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THE WELL-POSEDNESS OF THE GLOBAL SOLUTION FOR A DAMPED EULER-BERNOULLI EQUATION

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Abstract: In this paper, we study the well-posedness of the global solution to the following damped Euler-Bernoulli equation

$$u_{tt} + au_{xxxx} + 2bu_t + cu = f(u), \quad t \geq 0, \quad x \in [0, +\infty).$$

For the case $f(u) = u^2$, the existence and uniqueness of the global solution to an initial value problem of the equation are established in the space $C([0,+\infty), L^2([0,+\infty))) \cap C^1([0,+\infty), H^{-1}([0,+\infty)))$. For the case where $f(u)$ is a polynomial, we find that the well-posedness can be established in the Sobolev space $C([0,+\infty), H^s([0,+\infty))) \cap C^1([0,+\infty), H^{s-1}([0,+\infty)))$ $(s > \frac{1}{2})$.

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1. Introduction

Solutions to various Euler-Bernoulli equations associated with initial or initial

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boundary value problems have been obtained in different Banach or Sobolev space (see [3, 1, 6, 5, 2, 15]). The classical Euler-Bernoulli equation for the vibration of an infinite long beam can be written as

$$m\frac{\partial^2 u}{\partial t^2} + EI\frac{\partial^4 u}{\partial x^4} = F, \quad t \geq 0, \quad x \in [0, +\infty), \tag{1}$$

where $u(t,x)$ denotes the transverse displacement, m denotes the density of the beam, EI denotes the stiffness of the beam and F is the force exerted by the fluid on the beam (see [2, 7, 10, 9, 11]). If $F = F(t,x)$ satisfies certain assumptions, an Euler-Bernoulli beam may be exactly controlled by a force acting on one end of the beam. However, in real processes, the vibration of the beam is subject to damping and the forces acting on the beam may include the nonlinear terms due to the action from its surrounding fluid. Thus, study of the Euler-Bernoulli equation involving nonlinear effects become a main concern. In this paper, we consider a problem with F depending on the transverse displacement and its derivatives with respect to t. More specifically, we study the following damped Euler-Bernoulli beam equation

$$u_{tt} + au_{xxxx} + 2bu_t + cu = f(u), \quad t \geq 0, x \in [0, +\infty), \tag{2}$$

$$u(x,0) = \varepsilon\varphi(x), \ x \in [0, +\infty), \tag{3}$$

$$u_t(x,0) = \varepsilon\psi(x), \ x \in [0, +\infty), \tag{4}$$

where a, b and c are positive constants, ε is a small parameter. For the case $f(u) = u^2$, the existence and uniqueness of the global solution to an initial value problem of the equation are established in the space

$$C([0, +\infty), L^2([0, +\infty))) \cap C^1([0, +\infty), H^{-1}([0, +\infty))).$$

For the case where $f(u)$ is a polynomial, we find that the well-posedness can be established in the Sobolev space

$$C([0, +\infty), H^s([0, +\infty))) \cap C^1([0, +\infty), H^{s-1}([0, +\infty)))(s > \frac{1}{2}).$$

Various results on the existence and uniqueness of local and global solutions to nonlinear wave equations can be found from [12, 13, 14, 8]. However, as far as the authors know, the well-posedness of the global solution to the problem (2)-(4) has not been established before in the Sobolev space $C([0, +\infty), L^2([0, +\infty))) \cap C^1([0, +\infty), H^{-1}([0, +\infty)))$. The result of well-posedness in this paper shows that the motion and vibration or stability of the long beam may be controlled under certain assumptions on the parameter ε, the initial value functions $\varphi(x)$ and $\psi(x)$.

2. Preliminary

In this section, we define a Sobolev space and introduce two lemmas to be used in the proof of the theorem in Section 3.

Definition 1. If the function $u(t,x)$ is an even or odd function with respect to $x \in (-\infty,+\infty)$, we denote the corresponding Sobolev space by

$$X_s(\infty) = C([0,+\infty), H^s([0,+\infty))) \cap C^1([0,+\infty), H^{s-1}([0,+\infty))),$$

which is equipped with the following norm

$$|||u||| = (\|u\|_s + \|u_t\|_{s-1}),$$

where

$$\|u\|_s = \sup_{t\in[0,T]} \left(\int_{-\infty}^{+\infty} \langle \xi \rangle^{2s} |\widehat{u}(t,\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

$$\widehat{u}(t,\xi) = F[u(x,t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ix\xi} u(t,x) dx,$$

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}.$$

Lemma 2. *Suppose that the function $K(\xi,\eta)$ satisfies one of the following inequalities*

$$\operatorname*{Sup}_{\eta\in R^n} \int_{R^n} K^2(\xi,\eta) d\xi < \infty, \tag{5}$$

$$\operatorname*{Sup}_{\xi\in R^n} \int_{R^n} K^2(\xi,\eta) d\eta < \infty. \tag{6}$$

Then the following inequality holds

$$\| \int_{R^n} K(\xi,\eta) f(\xi-\eta) g(\eta) d\eta \|_{L^2(R^n)} < C \|f\|_{L^2(R^n)} \|g\|_{L^2(R^n)}.$$

The proof can be found in reference [13].

Lemma 3. *Suppose that $\xi,\eta \in R^1$, $s_1 \geq 0$, $s_2 \geq 0$, $s_1 + s_2 - (\frac{1}{2}+\delta) \leq 0$ ($\delta > 0$ is a constant) and*

$$| K(\xi,\eta) | \leq \frac{\langle \xi \rangle^{s_1+s_2-(\frac{1}{2}+\delta)}}{\langle \xi-\eta \rangle^{s_1} \langle \eta \rangle^{s_2}}. \tag{7}$$

Then $\| \int_{R^n} K(\xi,\eta) f(\xi-\eta) g(\eta) d\eta \|_{L^2(R^1)} < C \|f\|_{L^2(R^1)} \|g\|_{L^2(R^1)}$.

Proof. In the proof of this lemma, we use c to represent any positive constant independent of ξ and η, and let

$$\Omega_1 = \{(\xi,\eta) \mid\mid \xi-\eta \mid \geq \frac{1}{2} \mid \xi \mid\}, \quad \Omega_2 = \{(\xi,\eta) \mid\mid \xi-\eta \mid \leq \frac{1}{2} \mid \xi \mid\}.$$

If $(\xi,\eta) \in \Omega_1$, there must exists a positive constant c such that

$$\langle\xi\rangle \leq c\,\langle\xi-\eta\rangle\,, \tag{8}$$

from which we have

$$\mid K(\xi,\eta) \mid \leq \frac{\langle\xi\rangle^{s_1+s_2-(\frac{1}{2}+\delta)}}{\langle\xi-\eta\rangle^{s_1}\langle\eta\rangle^{s_2}} \leq \frac{c}{\langle\xi\rangle^{\frac{1}{2}+\delta-s_2}\,\rangle\eta\rangle^{s_2}} = I.$$

Since $\frac{1}{2}+\delta-s_2 \geq 0$, if $\langle\xi\rangle \geq \frac{1}{2}\langle\eta\rangle$, we have

$$I \leq \frac{c}{\langle\eta\rangle^{\frac{1}{2}+\delta-s_2+s_2}} = \frac{c}{\langle\eta\rangle^{\frac{1}{2}+\delta}}.$$

If $\langle\xi\rangle \leq \frac{1}{2}\langle\eta\rangle$, it follows

$$I \leq \frac{c}{\langle\xi\rangle^{\frac{1}{2}+\delta}}.$$

By Lemma 2, we know if $(\xi,\eta) \in \Omega_1$ the lemma holds.

If $(\xi,\eta) \in \Omega_2$, we have

$$\langle\xi-\eta\rangle \leq c\,\langle\xi\rangle\,,$$

$$\mid \xi \mid - \mid \eta \mid \leq \mid \xi-\eta \mid \leq \frac{1}{2} \mid \xi \mid,$$

$$\mid \eta \mid - \mid \xi \mid \leq \mid \xi-\eta \mid \leq \frac{1}{2} \mid \xi \mid.$$

Thus, we have $\frac{1}{2} \mid \xi \mid \leq \mid \eta \mid \leq \frac{3}{2} \mid \xi \mid$ and

$$\mid K(\xi,\eta) \mid \leq \frac{\langle\xi\rangle^{s_1+s_2-(\frac{1}{2}+\delta)}}{\langle\xi-\eta\rangle^{s_1}\langle\eta\rangle^{s_2}} \leq \frac{c}{\langle\xi-\eta\rangle^{s_1}\langle\xi\rangle^{\frac{1}{2}+\delta-s_1}}.$$

It then follows from $\frac{1}{2}+\delta-s_1 \geq 0$ that

$$\mid K(\xi,\eta) \mid \leq \frac{c}{\langle\xi-\eta\rangle^{s_1}\langle\xi\rangle^{\frac{1}{2}+\delta-s_1}} \leq \frac{c}{\langle\xi-\eta\rangle^{\frac{1}{2}+\delta}}.$$

By Lemma 2, we know that the Lemma 3 is valid in this case. □

Remark 1. The Shaulder Multiplying Lemma $H^s(R^n)\times H^s(R^n) \subset H^s(R^n)$ requires the assumption of index $s > \frac{n}{2}$. That is, if $u(x), v(x) \in H^s(R^n)$, $s > \frac{n}{2}$, then $u(x)v(x) \in H^s(R^n)$.

3. Theorem of Well-Posedness

Theorem 4. *Suppose* $f(u) = u^2$, $\varphi(x) \in H^\sigma([0,+\infty))$ *and* $\psi(x) \in H^{\sigma-1}([0,+\infty))$ *with* $0 \le \sigma \le \frac{1}{2}$, $a > 0$, $b > 0$, $c > b^2$. *If the parameter* ε *is sufficiently small, then there exists a unique solution* $u(t,x) \in C([0,+\infty), H^\sigma([0,+\infty))) \cap C^1([0,+\infty), H^{\sigma-1}([0,+\infty)))$ *to the problem defined by (2)-(4).*

Remark 2. If $\sigma = 0$, all other assumptions in Theorem 4 are satisfied, and thus there must exist a unique solution $u(t,x) \in C([0,+\infty), L^2([0,+\infty))) \cap C^1([0,+\infty), H^{-1}([0,+\infty)))$ to the damped Euler-Bernoulli equation (2) satisfying the initial conditions (3)-(4).

Proof of Theorem 4. For simplicity, throughout the proof of Theorem 4, we denote by C any positive constant independent of ε, but possibly depending on σ, a, b or c. We divide the proof of the theorem into two parts. Firstly, we prove that there exists a solution $u(x,t) \in X_s(\infty)$ to the problem defined by (2)-(4). Then, we prove that the solution is unique.

(a) Existence. In order to take Fourier transform, we make an even extension of $u(t,x)$ with respect to x to the interval $(-\infty, 0]$. Thus, taking the Fourier transform of equations (2)-(4), we get

$$\widehat{u}''(t,\xi) + 2b\widehat{u}'(t,\xi) + (a\xi^4 + c)\widehat{u}(t,\xi) = \widehat{u^2}(t,\xi), \tag{9}$$

$$\widehat{u}(\xi,0) = \varepsilon\widehat{\varphi}(\xi), \tag{10}$$

$$\widehat{u}_t(\xi,0) = \varepsilon\widehat{\psi}(\xi), \tag{11}$$

which give rise to the following solution

$$\widehat{u}(t,\xi) = \varepsilon e^{-bt}\left\{\left[\cos(\sigma_\xi t) + b\frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi) + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\}$$
$$- \frac{1}{\sigma_\xi}\int_0^t \exp\left[-b(t-\tau)\right]\sin[\sigma_\xi(t-\tau)]\widehat{u^2}(\tau,\xi)d\tau,$$

where $\sigma_\xi = \sqrt{a\xi^4 + c - b^2}$, $\quad c - b^2 > 0$. Denoting u_0 by

$$u_0 = \varepsilon F^{-1}\left[e^{-bt}\left\{\left[\cos(\sigma_\xi t) + b\frac{\sin(\sigma_\xi t)}{\sigma_\xi}\right]\widehat{\varphi}(\xi) + \frac{\sin(\sigma_\xi t)}{\sigma_\xi}\widehat{\psi}(\xi)\right\}\right], \tag{12}$$

where F^{-1} represents the inverse Fourier transform. Letting $0 \le \sigma \le \frac{1}{2}$, we have from (12) that

$$\|u_0\|_\sigma \le C\varepsilon\left(\|\varphi\|_\sigma + \|\psi\|_{\sigma-1}\right). \tag{13}$$

Now, by letting $A = C\left(\|\varphi\|_\sigma + \|\psi\|_{\sigma-1}\right)$, we have, from (13), that

$$\|u_0\|_\sigma \le \varepsilon A. \tag{14}$$

The sequence $\{\widehat{u_n}\}$ for $n = 1, 2, 3, \dots$ can thus be constructed as follows

$$\widehat{u_n}(t,\xi) = \widehat{u_0} - \frac{1}{\sigma_\xi}\int_0^t \exp\left[-b(t-\tau)\right]\sin\left[\sigma_\xi(t-\tau)\right] \widehat{u_{n-1}^2(\tau,\xi)})d\tau. \tag{15}$$

We know that

$$\int_0^t \exp\left[-b(t-\tau)\right]d\tau \quad \le \quad \frac{1}{b}\left(1-e^{-bt}\right) \quad \le \quad C \tag{16}$$

and

$$\begin{aligned}
&| \frac{\langle\xi\rangle^\sigma}{\sigma_\xi}\int_0^t \exp\left[-b(t-\tau)\right]\sin\left[\sigma_\xi(t-\tau)\right] \widehat{u_{n-1}^2(\tau,\xi)})d\tau \,| \\
&\le | \frac{\langle\xi\rangle^\sigma}{\sigma_\xi}\int_0^t \exp\left[-b(t-\tau)\right]\sin\left[\sigma_\xi(t-\tau)\right] \widehat{u_{n-1}^2(\tau,\xi)})d\tau \,| \\
&\le | \frac{\langle\xi\rangle^\sigma}{\sigma_\xi}\int_0^t \exp\left[-b(t-\tau)\right]\int \widehat{u_{n-1}}(\xi-\eta,\tau)\widehat{u_{n-1}}(\tau,\eta)d\eta d\tau \,| \\
&\le \frac{\langle\xi\rangle^\sigma}{\sigma_\xi}\int_0^t \exp\left[-b(t-\tau)\right] \\
&\times | \int \frac{1}{\langle\xi-\eta\rangle^\sigma\langle\eta\rangle^\sigma}\langle\xi-\eta\rangle^\sigma\,\widehat{u_{n-1}}(\xi-\eta,\tau)\,\langle\eta\rangle^\sigma\,\widehat{u_{n-1}}(\tau,\eta)d\eta d\tau \,| \,.
\end{aligned} \tag{17}$$

Let

$$\begin{aligned}
K(\xi,\eta) &= \frac{\langle\xi\rangle^\sigma}{\sigma_\xi}\times\frac{1}{\langle\xi-\eta\rangle^\sigma\langle\eta\rangle^\sigma} \\
&= \frac{1}{\sqrt{a\xi^4+c-b^2}}\frac{\langle\xi\rangle^\sigma}{\langle\xi-\eta\rangle^\sigma\langle\eta\rangle^\sigma}.
\end{aligned} \tag{18}$$

Note that there exist two positive constants c_1 and c_2 such that

$$c_1\langle\xi\rangle^2 \le \sqrt{a\xi^4+c-b^2} \le c_2\langle\xi\rangle^2\,.$$

Since $0 \le \sigma \le \frac{1}{2}$ and $\sigma + \sigma - (\sigma+2) = 2\sigma - (\frac{1}{2}+\sigma+\frac{3}{2}) \le 0$, we get from (15)-(18) and Lemma 3, that

$$\|u_n\|_\sigma \le \|u_0\|_\sigma + C\|u_{n-1}\|_\sigma^2. \tag{19}$$

Therefore, for $n = 1$, inequality (19) yields

$$\begin{aligned}
\|u_1\|_\sigma &\le \|u_0\|_\sigma + C\|u_0\|_\sigma^2 \\
&\le \varepsilon A + C(\varepsilon A)^2
\end{aligned}$$

$$\leq \varepsilon A + C(2\varepsilon A)^2. \tag{20}$$

Choosing ε sufficiently small such that

$$2C(2\varepsilon A) < 1, \tag{21}$$

we have, from (20), that

$$\|u_1\|_\sigma \leq 2\varepsilon A. \tag{22}$$

For $n = 2$, we have from (21) and (22) that

$$\|u_2\|_\sigma \leq \varepsilon A + C\|u_1\|_\sigma^2 < \varepsilon A + C(2\varepsilon A)^2. \tag{23}$$

Further using (21), we have

$$\|u_2\|_\sigma < 2\varepsilon A. \tag{24}$$

By induction, we get

$$\|u_n\|_\sigma < 2\varepsilon A, \quad n = 1, 2, 3, \cdots . \tag{25}$$

Following the same procedure as that for deriving (25), we obtain

$$|||u_n||| < 2\varepsilon A, \quad n = 1, 2, 3, \cdots . \tag{26}$$

From (21) and the assumption of the function f, we get

$$\|u_n - u_{n-1}||| \leq C \max(|||u_{n-1}|||, |||u_{n-2}|||)\ |(|||u_{n-1} - u_{n-2}|||)|, \tag{27}$$

By using (26), it follows that

$$\max(|||u_{n-1}|||, |||u_{n-2}|||)\ \leq C(2\varepsilon A). \tag{28}$$

Thus from (27) and (28), we obtain

$$\begin{aligned} |||u_n - u_{n-1}||| &\leq \quad C(2\varepsilon A)\ |(|||u_{n-1} - u_{n-2}|||)| \\ &\leq \quad [C(2\varepsilon A)]^{n-1}(4\varepsilon A). \end{aligned} \tag{29}$$

Furthermore

$$\begin{aligned} |||u_n||| &= \quad |||u_0 + \sum_1^n (u_i - u_{i-1})||| \\ &\leq \quad |||u_0||| + \sum_1^n |||u_i - u_{i-1}||| \\ &\leq \quad \varepsilon A + \sum_1^n [C(2\varepsilon A)]^{i-1}(4\varepsilon A). \end{aligned} \tag{30}$$

From (30) and noting that $C(2\varepsilon A) < 1$, it is clear that $\{u_n\}$ is uniformly convergent in the space $X_\sigma(\infty)$. Therefore there exists a function $u(t, x) \in X_\sigma(\infty)$ such that u_n uniformly converges to $u(x, t)$, i.e., a solution of the problem defined by (2)-(4).

(b) Uniqueness. Now we prove that the global solution $u(t,x) \in X_\sigma(\infty)$ of the problem defined by (2)-(4) is unique. Suppose that there exist two solutions $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$ of the problem, then both $|||u^{(1)}|||$ and $|||u^{(2)}|||$ are bounded in the space $X_\sigma(\infty)$. Letting

$$w(x,t) = u^{(1)}(t,x) - u^{(2)}(t,x),$$

we have

$$\widehat{w}(t,\xi) = -\frac{1}{(1+c\xi^2)\sigma_\xi} \times \int_0^t \exp\left[-\frac{b\xi^2}{1+c\xi^2}(t-\tau)\right] \sin\left[\sigma_\xi(t-\tau)\right] \widehat{F}(\tau,\xi)d\tau, \quad (31)$$

where $F(t,x) = f(u^{(1)}(t,x)) - f(u^{(2)}(t,x))$. It follows from (31) and Lemma 3 that

$$\int_{-\infty}^{+\infty} <\xi>^{2\sigma} |\widehat{w}(t,\xi)|^2 d\xi \leq C\int_0^t \|w\|_\sigma^2 \left(\max(\|u^{(1)}\|_\sigma, \|u^{(2)}\|_\sigma\right) d\tau \leq C\int_0^t \|w\|_\sigma^2 d\tau.$$

That is

$$\|w\|_\sigma^2 \leq C\int_0^t \|w\|_\sigma^2 d\tau. \quad (32)$$

By the Growall inequality, we have $w(t,x) \equiv 0$ in $X_\sigma(\infty)$. That is, $u^{(1)}(t,x) \equiv u^{(2)}(t,x)$ and hence there exists a unique solution $u(t,x) \in X_\sigma(\infty)$ of the problem defined by (2)-(4). □

In fact, if the Sobolev index $s > \frac{1}{2}$, using the Schaulder Lemma (see Remark 1) and following the same procedure for the proof of Theorem 3, we have the following theorem.

Theorem 5. *Suppose* $\varphi(x) \in H^s([0,+\infty))$ *and* $\psi(x) \in H^{s-1}([0,+\infty))$ *with* $s > \frac{1}{2}$, $a > 0$, $b > 0$, $c > b^2$. If the parameter ε *is sufficiently small, and* $f(u)$ *is a polynomial with* $f(0) = 0$, *then there exists a unique solution* $u(t,x) \in C([0,+\infty), H^s([0,+\infty))) \cap C^1([0,+\infty), H^{s-1}([0,+\infty)))$ *to the problem defined by (2)-(4).*

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