International Journal of Pure and Applied Mathematics
Volume 59 No. 2 2010, 191-193

ON THE X-RANKS OF POINTS OF $\mathbb{P}^4$, X A SURFACE

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Abstract: Here we look at the X-ranks of points of $\mathbb{P}^4$ with respect to a surface $X \subset \mathbb{P}^4$.

AMS Subject Classification: 14J99, 14N05
Key Words: ranks, X-rank, surface in $\mathbb{P}^4$

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Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denote the linear span. Hence $r_X(P) = 1$ if and only if $P \in X$. Since X is non-degenerate, the X-rank is defined and $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. If $\mathrm{char}(\mathbb{K}) = 0$, then $r_X(P) \leq n + 1 - \dim(X)$ for all $P \in \mathbb{P}^n$ (see [3], [4], Proposition 5.1). This inequality is in general sharp, but examples of varieties $X \subset \mathbb{P}^n$ such that $r_X(P) = n + 1 - \dim(X)$ for some $P \in \mathbb{P}^n$ seems to be very special (several space curves (see [5]), all rational normal curves (see [1], Theorem 4.1), for all $d > n \geq 3$ at least one smooth rational curve with degree d in $\mathbb{P}^n$, a cuspidal curve of degree $n+1$ in $\mathbb{P}^n$ (see [1], Proposition 2)). To the best of my knowledge all higher dimensional examples are obtained taking the cone of a one-dimensional example. Here we study the case of surfaces in $\mathbb{P}^4$.

Received: January 5, 2010 © 2010 Academic Publications

For any $O \in \mathbb{P}^5$ let $\ell_O : \mathbb{P}^5 \backslash \{O\} \to \mathbb{P}^4$ denote the linear projection from O.

The following result is a variation on the theme of [1], §1.

Theorem 1. *Let $Y \subset \mathbb{P}^5$ be an integral and non-degenerate surface. Assume that Y is not the Veronese surface. Then there is a non-empty open subset U of $\mathbb{P}^5$ such that $r_{\ell_O(Y)}(P) \le 2$ for all $O \in U$ and all $P \in \mathbb{P}^4$.*

Remark 1. Take Y as in Theorem 1. If Y is smooth, then we may take $U = \mathbb{P}^5 \backslash TY$.

If Y is the Veronese surface, then we may prove the following result.

Proposition 1. *Let $X \subset \mathbb{P}^4$ be a isomorphic projection of a Veronese surface of $\mathbb{P}^5$. Then $r_X(P) = 2$ for all $P \in \mathbb{P}^4 \backslash X$.*

The Veronese surface has degree 4. Minimal degree surfaces in $\mathbb{P}^4$, i.e. degree 3 non-degenerate integral surfaces, are either cones over a rational normal curve of $\mathbb{P}^3$ or they are smooth. If X is a cone over a rational normal curve of $\mathbb{P}^3$, then the set $\{P \in \mathbb{P}^4 : r_X(P) = 3\}$ is non-empty and it has dimension 3 (hence it is huge). In the other case we prove the following result.

Proposition 2. *Let $X \subset \mathbb{P}^4$ be a smooth and non-degenerate surface of degree 3. Then $r_X(P) \le 2$ for all $P \in \mathbb{P}^4$.*

Proof. All surfaces X as in the statement are projectively equivalent and isomorphic to the Hirzebruch surface F_1. Fix $P \in \mathbb{P}^4 \backslash X$. Let $\ell_P : \mathbb{P}^4 \backslash \{P\} \to \mathbb{P}^3$ be the linear projection from P. Set $v_P := \ell_P | X$. Notice that $r_X(P) \ge 3$ if and only if v_P is injective. Set $Y := v_P(X)$. Since $\deg(v_P) \cdot \deg(Y)$ and Y spans $\mathbb{P}^3$, $\deg(v_P) = 1$ and Y is a non-normal cubic surface. Thus Y has a line T as its singular locus. Assume that v_P is injective. Thus $T = v_P(D)$ for some line $D \subset X$. The lines contained in X are the fiber of the ruling π of X and the section h with negative self-intersection of the ruling. Snce $P \notin X$, T is a cuspidal line if and only if $T_Q X = \langle D \cup \{P\} \rangle$ for all $Q \in D$. This is not the case if D is a fiber of π, because X is not a cone. This is not the case if $D = h$ because $T_Q X = \langle D \cup F_Q \rangle$ for all $Q \in h$, where F_Q denote the fiber of π containing Q. □

Proof of Theorem 1. Since Y is not the Veronese surface, $\mathbb{P}^5$ is the secant variety of Y. Hence there is a non-empty open subset U of $\mathbb{P}^5$ such that $r_Y(O) = 2$ for all $O \in U$. Fix $O \in U$ and set $X := \ell_O(Y)$. Fix $P \in \mathbb{P}^4 \backslash X$. Take $A \in \mathbb{P}^5 \backslash (\{O\} \cup Y)$ such that $\ell_O(A) = P$. Fix $B \in \mathbb{P}^5 \backslash \{O\}$ and set $E := (\langle \{O, B\} \rangle \cap U) \backslash \{O\}$. Since $O \in U$, E is a non-empty open subset of the line $\langle \{O, B\} \rangle$. Fix any $Q \in E$. Since $Q \in U$, there is $S' \subset Y$ such that $\sharp(S') = 2$

and $Q \in \langle S' \rangle$. Set $S := \ell_O(S') \subset X$. Since $P = \ell_O(Q)$, we have $P \in \langle S \rangle$. Thus $r_X(P) \leq 2$. □

Proof of Proposition 1. Let $Y \subset \mathbb{P}^5$ be the Veronese embedding of $\mathbb{P}^2$, i.e. the image of $\mathbb{P}^2$ by the complete and very ample linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$. Y is unique, up to a projective transformation and its secant variety $\mathrm{Sec}(Y)$ is equal to its tangent developable TY of Y and hence a hypersurface of $\mathbb{P}^5$. For every $O \in \mathbb{P}^5$ let $\ell_O : \mathbb{P}^5 \backslash \{O\} \to \mathbb{P}^4$ denote the linear projection from O. For all $O \in \mathbb{P}^5 \backslash \mathrm{Sec}(Y)$ set $X_O := \ell_O(Y)$. Fix $O \in \mathbb{P}^5 \backslash \mathrm{Sec}(Y)$ such that X is projectively equivalent to the surface X_O. Fix $P \in \mathbb{P}^4 \backslash X_O$. Take $A \in \mathbb{P}^5 \backslash (\{O\} \cup Y)$ such that $\ell_O(A) = P$. Since TY is a hypersurface, there is $B \in \langle \{O, A\} \rangle \cap TY$. The case $n = d = 2$ of [2], Theorem 4.2, gives that existence of $S' \subset Y$ such that $\sharp(S') = 2$ and $B \in \langle S' \rangle$. Set $S := \ell_O(S') \subset X_O$. Since $\ell_O|Y$ is injective, $\sharp(S) = 2$. Since $P = \ell_O(B)$, we have $P \in \langle S \rangle$. Thus $r_{X_O}(P) \leq 2$. □

Question 1. Is there a smooth and non-degenerate surface $X \subset \mathbb{P}^4$ such that there is $P \in \mathbb{P}^4$ with $r_X(P) = 3$?

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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