Robustness estimating of optimal stopping problem with unbounded revenue and cost functions

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Abstract: We study the stability of the optimal stopping problem for a discrete-time Markov process on a general space state $X$. Revenue and cost functions are allowed to be unbounded. The stability (robustness) is understood in the sense that an unknown transition probability $p(\cdot|x)$, $x \in X$, is approximated by the known one $\tilde{p}(\cdot|x)$, $x \in X$, and the stopping rule $\tilde{\tau}_n$, optimal for the process governed by $\tilde{p}$ is applied to the original process represented by $p$. The criteria of stopping rule optimization is the total expected return. We give an upper bound for the decrease of the return due to the replacement of the unknown optimal stopping rule $\tau_n$ by its approximation $\tilde{\tau}_n$. The bound is expressed in terms of the weighted total variation distance between the transition probabilities $p$ and $\tilde{p}$.

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1. Introduction

In the paper [21] the robustness (stability) of optimal stopping of a discrete-time Markov process was investigated in the case of bounded revenue and cost functions. The aim of the present work is to study the stability of such problem...
allowing unbounded payoffs. Bearing in mind the standard optimal stopping problem (see, for instance, [16], [18]), let \( \{x_t\} \equiv \{x_t, t = 0, 1, \ldots \} \) be a Markov process on a Borel state space \((X, \mathcal{B}_X)\) specified by a transition probability \(p(B|x), x \in X, B \in \mathcal{B}_X\) (the Borel \(\sigma\)-algebra). We assume that \(p\) is not available to the controller, and it is substituted by a given approximating transition probability \(\tilde{p}(B|x), x \in X, B \in \mathcal{B}_X\) (the last one could be obtained, for example, by means of statistical estimation). The “approximating” Markov process governed by \(\tilde{p}\) will be denoted by \(\{\tilde{x}_t\} \equiv \{\tilde{x}_t, t = 0, 1, \ldots\} \).

Assuming that the optimal stopping rule \(\tilde{\tau}_*\) for the process \(\{\tilde{x}_t\}\) can be found, it is applied to the process \(\{x_t\}\). Thus \(\tilde{\tau}_*\) is used as an available approximation to an unavailable stopping rule \(\tau_*\) optimal for the process \(\{x_t\}\). By robustness (stability) estimating we mean finding upper bounds for the decrease of the total payoff due to using \(\tilde{\tau}_*\) instead of \(\tau_*\) to control (to “stop”) the “original” Markov process \(\{x_t\}\). Under assumptions introduced in Section 2, in Sections 3 and 4 we obtain the bound for the mentioned payments change. This bound is a constant (depending on an initial state and on certain properties the processes under consideration) times the following distance between the transition probabilities:

\[
\sup_{x \in X} \frac{1}{V(x)} \sup_{|\varphi| \leq V} \left| \int_X \varphi(y)p(dy|x) - \int_X \varphi(y)\tilde{p}(dy|x) \right|,
\]

where \(V\) is a stochastic Lyapunov function introduced in the next section.

### 2. Assumptions and Setting of Problem

Let \(x_0 \in X\) be arbitrary, but fixed initial state, common for the processes \(\{x_t\}\) and \(\{\tilde{x}_t\}\), defined by the transition probabilities \(p\) and \(\tilde{p}\), respectively. We will denote by \(P_{x_0}, \tilde{P}_{x_0}\) and by \(E_{x_0}, \tilde{E}_{x_0}\) the corresponding probability measures on the trajectory space \(\Omega = X^\infty\), and the expectations with respect to these measures. Also, let \(T_{x_0}\) and \(\tilde{T}_{x_0}\) denote the classes of all (correspondingly, \(P_{x_0}\), \(\tilde{P}_{x_0}\)) almost surely finite stopping times with respect to the natural filtration on \(X^\infty\).

Two measurable functions:

\[
c_0 : X \to [0, \infty), \quad r : X \to [0, \infty)
\]

are given, where \(c_0(x)\) represents a payment for keeping on observing the process when \(x_t = x\), and \(r(x)\) expresses a reward for stopping the process when \(x_t = x\). Note that the functions \(c_0\) and \(r\) are common for the problem of stopping optimization for both processes \(\{x_t\}\) and \(\{\tilde{x}_t\}\).
To deal with a minimization problem we will use the total expected cost \((\tau \in T, x_0)\):

\[
E_{x_0} \left[ \sum_{t=0}^{\tau-1} c_0(x_t) - r(x_\tau) \right]
\]  

(2.1)
as a criterion of stopping time optimization (for the process \(\{x_t\}\), and the similar criterion for the process \(\{\tilde{x}_t\}\)). However, the expectation in (2.1) could be undefined. Therefore, we first impose certain restrictions on the functions \(c_0, r\), on the processes \(\{x_t\}\), \(\{\tilde{x}_t\}\), and on the class of stopping times under consideration.

Let \(V : X \to [1, \infty)\) be a given measurable function, and \(B_V\) stand for the Banach space of all measurable functions \(u : X \to \mathbb{R}\) with a finite norm:

\[
||u||_V := \sup_{x \in X} \frac{|u(x)|}{V(x)}.
\]

**Assumption 1.** \(c_0, r \in B_V\).

**Assumption 2.** (a) Both Markov chains \(\{x_t\}\) and \(\{\tilde{x}_t\}\) are aperiodic and \(\psi\)-irreducible.

(b) There exist numbers \(p > 1, \lambda < 1, b < \infty, d < \infty\), and small sets \(C, \tilde{C}\) such that

\[
\int_X V^p(y)p(dy|x) \leq \lambda V^p(x) + bI_C(x), \quad x \in X,
\]  

(2.2)
\[
\int_X V^p(y)p(dy|x) \leq \lambda V^p(x) + bI_{\tilde{C}}(x), \quad x \in X
\]  

(2.3)
(here \(I_A\) is the indicator function).

(c)

\[
\int_X V(y)p(dy|x) \leq dV(x), \quad x \in X,
\]  

(2.4)
\[
\int_X V(y)p(dy|x) \leq dV(x), \quad x \in X.
\]  

(2.5)

The standard definitions related to discrete-time Markov processes \((\psi\)-irreducibility, recurrence, small sets, etc.) can be found, for instance, in [10]. From Assumption 2, (a), (b), it follows that both Markov chains \(\{x_t\}\) and \(\{\tilde{x}_t\}\) are positive Harris recurrent with unique invariant probabilities, which we will denote, respectively, by \(\pi\) and \(\tilde{\pi}\).

**Assumption 3.** The invariant probability measures \(\pi\) and \(\tilde{\pi}\) are equiva-
Let $\mathcal{B}_+$ denote the family of all Borel subsets $B \subset X$ for which $\pi(B) > 0$ (and also $\tilde{\pi}(B) > 0$ due to Assumption 3), and $T$ denote the set of all stopping time of the form

$$\tau \equiv \tau_B := \inf\{n \geq 0 : x_t \in B\},$$

where $B \in \mathcal{B}_+$. Replacing in (2.6) $x_t$ by $\tilde{x}_t$ we define the set of stopping times $\tilde{T}$.

Remark 1. Since $\pi$ and $\tilde{\pi}$ are maximal irreducibility measures, respectively, for the processes $\{x_t\}$ and $\{\tilde{x}_t\}$ (Theorem 3.6 in [20]), every $\tau \in T$, $\tilde{\tau} \in \tilde{T}$ is almost surely finite. Elements of $T$ and $\tilde{T}$ will be called stopping rules.

Assumption 4. For each $B \in \mathcal{B}_+$

$$\sup_{x \in X \setminus B} P_x(x_1 \notin B, \ldots, x_n \notin B) \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$\sup_{x \in X \setminus B} \tilde{P}_x(\tilde{x}_1 \notin B, \ldots, \tilde{x}_n \notin B) \to 0 \quad \text{as} \quad n \to \infty.$$  

Remark 2. For every fixed initial state $x \in X$, the convergence in (2.7) and (2.8) (without supremum) follows from the strong law of large numbers for $\{x_t\}$ and $\{\tilde{x}_t\}$ (see [10], Chapter III). Sufficient conditions for the uniform (over $x \in X$) convergence are given, for example, in the paper [22]. In particular, Doeblin’s condition yields the fulfillment of Assumption 4.

The corollary of Section 4 ensures that, for each $\tau \in T$, $\tilde{\tau} \in \tilde{T}$, $x_0 \in X$, the following expected total costs are well-defined and finite:

$$W(x_0, \tau) := E_{x_0} \left[ \sum_{t=0}^{\tau-1} c_0(x_t) - r(x_\tau) \right],$$

and

$$\tilde{W}(x_0, \tilde{\tau}) := \tilde{E}_{x_0} \left[ \sum_{t=0}^{\tilde{\tau}-1} c_0(\tilde{x}_t) - r(\tilde{x}_{\tilde{\tau}}) \right].$$

Taking into account the structure of optimal stopping rules in typical examples (see, for instance, [13], [16], [18], [21]), we will restrict the optimization problem to the classes $T$ and $\tilde{T}$. Correspondingly, we call a stopping rule $\tau^* \in T$ optimal for $\{x_t\}$ if

$$W^*(x_0) := \inf_{\tau \in \tilde{T}} W(x_0, \tau) = W(x_0, \tau^*)$$

for all $x_0 \in X$. 

Similarly, $\tilde{\tau}_s \in \tilde{T}$ is optimal for $\{\tilde{x}_t\}$ if
\begin{equation}
\tilde{W}_s(x_0) := \inf_{\tau \in \tilde{T}} \tilde{W}(x_0, \tau) = \tilde{W}(x_0, \tilde{\tau}_s)
\end{equation}
for all $x_0 \in X$.

**Assumption 5.** Optimal stopping rules $\tau_s$ and $\tilde{\tau}_s$ exist.

**Remark 3.** Certain sufficient conditions of the existence of optimal stopping rules are well-known for models with bounded functions $c_0$ and $r$. In the case of unbounded payoffs some conditions could be found using results on general Markov control processes with a total expected cost (see [2], [8], [14]). However this is not the aim of the present paper. The example given in Section 5 shows that optimal stopping rules cannot exist even for very simple Markov processes.

In accordance with the approach outlined in the introduction, the stopping rule $\tilde{\tau}_s$ is applied to the “original” process $\{x_t\}$ (as a reasonable approximation to the unknown $\tau_s$). The total cost excess due to such replacement is measured by the following *stability index* (see [4], [5], [6], [7], [21]):
\begin{equation}
\Delta(x_0) := W_s(x_0, \tilde{\tau}_s) - W_s(x_0, \tau_s) \geq 0.
\end{equation}

For the case of bounded functions $c_0$ and $r$ in paper [21] the examples of unstable optimal stopping problems are given. That is, the examples in which $\Delta(x_0)$ does not approach zero in spite of
\[ \sup_{x \in X} ||p(\cdot|x) - \tilde{p}(\cdot|x)|| \to 0 \] (where $|| \cdot ||$ is the total variation norm).

### 3. Main Result

**Theorem.** Suppose that Assumptions 1-5 hold. Then there exists a constant $K < \infty$ such that
\begin{equation}
\Delta(x_0) \leq K V(x_0) \sup_{x \in X} \frac{1}{V(x)} ||p(\cdot|x) - \tilde{p}(\cdot|x)||_V,
\end{equation}
where
\[ ||p(\cdot|x) - \tilde{p}(\cdot|x)||_V := \sup_{|\varphi| \leq V} \left| \int_X \varphi(y)[p(dy|x) - \tilde{p}(dy|x)] \right|. \]

The evaluation of the constant $K$ in inequality (3.1) is beyond the scope of this paper. Upper bounds for $K$ could be given when using some lower bounds of invariant probabilities of “optimal stopping sets” $S$ and $\tilde{S}$ defined in Section 4. To bound $K$ one would also need estimates of the rate of convergence of the
distributions of a Markov chain to its invariant distribution (for instance, given in [11], [15], [17]) and the explicit approximations of, so-called, taboo transition probabilities (given, for example, in [22]). In the case of bounded $c_0$ and $r$ the constant on the right-hand side of (3.1) (with $V \equiv 1$) is evaluated in [21]. Note that $K$ in (3.1) is not an absolute constant, and it can depend on certain characteristics of $p$ and $\tilde{p}$.

In Section 5 we consider a simple example of application of the stability inequality (3.1) to the asset selling problem.

4. Subsidiary Results and the Proofs

We introduce Markov decision processes (MDP’s) (see [1], [3], [8]) which represent the optimal stopping problems under consideration. MDP $\{z_t, a_t\} \equiv \{z_t, a_t; \ t = 0, 1, 2, \ldots\}$ that corresponds to $\{x_t\}$ is specified by $(Z, A, q, c)$.

Here:

— $Z := X \cup \{\infty\}$ is the state space (the state “$\infty$” is absorbing where the process passes once being stopped);

— $A = \{1, 2\}$ is the action set, where the action $a = 1$ prescribes to stop $\{x_t\}$ and the action $a = 2$ requires to continue observation about $\{x_t\}$;

— the transition probability $q(D|z, a)$, where $D \in \mathcal{B}_Z$ (the Borel $\sigma$-algebra), $z \in Z, a \in A$, is defined as follows:

\[
q(D|z, 2) := \begin{cases} 
    p(D'|z) & \text{if } z \in X, \\
    1 & \text{if } z = \infty \text{ and } \infty \in D, \\
    0 & \text{if } z = \infty \text{ and } \infty \notin D,
\end{cases}
\]

(4.1)

where $D' = D\setminus\{\infty\}$;

\[
q(D|z, 1) := \begin{cases} 
    1 & \text{if } \infty \in D, z \in Z, \\
    0 & \text{if } \infty \notin D, z \in Z;
\end{cases}
\]

(4.2)

— finally, the one-step cost function $c : Z \times A \to \mathbb{R}$ is given by expression:

\[
c(z, 2) := \begin{cases} 
    c_0(z) & \text{if } z \in X, \\
    0 & \text{if } z = \infty;
\end{cases}
\]

(4.3)

\[
c(z, 1) := \begin{cases} 
    -r(z) & \text{if } z \in X, \\
    0 & \text{if } z = \infty,
\end{cases}
\]

(4.4)

(to stop at the state $z \in X$ one gains $r(z)$ or “pays” $-r(z)$).

Similarly, using $\tilde{p}$ instead of $p$ the “approximating” MDP $\{\tilde{z}_t, \tilde{a}_t\}$ specifying by $(Z, A, \tilde{q}, c)$ is defined.
Each stopping rule \( \tau = \tau_B, B \in \mathcal{B}_+, B \subset X \), defines a stationary control policy \( f = \{ f, f, \ldots \} \) for the process \( \{ z_t, a_t \} \) (for the definition, see, e.g. [3]). The application of \( f \) at time \( t \geq 0 \) consists in the following actions:

\[
f(z_t) := \begin{cases} 1 & \text{if } z_t \in B, \\ 2 & \text{if } z_t \notin B. \end{cases}
\] (4.5)

Similarly, every stopping rule \( \tilde{\tau} = \tilde{\tau}_B, B \in \mathcal{B}_+, B \subset X \), determines the stationary control policy \( \tilde{f} = \{ \tilde{f}, \tilde{f}, \ldots \} \) for the process \( \{ \tilde{z}_t, \tilde{a}_t \} \) in such a way that

\[
\tilde{f}(\tilde{z}_t) := \begin{cases} 1 & \text{if } \tilde{z}_t \in B, \\ 2 & \text{if } \tilde{z}_t \notin B. \end{cases}
\] (4.6)

Observing that any \( f \) or \( \tilde{f} \) in (4.5), (4.6) is uniquely determined by \( B \in \mathcal{B}_+ \), we denote the class of all such stationary policies by \( F \), and for any \( f \in F \) we define on the Banach space \( \bar{B}_V \) two operators \( T_f \) and \( \tilde{T}_f \):

\[
T_f u(z) := c(z, f(z)) + \int_Z u(y)q(dy|z, f(z)),
\] (4.7)

\[
\tilde{T}_f u(z) := c(z, f(z)) + \int_Z u(y)\tilde{q}(dy|z, f(z)),
\] (4.8)

where \( u \in \bar{B}_V, z \in Z \). Here and further on \( \bar{B}_V \) stands for the Banach space consisting of all functions \( u : Z \rightarrow R \) such that \( u(\infty) = 0 \), and such that the restriction of \( u \) on \( X \) belongs to \( B_V \). Consequently, for \( u \in \bar{B}_V \), \( \|u\|_V = \sup_{x \in X} \frac{|u(x)|}{V(x)} \).

The following assertion is a key step in the proof of the main inequality (3.1).

**Lemma 1.** Suppose that Assumptions 1-4 hold. Then for every \( f \in F \) we have the following:

(a)

\[
T_f \bar{B}_V \subset \bar{B}_V, \quad \tilde{T}_f \bar{B}_V \subset \bar{B}_V,
\]

moreover,

\[
\|T_f u\|_V \leq l + d\|u\|_V, \quad u \in \bar{B}_V; \tag{4.9}
\]

\[
\|\tilde{T}_f u\|_V \leq l + d\|u\|_V, \quad u \in \bar{B}_V; \tag{4.10}
\]

where \( l = \max\{\|c_0\|_V, \|r\|_V\} \).

(b) There exists an integer \( N = N(f) \) such that for every \( u, v \in \bar{B}_V \),

\[
\|T_f^N u - \tilde{T}_f^N v\|_V \leq 0.5\|u - v\|_V, \tag{4.11}
\]

\[
\|\tilde{T}_f^N u - \tilde{T}_f^N v\|_V \leq 0.5\|u - v\|_V. \tag{4.12}
\]
Proof. (a) We confine ourselves to the proof of (4.9) and (4.11). By (4.1), (4.3), (4.4), (4.7) and by the fact that $u(\infty) = 0$, we get

$$T_f u(\infty) = c(\infty, f(\infty)) + \int_X u(x)q(dx|\infty, f(\infty)) = 0,$$

because of $q(X|\infty, \cdot) = 0$.

Then, in view of (4.1), Assumption 1, (2.4), (4.3) and (4.4) we get:

$$||T_f u||_V = \sup_{x \in X} \frac{1}{V(x)} |c(x, f(x)) + \int_X u(y)p(dy|x)|$$

$$\leq \max\{||c_0||_V, ||r||_V\} + \sup_{x \in X} \frac{||u||_V}{V(x)} \int_X V(y)p(dy|x).$$

(b) For every arbitrary but fixed $n \geq 1$, iterating (4.7) we get that

$$T^n_f u(z) = R_{f,n} c(z, f) + \int_Z u(y)q^n(dy|z, f), \quad (4.13)$$

$z \in Z, u \in \tilde{B}_V$. In equation (4.13) the term $R_{f,n} c$ does not contain the function $u$ (and does not depend on $u$).

Therefore, since $u(\infty) = v(\infty) = 0$,

$$||T^n_f u - T^n_f v||_V = \sup_{x \in X} \frac{1}{V(x)} \left| \int_Z [u(z) - v(z)]q^n(dz|x, f) \right|$$

$$\leq ||u - v||_V \sup_{x \in X} \int_Z \frac{V(z)}{V(x)} q^n(dz|x, f), \quad (4.14)$$

where

$$\overline{V}(z) = \begin{cases} 0 & \text{if } z = \infty, \\ V(z) & \text{if } z \in X. \end{cases}$$

For each $x \in X$, $n \geq 1$, let $\eta_{n,x}$ be a random element in $(Z, \mathcal{B}_Z)$ with the distribution $q^n(\cdot|x, f)$. Also for every $x \in X$, $n \geq 1$, we define a nonnegative random variable

$$\xi_{n,x} := \frac{\overline{V}(\eta_{n,x})}{V(x)}. \quad (4.15)$$

We show that the family of random variables $\{\xi_{n,x}; x \in X, n \geq 1\}$ is uniformly integrable. For this it is enough to verify that for $p > 1$

$$\sup_{x \in X, n \geq 1} E(\xi_{n,x}^p) < \infty.$$

But

$$E(\xi_{n,x}^p) = \int_Z \frac{V^n(z)}{V^p(x)} q^n(dz|x, f).$$
Indeed, for each $x \in X, D \subset X$,
\[
q^n(D|x, f) \leq p^n(D|x)
\]
because on the $n$-passage from $x$ to $D$ the process $\{x_t\}$ can visit the set $B \in \mathcal{B}_+$. But $B$ defines the policy $f$ (see (4.5)) and the corresponding stopping rule $\tau$, which prescribes to leave $X$ as soon as the process enters $B$. It is known (see Proposition 3.13 in [20]) that Assumption 2, particularly inequality (2.2), provides that
\[
\sup_{x \in X} \frac{1}{V^p(x)} \left| \int_X V^p(y)p^n(dy|x) - \int_X V^p(y)\pi(dy) \right| \rightarrow 0 \quad (4.17)
\]
as $n \rightarrow \infty$, and that $\int_X V^p(y)\pi(dy) < \infty$.

Remembering that $V \geq 1$, from (4.17) it follows that the right-hand side of (4.16) is bounded uniformly on $x \in X, n \geq 1$.

In view of (4.15) the last integral in (4.14) can be rewritten as $E\xi_{n,x}$, and by virtue of the uniform integrability
\[
\sup_{x \in X, n \geq 1} |E\xi_{n,x} - E\{\xi_{n,x}; \xi_{n,x} \leq L\}| \rightarrow 0 \quad \text{as} \quad L \rightarrow \infty.
\]

Therefore, for every $\epsilon > 0$, we can choose $L$ in such a way that in (4.14), for all $n \geq 1$,
\[
\sup_{x \in X} \int_X \frac{V(y)}{V(x)} q^n(dy|x, f) \leq \sup_{x \in X} \int_{y: \frac{V(y)}{V(x)} \leq L} \frac{V(y)}{V(x)} q^n(dy|x, f) + \epsilon \\
\leq L \sup_{x \in X} q^n(X|x, f) + \epsilon. \quad (4.18)
\]

On the other hand, by definitions of $f$ and of the corresponding stopping rule $\tau_B$, the process $\{z_t\}$ passes in the absorbing state “$\infty$” as soon as $\{x_t\}$ reaches the set $B$. Therefore, for $x \notin B, n \geq 2$,
\[
q^n(X|x, f) = P_x(x_1 \notin B, \ldots, x_{n-1} \notin B). \quad (4.19)
\]

Finally, it suffices to compare (4.14) and (4.18) and to apply Assumption 4. 

For each stationary policy $f \in \mathcal{F}$ we introduce the total expected costs for the control processes $\{z_t, a_t\}$ and $\{\tilde{z}_t, \tilde{a}_t\}$:
\[
H(z, f) := E^f \sum_{t=0}^{\infty} c(z_t, f(z_t)); \quad (4.20)
\]
\[ \widetilde{H}(z,f) := \widehat{E}_z^f \sum_{t=0}^{\infty} c(\tilde{z}_t, f(\tilde{z}_t)), \]

where \( z \in X = Z \setminus \{\infty\} \) is a given initial state. Also we set:
\[ H(\infty, f) = \widehat{H}(\infty, f) := 0. \]

**Remark 4.** We use symbols \( E^f_z, \widehat{E}^f_z \) to denote expectations of functionals on trajectories of the corresponding control Markov processes.

**Lemma 2.** (a) \( H, \widehat{H} \in \bar{B}_V \).
(b) \( H \) and \( \widehat{H} \) are the unique solutions of operator equations
\[ H = T_f H, \]
\[ \widehat{H} = \widehat{T}_f \widehat{H}. \]

**Proof.** (a) The mapping \( Q : \bar{B}_V \to \bar{B}_V \) defined as
\[ Qu(z) := \int_Z u(y)q(dy|z,f), \quad u \in \bar{B}_V, \]

is linear bounded operator on \( (\bar{B}_V, \| \cdot \|_V) \) (the boundedness was shown in the proof of Lemma 1).

By Fubini’s theorem and the Markov property,
\[ QQu(z) = \int_Z q(ds|z,f) \int_Z u(y)q(dy|s,f) = \int_Z u(y)q^2(dy|z,f). \]

Similarly, for any \( n \geq 1 \),
\[ Q^n u(z) = \int_Z u(y)q^n(dy|z,f), \quad u \in \bar{B}_V. \]

From (4.18), (4.19) it follows that for every \( \delta > 0 \) there exists \( n \geq 1 \) such that
\[ \|Q^n\|_V \leq \delta. \]

Thus for every \( m \geq 1 \),
\[ \|Q^{nm}\|_V \leq \delta^m. \]

Using these inequalities it is easy to see that there exist constants \( M < \infty, \gamma < 1 \) such that
\[ \|Q^n\|_V \leq M\gamma^n, \quad n = 0, 1, 2, \ldots \]

(4.25)

On the other hand, in (4.20) \( c \in \bar{B}_V \) (Assumption 1) and for each \( z \neq \infty \),
\[ \frac{1}{V(z)}|H(z,f)| \leq E^f_z \sum_{t=0}^{\infty} \frac{1}{V(z)}|c(z_t, f)| \]
\[
= \sum_{t=0}^{\infty} \frac{1}{V(z)} E^f_{x_0} |c(z_t, f)| = \sum_{t=0}^{\infty} \frac{1}{V(z)} \int_Z |c(z_t, f)| q^t(dy|z, f) \\
\leq L \sum_{t=0}^{\infty} ||Q^t||v \leq \frac{LM}{1 - \gamma}
\]
due to (4.25) \(L < \infty\) is a certain constant. Also we have that \(H(\infty) = 0\).

(b) Since the sum in (4.20) is an integrable function, the standard arguments of dynamic programming show that \(H\) satisfies equation (4.22). This solution of (4.22) is unique because of the contractive property proved in Lemma 1.

**Corollary.** For each \(x_0 \in X, \tau \in T, \tilde{\tau} \in \tilde{T}\) the expectations in (2.9), (2.10) exist, and the total expected costs \(W(x_0, \tau)\) and \(\tilde{W}(x_0, \tilde{\tau})\) are finite.

**Proof.** For \(\tau = \tau_B \in T\), the definitions in (2.6), (4.5) imply that for the corresponding policy \(f\) in (4.20) we get:

\[
H(z, f) = E^f_{x_0} \left[ \sum_{t=0}^{\tau-1} c(z_t, f(z_t)) + c(z_\tau, f(z_\tau)) \right]
\]
(because of \(c(z_t, \cdot) = 0\) for \(t > \tau\) according to (4.3), (4.4)).

Then, for \(t \leq \tau - 1, f(z_t) = 2, z_t \in X \setminus B\). Thus, by (4.3), \(c(z_t, f(z_t)) = c_0(z_t)\). For \(t = \tau, f(z_\tau) = 1\) (stopping) and \(z_\tau \in B \subset X\), and by (4.4), \(c(z_\tau, f(z_\tau)) = -r(z_\tau)\).

Consequently, for \(z = x_0\), we have in (4.20):

\[
H(z, f) = E^f_{x_0} \left[ \sum_{t=0}^{\tau-1} c_0(z_t) - r(z_\tau) \right].
\]

On the other hand, since \(z_t \in X \setminus B, t = 0, 1, 2, \ldots, \tau - 1,\) according to (4.1), the expectation \(E^f_{x_0}\) of any functional of the part of trajectory \(z_0, z_1, \ldots, z_\tau\) coincides with the expectation \(E_{x_0}\). Thus

\[
W(x_0, \tau) = H(x_0, f), \quad (4.26)
\]
and the assertion of Corollary follows from Lemma 2.

**The Proof of the Stability Inequality (3.1).** To prove inequality (3.1) we first of all rewrite the stability index defined in (2.13). Let \(\tau_s \in T\) and \(\tilde{\tau}_s \in T\) be the optimal stopping rules, respectively, for the processes \(\{x_t\}\) and \(\{\tilde{x}_t\}\) (see Assumption 5). The corresponding “stopping sets” (defined in (2.6)) we denote by \(S, \tilde{S} \in B_+\), and the corresponding stationary policies (see (4.5)) we denote by \(f_s\) and \(\tilde{f}_s\), respectively (recall that \(f_s, \tilde{f}_s \in \mathcal{F}\)). Using (2.13) and (4.26) we get that
\[ \Delta(x_0) = H(x_0, \tilde{f}_*) - H(x_0, f_*). \quad (4.27) \]

**Remark 5.** Applying the technique of optimality equations (see, e.g. [16], [18]), it could be shown under certain conditions that \( S = \{ x \in X : W_*(x) = -r(x) \} \). We do not use this fact.

It is easy to see (e.g. [21]) that

\[ \Delta(x_0) \leq 2 \max_{f \in [f_*, f_*]} \left| H(x_0, f) - \tilde{H}(x_0, f) \right|, \quad (4.28) \]

where the functions \( H \) and \( \tilde{H} \) were defined in (4.20), (4.21).

Let us evaluate \(|H(x_0, f) - \tilde{H}(x_0, f)|\), for example, for \( f = f_* \). Using the notation:

\[ x := x_0, \quad H_f(\cdot) := H(\cdot, f), \quad \tilde{H}_f(\cdot) := \tilde{H}(\cdot, f), \]

by (4.22), (4.23) we obtain:

\[
\begin{align*}
\frac{1}{V(x)} \left| \tilde{H}_f(x) - H_f(x) \right| &= \frac{1}{V(x)} \left| \tilde{T}^N_f \tilde{H}_f(x) - T^N_f H_f(x) \right| \\
&\leq \frac{1}{V(x)} \left| \tilde{T}^N_f \tilde{H}_f(x) - \tilde{T}^N_f H_f(x) \right| + \frac{1}{V(x)} \left| \tilde{T}^N_f H_f(x) - T^N_f H_f(x) \right|, \quad (4.29)
\end{align*}
\]

where \( N = N(f) \) is the integer from Lemma 1. The last inequality implies that

\[
\begin{align*}
\frac{1}{V(x)} \left| \tilde{H}_f(x) - H_f(x) \right| &\leq \| \tilde{H}_f - H_f \|_V \\
&\leq \| \tilde{T}^N_f \tilde{H}_f - \tilde{T}^N_f H_f \|_V + \| \tilde{T}^N_f H_f - T^N_f H_f \|_V. \quad (4.30)
\end{align*}
\]

In view of (4.12),

\[
\| \tilde{H}_f - H_f \|_V \leq 2 \| \tilde{T}^N_f H_f - T^N_f H_f \|_V. \quad (4.31)
\]

Applying the induction and the Fubini Theorem we obtain from (4.7) and (4.8) that

\[
T^N_f H_f(z) = c(z, f(z)) + \int_z c(y, f(y))q(dy|z, f) + \cdots
\]

\[
+ \int_z c(y, f(y))q^{N-1}(dy|z, f) + \int_z H_f q^N(dy|z, f).
\]

The analogous expression can be written for \( \tilde{T}^N_f H_f(z) \). Thus,

\[
\tilde{T}^N_f H_f - T^N_f H_f = \sum_{n=1}^{N-1} [Q^n - \tilde{Q}^n] c + [Q^N - \tilde{Q}^N] H_f, \quad (4.32)
\]

where the bounded linear operator \( \tilde{Q} : \tilde{B}_V \to \tilde{B}_V \) is defined similar to (4.24).
ROBUSTNESS ESTIMATING OF OPTIMAL STOPPING...

(Using \( \tilde{q} \) in place of \( q \)).

By Assumption 1 and Lemma 2, \( c \in \bar{B}_V \) and \( H_f \in \bar{B}_V \).

Thus from (4.32) it follows that the norm \( ||\tilde{T}^N_f H_f - T^N_f H_f||_V \) is bounded by a finite number of the summands of the type:

\[
||Q^n - \tilde{Q}^n||_V ||u||_V, \text{ with a certain } u \in \bar{B}_V.
\]

But

\[
||Q^n - \tilde{Q}^n||_V \leq ||Q Q^{n-1} - \tilde{Q} Q^{n-1}||_V + ||\tilde{Q} Q^{n-1} - \tilde{Q} \tilde{Q}^{n-1}||_V
\]

\[
\leq ||Q - \tilde{Q}||_V ||Q^{n-1}||_V + ||\tilde{Q}||_V ||Q^{n-1} - \tilde{Q}^{n-1}||_V. \quad (4.33)
\]

Using (4.25) (and the similar inequality for \( ||\tilde{Q}^n||_V \)) and the induction, we show that the right-hand side of the last inequality is less than a constant times

\[
||Q - \tilde{Q}||_V = \sup_{x \in X} \frac{1}{V(x)} \sup_{|\phi| \leq V} \left| \int_Z \phi(y) [q(dy|z,f) - \tilde{q}(dy|z,f)] \right|
\]

\[
= \sup_{x \in X} \frac{1}{V(x)} \sup_{|\phi| \leq V} \left| \int_X \phi(y) [p(dy|x) - \tilde{p}(dy|x)] \right|, \quad (4.34)
\]

where the last equality holds because of \( \phi(\infty) = 0 \), and (4.1), (4.2). Comparing relationships (4.28)-(4.34) we finally obtain the desired stability inequality (3.1).

5. Simple Example of Application of Robustness Inequality: The Asset Selling Problem

In fact inequality (3.1) could be applied to evaluate the stability of optimal stopping of any discrete-time Markov process for which Assumptions 1-5 hold. Many important applied processes for which ergodicity Assumptions 2-4 are satisfied can be found in the book [10]. In the present paper we do not carry out the task to consider some serious applications. We only mention the well-known problem of asset selling. In the last years this problem has received attention in connection with application in insurance and finance (see [9], [13], [19]).

The classical version of the problem is reduced to observation about a sequence \( \{x_1, x_2, \ldots, x_t, \ldots\} \) of non-negative random variables (offers for an asset). Thus, stopping at time \( t \) (baying the asset) supposes the revenue \( r(x_t) \equiv x_t \), while the decision to expect future offers results the payment \( c_0(x_t) \) (where \( c_0 \) is a, so-called, one-step holding cost). Usually (see, e.g., [16]) it is supposed that the random variables \( \{x_t\} \) are uniformly bounded, and \( c_0 \) does not depend on...
We consider a more general situation where \( x_t, t \geq 1 \) are i.i.d. random variables taking values in \( X = [0, \infty) \), and the revenue \( r(x), x \in X \), and the holding cost \( c_0(x), x \in X \) are nonnegative measurable functions, which can be either bounded or unbounded. In the framework of our stability problem we also consider an approximating offer process \( \{ \tilde{x}_t, t = 1, 2, \ldots \} \) which is represented by i.i.d. random variables with values in \( [0, \infty) \).

Let \( V : [0, \infty) \to [1, \infty) \) be a given nondecreasing measurable function such that \( V(0) = 1 \). We suppose the following:

(a) \( \max\{r(x), c_0(x)\} \leq V(x), x \in [0, \infty) \).

(b) There exist \( p > 1 \) such that \( EV^p(x_1) < \infty, EV^p(\tilde{x}_1) < \infty \).

(c) The random variables \( \{x_t\}, \{\tilde{x}_t\} \) have densities \( \rho \) and \( \tilde{\rho} \) respectively.

(d) The densities \( \rho \) and \( \tilde{\rho} \) have a common support, which is either a segment \( [a, b] \subset [0, \infty) \) or \( [0, \infty) \).

It is very easy to verify that the hypotheses (a)-(d) yield the fulfilment of Assumptions 1-4 (with the Lyapunov function \( V \)). The invariant probabilities \( \pi \) and \( \tilde{\pi} \) are, respectively, the distributions of \( x_t \) and \( \tilde{x}_t \). Meanwhile, we must make Assumption 5 in the present example.

Remark 6. (i) Suppose for a moment that \( c_0 \equiv 0 \), \( \rho(x) > 0 \) for all \( x \in [0, \infty) \) and \( r(x) \to \infty \) as \( x \to \infty \). Then, as it is easy to see, an optimal stopping rule \( \tau_\ast \) for \( \{x_t\} \) does not exist (since at any time \( t \) is “preferable” to continue observations).

(ii) On the other hand, if \( r \) is a bounded function and \( \inf_{x \in [0, \infty)} c_0(x) > 0 \), it is not difficult to show the existence of the optimal stopping rule \( \tau_\ast \) (which is in the class \( T \)).

(iii) As it was shown in the counterexample in [21] the coincidence the supports of the random variables \( x_t \) and \( \tilde{x}_t \) (which yields the fulfilment of Assumption 3) is essential to obtain the stability of the stopping optimization in the present example.

It is convenient to assume that an initial state of the offer processes \( \{x_t\} \) and \( \{\tilde{x}_t\} \) is \( x_0 = \tilde{x}_0 = 0 \). One can verify that in the particular case considered,

\[
||p(\cdot|x) - \tilde{p}(\cdot|x)||_V = \int_0^\infty V(y) |\rho(y) - \tilde{\rho}(y)| \, dy.
\]
Thus the application of (3.1) gives the following stability inequality:

$$\Delta(0) \leq K \int_{0}^{\infty} V(y) |\rho(y) - \tilde{\rho}(y)| \, dy.$$ 

References


