\|_\Phi$ and $\psi_{i,x} : X \to R$ such that $\psi_{i,x}(y) = \|x - F_i(y)\|_\Phi$ for any $x \in X$. To prove $G(x)$ is closed we need to show that θ_i is lower semi-continuous on X, while the function $\psi_{i,x}$ is upper semi-continuous on X. Let $r \in R$ be fixed. Because of the compactness of each $F_i(y)$, we have:

(a) $\cap\{y \in X : \sum_{i=1}^n \theta_i(y) > r\}$
$= \cap\{y \in X : (y, F_i(y)) \subset \{(z, v) \in X \times E : \|z - v\|_\Phi > r\}\}.$

(b) $\cap\{y \in X : \sum_{i=1}^n \psi_{i,x}(y) < r\}$
$= \cap\{y \in X : (y, F_i(y)) \cap \{v \in E : \|z - v\|_\Phi < r\} \neq \phi\}.$

The first set (a) is open since the multi-functions $y \to (y, F_i(y))$ is upper semi-continuous and the set $\{(z, v) \in X \times E : \|z - v\|_\Phi > r\}$ is open. The second set (b) is open since F_i is lower semi-continuous and the set $\{v \in E : \|z - v\|_\Phi < r\}$ is open. Thus θ_i is l.s.c. on X and $\psi_{i,x}$ is u.s.c. on X for any $x \in X$. Therefore, each $G(x)$ is closed in X.

Now assume that property (i) does not hold, i.e. for each $y \in X \setminus K$ there is an $x \in K_y$ such that $yG(x)$, or equivalently, $\{y \in X : y \in \cap\{G(x) : x \in K_y\}\} \subset K$. Therefore, $\cap\{G(x) : x \in X\} \neq \phi$. This means that there exist $y_0 \in X$ such that $\sum_{i=1}^{n} \|y_0 - F_i(y_0)\|_\Phi = \inf_{x \in K_y} \sum_{i=1}^{n} \|x - F_i(y_0)\|_\Phi$ for all $x \in X$ such a y_0 obviously lies in X and thus (ii) is satisfied. □

Theorem 3.7. *Let X be a convex set in a normed space E and $F_i : X \to K_{co}(E)$, $i = 1, 2, \dots, n$ are continuous multi-functions . Assume that there exist a nonempty compact convex set $K \subset X$ such that $F_i(y) - y \subset T_K(y) \cup RF_{i,K}(y)$, $y \in X \setminus K$. Then for each Φ, there exist $y_0 \in X$ such that*

$$\sum_{i=1}^{n} \|y_0 - F_i(y_0)\|_\Phi = \inf\{\sum_{i=1}^{n} \|x - F_i(y_0)\|_\Phi : x \in X\}.$$

Proof. By Theorem 3.6, either there exist $y_1 \in X \setminus K$ such that $\sum_{i=1}^{n} \|y_1 - F_i(y_1)\|_\Phi = \inf_{x \in K_{y_1}} \sum_{i=1}^{n} \|x - F_i(y_1)\|_\Phi$, or there exist $y_0 \in K$ such that $\sum_{i=1}^{n} \|y_0 - F_i(y_0)\|_\Phi = \inf_{x \in X} \sum_{i=1}^{n} \|x - F_i(y_0)\|_\Phi$. Clearly, it is sufficient to consider only first situation. In this case, there exist a point $y_1 \in X \setminus K$ such that

$$F_i(y_1) \cap n_K(y_1, \|.\|_\Phi) \neq \phi.$$

Since by assumption $F_i(y_1) \subset y_1 + T_K(y_1) \cup RF_{i,K}(y_1)$ it follows from Lemma 3.5 that

$$\|v_1 - x\|_\Phi + \|v_2 - x\|_\Phi = 0$$

for some $v_1, v_2 \in F(y_1)$.

Hence

$$\sum_{i=1}^{n} \|y_1 - F_i(y_1)\|_\Phi = 0 = \inf\{\sum_{i=1}^{n} \|x - F_i(y_1)\|_\Phi : x \in X\},$$

and the theorem is proved. □

References

[1] H.T. Banks, M.Q. Jacobs, A differential calculus for multifunctions, *J. Math. Anal. Appl.*, **29** (1970), 246-272.

[2] D. Delbosco, Simultaneous approximation, Ky Fan theorems and approximatively compact sets, *Indian J. of Math.*, **37**, No. 1 (1995), 69-77.

[3] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Annalen.*, **142** (1961), 305-310.

[4] Ky Fan, Extension of two fixed point theorems of F.E. Browder, *Math. Z.*, **112** (1969), 234-240.

[5] B. Knaster, K. Kuratowski, S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, *Fund. Math.* (1929), 132-137.

[6] Marc Lassonde, On the use of KKM multifunction in fixed point thery and related topics, *J. of Math. Anal. and Appl.*, **97** (1983), 151-201.

[7] K. Nikodem, *K-Convex and K-Concave Set-Valued Functions*, Politechnika, Lodzka (1989).

[8] Vasile Postolica, A survey on best approximation in locally convex spaces, *Nonlinear Forum*, **10**, No. 1 (2005), 37-53

[9] S.P. Singh, B. Watson, P. Shrivastav, *Fixed Point Theory and Best Approximation: The KKM-Map Principle*, Kluwer Academic Press (1997).

[10] G.X.Z. Yuan, *KKM Theory and Application in Nonlinear Analysis*, Marcel Dekker, New York (1999).

[11] Zoran D Mitrovic, Simultaneous approximation for multivalued mappings, *Indian J. of Mathematics*, **46**, No. 1 (2004), 79-85.