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QUINTIC SPLINE METHOD FOR SOLVING TWO POINT BOUNDARY VALUE PROBLEMS

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Abstract: We use uniform quintic spline polynomial functions to develop a new spline method for solving second order boundary value problems. The present method is of order four and is capable of producing approximations for the solution as well as its first, second, third, fourth and fifth derivatives over the range of integration. The convergence analysis of the method is discussed and a bound for the error is derived. Numerical example and comparison with other fourth order spline methods are presented to demonstrate our conclusion.

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1. Introduction

We consider using quintic spline functions to form smooth approximate solutions of the two point boundary value problem

$$\begin{aligned} &y''(x) = p(x)y(x) + q(x), \quad a \le x \le b, \\ &y(a) = \alpha, \quad y(b) = \beta, \end{aligned} \tag{1.1}$$

where $p(x)$ and $q(x)$ are continuous functions on $[a, b]$ and a, b, α, β are arbitrary real finite constants. The problems of the form (1.1) arise in theory which

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describes the deflection of plates and a variety of other scientific applications. In general it is not possible to obtain the analytical solution of (1.1) for arbitrary choices of $p(x)$ and $q(x)$. We usually resort to some numerical methods for obtaining an approximate solution of (1.1).

The possibility of using spline polynomials for obtaining a smooth approximate solution of problem (1.1) was first discussed in 1967 by Ahlberg, Nilson and Walsh [1]. Following this several spline methods for solving various second order boundary value problems have been developed by different authors, see, for example [1-9] and the references therein. In this paper, we shall use quintic spline functions to derive some consistency relations. These relations are then used to develop a numerical method for obtaining smooth approximations for the solution $y(x)$ of (1.1) as well as its first, second, third, fourth and fifth derivatives at every point of the integration range. We shall show that the present method is of order four.

We remark here that the present method can be used to produce smooth approximations for the solution and its derivatives of the nonlinear boundary value problems of the form

$$\begin{aligned} &y''(x) = \Phi(x, y), \quad a \leq x \leq b, \\ &y(a) = \alpha, \quad y(b) = \beta, \end{aligned} \tag{1.2}$$

where $\Phi(x, y)$ is a nonlinear function of y. The error analysis given in Section 2 can be generalized for the nonlinear case (1.2).

2. The Quintic Spline Method

We develop a smooth approximate solution of (1.1) using quintic spline functions. For this purpose, we discretize the interval $[a, b]$ using equally spaced points $x_i = a + ih$, $i = 0, 1, 2, \ldots, n$, $x_0 = a$, $x_n = b$ and $h = (b - a)/n$, where n is a positive integer. Also, let $y(x)$ be the exact solution of the problem (1.1) and s_i be an approximation to $y_i = y(x_i)$ obtained by the quartic $Q_i(x)$ passing through the points (x_i, s_i) and (x_{i+1}, s_{i+1}). We write $Q_i(x)$ in the form

$$Q_i(x) = a_i(x-x_i)^5 + b_i(x-x_i)^4 + c_i(x-x_i)^3 + d_i(x-x_i)^2 + e_i(x-x_i) + f_i \tag{2.1}$$

for $i = 0, 1, 2, \ldots, n-1$. Then the quintic spline defined by $s(x) = Q_i(x)$, $\quad i = 0, 1, 2, \ldots, n-1$, and $s(x) \in C^4[a, b]$.

Our aim is to develop explicit expressions for the four coefficients in (2.1). To do this we first designate

$$Q_i(x_{i+\frac{1}{2}}) = s_{i+\frac{1}{2}},$$

$$\begin{aligned} Q_i'(x_i) &= D_i, \\ Q_i''(x_{i+\frac{1}{2}}) &= F_{i+\frac{1}{2}}, \\ Q_i'''(x_i) &= T_i, \\ Q_i^{(iv)}(x_{i+\frac{1}{2}}) &= W_{i+\frac{1}{2}}, \\ Q_i^{(v)}(x_i) &= M_i, \end{aligned} \tag{2.2}$$

for $i = 0, 1, \ldots, n-1$.

From conditions (2.2) we determine the five coefficients in (2.1) as functions of $s_{i+\frac{1}{2}}$, D_i, $F_{i+\frac{1}{2}}$, T_i , W_{i+1} and M_i in the form

$$\begin{aligned} &a_i = \frac{1}{120}M_i, \quad b_i = \frac{1}{24}W_{i+\frac{1}{2}} - \frac{1}{48}hM_i, \\ &c_i = \frac{1}{6}T_i, \quad d_i = \frac{1}{2}F_{i+\frac{1}{2}} - \frac{1}{4}hT_i - \frac{1}{16}h^2W_{i+\frac{1}{2}} + \frac{1}{48}h^3M_i, \\ &e_i = D_i, \end{aligned} \tag{2.3}$$

$$f_i = s_{i+\frac{1}{2}} - \frac{1}{2}hD_i - \frac{1}{8}h^2F_{i+\frac{1}{2}} + \frac{1}{24}h^3T_i + \frac{5}{384}h^4W_{i+\frac{1}{2}} - \frac{1}{240}h^5M_i$$

for $i = 0, 1, 2, \ldots, n-1$. The details of derivation of these equations are given in [3].

Now from the continuity of quintic spline $s(x)$ and its derivatives up to order two at the point (x_i, s_i) where the two quartics $Q_{i-1}(x)$ and $Q_i(x)$ join, we can have

$$Q_{i-1}^{(m)}(x_i) = Q_i^{(m)}(x_i), \quad m = 0, 1, 2, 3, 4. \tag{2.4}$$

Using (2.1)-(2.4) we get the following consistency relations

$$\begin{aligned} h[D_i + D_{i-1}] = {} & 2[s_{i+\frac{1}{2}} - s_{i-\frac{1}{2}}] - \frac{1}{4}h^2[F_{i+\frac{1}{2}} + 3F_{i-\frac{1}{2}}] \\ & + \frac{1}{12}h^3[T_i + T_{i-1}] + \frac{1}{192}h^4[5W_{i+\frac{1}{2}} + 3W_{i-\frac{1}{2}}] \\ & - \frac{1}{120}h^5[M_i + M_{i-1}], \end{aligned} \tag{2.5}$$

$$h[D_i - D_{i-1}] = h^2F_{i-\frac{1}{2}} + \frac{1}{24}h^4W_{i-\frac{1}{2}}, \tag{2.6}$$

$$h[T_i + T_{i-1}] = 2[F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}] - \frac{1}{4}h^2[W_{i+\frac{1}{2}} + 3W_{i-\frac{1}{2}}] + \frac{1}{12}h^3[M_i + M_{i-1}], \tag{2.7}$$

and

$$h[T_i - T_{i-1}] = h^2W_{i-\frac{1}{2}}, \tag{2.8}$$

$$h[M_i + M_{i-1}] = 2[W_{i+\frac{1}{2}} - W_{i-\frac{1}{2}}], \tag{2.9}$$

From (2.5)-(2.9), we have

$$hD_i = s_{i+\frac{1}{2}} - s_{i-\frac{1}{2}} - \frac{1}{24}h^2[F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}] + \frac{1}{5760}h^4[W_{i+\frac{1}{2}} - W_{i-\frac{1}{2}}], \tag{2.10}$$

$$h^2F_{i-\frac{1}{2}} = s_{i-\frac{3}{2}} - 2s_{i-\frac{1}{2}} + s_{i+\frac{1}{2}} - \frac{1}{1920}h^4[W_{i-\frac{3}{2}} + 156W_{i-\frac{1}{2}} + W_{i+\frac{1}{2}}] \tag{2.11}$$

$$h^3T_i = h^2[F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}] - \frac{1}{24}h^4[W_{i+\frac{1}{2}} - W_{i-\frac{1}{2}}], \tag{2.12}$$

$$h^4W_{i-\frac{1}{2}} = 24[s_{i-\frac{3}{2}} - 2s_{i-\frac{1}{2}} + s_{i+\frac{1}{2}}] - h^2[F_{i-\frac{3}{2}} + 22F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}], \tag{2.13}$$

and

$$h^2[F_{i-\frac{3}{2}} - 2F_{i-\frac{1}{2}} + F_{i+\frac{1}{2}}] = \frac{1}{24}h^4[W_{i-\frac{3}{2}} + 22W_{i-\frac{1}{2}} + W_{i+\frac{1}{2}}]. \tag{2.14}$$

The elimination of $W_{i+\frac{1}{2}}$ from (2.13) and (2.14) gives the following consistency relation

$$s_{i-\frac{5}{2}} + 2s_{i-\frac{3}{2}} - 6s_{i-\frac{1}{2}} + 2s_{i+\frac{1}{2}} + s_{i+\frac{3}{2}} = \frac{h^2}{24}[F_{i+\frac{3}{2}} + 32F_{i+\frac{1}{2}} + 78F_{i-\frac{1}{2}} + 32F_{i-\frac{3}{2}} + F_{i-\frac{5}{2}}] \tag{2.15}$$

for $i = 2, 3, \ldots, n-2$, where $F_i = p_i s_i + q_i$ with $p_i = p(x_i)$ and $q_i = q(x_i)$. See [3] for more details.

The recurrence relation (2.15) gives $(n-4)$ linear equations in the n unknowns $s_{i+\frac{1}{2}}$, $i = 0, 1, 2, \ldots, n-1$. We need four more equations two at each end of the range of integration, for direct computation of $s_{i+\frac{1}{2}}$. These four equations are given by

$$8s_0 - 11s_{\frac{1}{2}} + 2s_{\frac{3}{2}} + s_{\frac{5}{2}} = \frac{h^2}{24}[-0.8F_0 + 62F_{\frac{1}{2}} + 34F_{\frac{3}{2}} + 0.8F_{\frac{5}{2}}], \quad \text{for } i = 1, \tag{2.16}$$

$$4s_{\frac{1}{2}} - 7s_{\frac{3}{2}} + 2s_{\frac{5}{2}} + s_{\frac{7}{2}} = \frac{h^2}{24}[8F_{\frac{1}{2}} + 82F_{\frac{3}{2}} + 28F_{\frac{5}{2}} + 2F_{\frac{7}{2}}], \quad \text{for } i = 2, \tag{2.17}$$

$$s_{n-\frac{7}{2}} + 2s_{n-\frac{5}{2}} - 7s_{n-\frac{3}{2}} + 4s_{n-\frac{1}{2}} = \frac{h^2}{24}[2F_{n-\frac{7}{2}} + 28F_{n-\frac{5}{2}} + 82F_{n-\frac{3}{2}} + 8F_{n-\frac{1}{2}}], \quad \text{for } i = n-1, \tag{2.18}$$

$$s_{n-\frac{5}{2}} + 2s_{n-\frac{3}{2}} - 11s_{n-\frac{1}{2}} + 8s_n = \frac{h^2}{24}[0.8F_{n-\frac{5}{2}} + 34F_{n-\frac{3}{2}} + 62F_{n-\frac{1}{2}} - 0.8F_n], \quad \text{for } i = n. \tag{2.19}$$

The quintic spline solution (2.1) of problem (1.1) is based on the linear equations given by (2.15)-(2.19). The determination of $s_{i+\frac{1}{2}}$, $i = 0, 1, 2, \ldots, n-1$, can be effected by solving the system of linear equations defined by (2.15)-(2.19). Having the values of $s_{i+\frac{1}{2}}$, $i = 0, 1, 2, \ldots, n-1$, we can compute $F_{i-\frac{1}{2}}$, $i = 1, 2, \ldots, n$ using the differential equation in (1.1). Note that F_0 and F_n can also be computed using the differential equation in (1.1) since both s_0 and s_n are known. We next compute $W_{i+\frac{1}{2}}$, $i = 0, 1, 2, \ldots, n-1$ using (2.13), and $W_{\frac{1}{2}}$ and $W_{n-\frac{1}{2}}$ can be computed by (2.14). Then we use equations (2.10) and (2.12) to compute the values of D_i and T_i, for $i = 1, 2, \ldots, n-1$, respectively. The values of D_0 and T_0 are computed by (2.6) and (2.8), respectively. Having the value of D_0 we can compute the value of M_0 from the following equation

$$M_0 = (p_0^2 + 3p_0'')D_0 + (4p_0p_0' + p_0'')s_0 + 3p_0'q_0 + p_0q_0' + q_0''.$$

Next we can compute the values of M_i for $i = 1, 2, \ldots, n-1$ using relation (2.9). We remark that the knowledge of $s_{i+\frac{1}{2}}$, D_i, $F_{i+\frac{1}{2}}$, T_i, $W_{i+\frac{1}{2}}$ and M_i for $i = 0, 1, 2, \ldots, n-1$, enables us to write down $Q_i(x)$, $i = 0, 1, 2, \ldots, n-1$ as given by (2.1). Thus, the quintic spline $s(x)$ approximating the solution of (1.1) is determined.

Using Taylor series expansions we can easily show that the local truncation errors $t_i, i = 1, 2, \ldots, n$ associated with the difference equations (2.16), (2.17), (2.15), (2.18) and (2.19) are given by

$$t_i = \begin{cases} \dfrac{164}{15360}h^6y^{(6)}(\zeta_1) + O(h^7), \quad a < \zeta_1 < x_{\frac{5}{2}}, & i = 1, \\ -\dfrac{320}{15360}h^6y^{(6)}(\zeta_2) + O(h^7), \quad a < \zeta_2 < x_{\frac{7}{2}}, & i = 2, \\ -\dfrac{13}{15360}h^6y^{(6)}(\zeta_i) + O(h^7), \quad x_{i-\frac{5}{2}} < \zeta_i < x_{i+\frac{3}{2}}, & 3 \le i \le n-2, \\ -\dfrac{320}{15360}h^6y^{(6)}(\zeta_{n-1}) + O(h^7), \quad x_{n-\frac{7}{2}} < \zeta_{n-1} < b, & i = n-1, \\ \dfrac{164}{15360}h^6y^{(6)}(\zeta_n) + O(h^7), \quad x_{n-\frac{5}{2}} < \zeta_n < b, & i = n, \end{cases}$$

and using standard convergence analysis we can get $\|\mathbf{e}\| \le KM_6h^4$, where $\mathbf{e} = (e_{i+\frac{1}{2}})$ is an n-dimensional column vector with $e_{i+\frac{1}{2}} = y_{i+\frac{1}{2}} - s_{i+\frac{1}{2}}$ is the discretization error, $M_6 = \max_x |y^{(6)}|$, and K is a constant. This indicate that the method is a fourth order convergent process. Also, it can be shown that $\max_i |y_i'' - s_i''| \le K \max_x |p(x)| M_6h^4$. Thus, the order of accuracy for the computed approximation of the second derivative is four. See [3] for the details. Furthermore, using Taylor series expansions we can show that the local truncation errors associated with the consistency relations (2.11), (2.10), (2.14) and

(2.9) are

$$-\frac{1}{3}h^5 y_i^{(5)} + O(h^6), \quad \frac{7}{576}h^7 y_i^{(7)} + O(h^8),$$
$$-\frac{1}{1440}h^6 y_i^{(6)} + O(h^7), \quad \text{and} \quad - h^6 y_i^{(6)} + +O(h^7),$$

respectively, which indicate that the accuracy involved with the computed approximate values of y' and y''' are of order four and that of $y^{(iv)}$ is two while that of $y^{(v)}$ is of order one. These results are supported by the numerical experiments given in the next section.

3. Numerical Results

In this section, we use the quartic spline method derived in the previous sections to solve a numerical example and compare the numerical results of our method with that of other fourth order spline methods.

We consider the boundary value problem

$$\begin{aligned} y'' &= \frac{2}{x^2}y - \frac{1}{x}, \\ y(2) &= 0 \quad \text{and} \quad y(3) = 0, \end{aligned} \tag{3.1}$$

for which the analytical solution $y(x) = \frac{1}{38}[-5x^2 + 19x - \frac{36}{x}]$.

The problem (3.1) was solved using the quartic spline method described in Section 2 with a variety of h values and the spline solution (2.1) was compared with the exact solution. The observed maximum errors in absolute values associated with $y_i^{(m)}$, $m = 0, 1, 2, 3, 4, 5$, are given in Tables 1 and 2. From these tables we can notice that if the stepsize h is reduced by a factor 1/2, then the maximum errors $\max_i |y_i^{(m)} - s_i^{(m)}|$, $m = 0, 1, 2, 3$ are approximately reduced by a factor 1/16. Thus, the numerical results confirm that our present method is a fourth-order convergent process, and that the fourth order accuracy holds for y', y'' and y''' as predicted in Section 3. Also, the numerical results given in these table show that $\max_i |y_i^{(m)} - s_i^{(m)}|$ $m = 3, 4$ are reduced by a factor 1/4 for $m = 3, 4$ and by a factor 1/2 when $m = 5$ as h is reduced by a factor of 1/2, which confirm that our quintic spline method gives second order accurate approximations for the third and fourth derivatives and first order approximation for the fifth derivative.

In [5] Al-Said has introduced a quartic spline method for solving second order boundary value problems of the form (1.1). The problem (3.1) was solved

h	$\max_i \lvert y_i - s_i \rvert$	$\max_i \lvert y'_i - s'_i \rvert$	$\max_i \lvert y''_i - s''_i \rvert$
$\frac{1}{8}$	9.67×10^{-8}	2.99×10^{-7}	3.47×10^{-8}
$\frac{1}{16}$	6.90×10^{-9}	2.08×10^{-8}	$2./49 \times 10^{-9}$
$\frac{1}{32}$	4.49×10^{-10}	1.35×10^{-9}	1.64×10^{-10}

Table 1: Observed maximum errors for problem (3.1)

h	$\max_i \lvert y'''_i - s'''_i \rvert$	$\max_i \lvert y^{(iv)}_i - s^{(iv)}_i \rvert$	$\max_i \lvert y^{(v)}_i - s^{(v)}_i \rvert$
$\frac{1}{8}$	2.49×10^{-5}	4.30×10^{-3}	2.59×10^{-1}
$\frac{1}{16}$	3.59×10^{-6}	1.10×10^{-4}	1.60×10^{-1}
$\frac{1}{32}$	4.81×10^{-7}	3.49×10^{-5}	8.08×10^{-2}

Table 2: Observed maximum errors for problem (3.1)

	Quartic spline [5]	Quartic spline [8]	Quintic spline [9]
$\max_i \lvert y_i - s_i \rvert$	1.55×10^{-7}	1.74×10^{-7}	1.13×10^{-7}
$\max_i \lvert y'_i - s'_i \rvert$	7.53×10^{-7}	2.72×10^{-6}	3.03×10^{-5}
$\max_i \lvert y''_i - s''_i \rvert$	5.28×10^{-8}	6.28×10^{-8}	1.69×10^{-8}
$\max_i \lvert y'''_i - s'''_i \rvert$	1.58×10^{-3}	2.90×10^{-3}	3.95×10^{-3}
$\max_i \lvert y^{(iv)}_i - s^{(iv)}_i \rvert$	1.54×10^{-3}	———	6.85×10^{-3}

Table 3: Observed maximum errors for problem (3.1) when $h = \frac{1}{8}$

in [5] with different h values and fourth order approximations for the solution and its first and second derivatives, and second order accurate approximations for the third and fourth derivatives were computed. Also, this problem was solved in [8], [9] using fourth order quartic and quintic spline methods. For the sake of comparison between our and their methods, we list some of their numerical results in Table 3 for $h = 1/8$. An asterisk is to indicate that the numerical results are taken from the specified paper. It is clear from Tables 1, 2, and 3 that our present quintic spline method gives better results than the other methods. We mentioned here in passing that our present method is capable producing approximations for the fifth derivative while the other do not.

4. Conclusion

We introduced a new quintic spline method for solving second order boundary value problems. The present method enables us to approxiate the solution as well as its first, second, third, fourth and fifth derivatives at every point of the range of integration. The order of the method is four, and it was shown that the fourth order accuracy holds for approximating $y(x)$, $y'(x)$, $y''(x)$ and $y'''(x)$. Also, it was shown that this method produces second order approximations for the fourth derivative and first order approximation for the fifth derivative.

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