INTERVAL-VALUED H-FUZZY RELATIONS

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Abstract: We introduce the category \textsc{IVRel}(H) consisting of interval-valued H-fuzzy relational spaces and relation preserving mappings between them and we study structures of the category \textsc{IVRel}(H) in the viewpoint of the topological universe introduced by Nel. Thus we show that \textsc{IVRel}(H) satisfies all the conditions of a topological universe over \textsc{Set} except the terminal separator property and \textsc{IVRel}(H) is Cartesian closed over \textsc{Set}. Furthermore, we study some relations among \textsc{Rel}(H), \textsc{IRel}(H) and \textsc{IVRel}(H).

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1. Introduction

Nel [20] introduced the notion of a topological universe which implies a Cartesian closed and a concrete quasitopos [1]. Every topological universe satisfies all the properties of a topos except one condition on the subobject classifier. The notion of a topological universe has already been put to effective use in
several areas of mathematics in [18], [19], [21]. In 1980, Cerruti [5] introduced the category of L-fuzzy relations and investigated some of its properties. After that time, Hur [11] introduced the category $\text{Rel}(H)$ of the fuzzy relational spaces with a complete Heyting algebra $H$ as a codomain and he studied the category $\text{Rel}(H)$ in the sense of a topological universe. Moreover, by using the concept of intuitionistic fuzzy set introduced by Atanassov [2], Hur et al [13] investigated the category $\text{IVRel}(H)$ consisting of intuitionistic H-fuzzy relational spaces and morphisms between them in a topological universe viewpoint.


In this paper, we introduce the category $\text{IVRel}(H)$ of interval-valued H-fuzzy relational spaces and study the category $\text{IVRel}(H)$ in the sense of a topological universe. In particular, we show that $\text{IVRel}(H)$ satisfies all the conditions of a topological universe over $\text{Set}$ except the terminal separator property. And $\text{IVRel}(H)$ is shown to be Cartesian closed over $\text{Set}$. Furthermore, we study some relations among $\text{Rel}(H)$, $\text{IRel}(H)$ and $\text{IVRel}(H)$.

2. Preliminaries

In this section, we will introduce some basic definitions and well-known results which are needed in the next sections.

**Definition 2.1.** (see [9]) Let $A$ be a concrete category and let $\Gamma$ be a class.

1. A source in $A$ is a pair $(X, (f_\alpha)_\Gamma)$ (simply, $(X, f_\alpha)$ or $(f_\alpha)_\Gamma$), where $X$ is an $A$-object and $(f_\alpha : X \to X_\alpha)_\Gamma$ is a family of $A$-morphisms each with domain $X$. In this case, $X$ is called the domain of the source and the family $(X_\alpha)_\Gamma$ is called the codomain of the source.

2. A source $(X, f_\alpha)$ is called a *mono-source* providing that the $f_\alpha$ can be simultaneously cancelled from the left; i.e., providing that for any pair $Y \xrightarrow{r} X$, of morphisms such that $f_\alpha \circ r = f_\alpha \circ s$ for each $\alpha \in \Gamma$, it follows that $r = s$.

**Dual Notions.** Sink in $A$ and epi-sink.
Definition 2.2. (see [17]) Let $A$ be a concrete category and let $((Y_\alpha, \xi_\alpha))_\Gamma$ be a family of objects in $A$ indexed by a class $\Gamma$. For any set $X$, let $(f_\alpha : X \to Y_\alpha)_\Gamma$ be a source of mappings indexed by $\Gamma$. An $A$-structure $\xi$ on $X$ is said to be initial with respect to $(X, (f_\alpha), ((Y_\alpha, \xi_\alpha)))$ providing that the following conditions hold:

1. For each $\alpha \in Z$, $f_\alpha : (X, \xi) \to (Y_\alpha, \xi_\alpha)$ is an $A$-morphism.

2. If $(Z, \rho)$ is an $A$-object and $g : Z \to X$ is a mapping such that for each $i \in Z$, the mapping $f_\alpha \circ g : (Z, \rho) \to (Y_\alpha, \xi_\alpha)$ is an $A$-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an $A$-morphism. In this case, $(f_\alpha : (X, \xi) \to (Y_\alpha, \xi_\alpha))_\Gamma$ is called an initial source in $A$.

Dual Notions. Final structure and final sink.

Definition 2.3. (see [17]) A concrete category $A$ is said to be topological over $\text{Set}$ providing that for each set $X$, for any family $((Y_\alpha, \xi_\alpha))_\Gamma$ of $A$-objects, and for any source $(f_\alpha : X \to Y_\alpha)_\Gamma$ of mappings, there exists a unique $A$-structure $\xi$ on $X$ which is initial with respect to $(X, (f_\alpha), ((Y_\alpha, \xi_\alpha)))$.

Dual Notions. Cotopological category.

Result 2.A. (see [17, Theorem 1.5]) A concrete category $A$ is topological if and only if $A$ is cotopological.

Result 2.B. (see [17, Theorem 1.6]) Let $A$ be a topological category over $\text{Set}$. Then $A$ is complete and cocomplete.

Definition 2.4. (see [8]) A category $A$ is called Cartesian closed providing that the following conditions hold:

1. For any $A$-objects $A$ and $B$, there exists a product $A \times B$ in $A$.

2. Exponential exists in $A$, i.e., for any $A$-object $A$, the functor $A \times - : A \to A$ has a right adjoint, i.e., for any $A$-object $B$, there exists an $A$-object $B^A$ and an $A$-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any $A$-object $C$ and any $A$-morphism $f : A \times C \to B$, there exists a unique $A$-morphism $\exists ! A \times f : C \to B^A$ such that the diagram

\[
\begin{array}{ccc}
A \times B^A & \xrightarrow{e_{A,B}} & B \\
\downarrow \exists ! A \times f & & \downarrow f \\
A \times C & & \\
\end{array}
\]
commutes.

**Definition 2.5.** (see [17]) Let $A$ be a concrete category.

1. The $A$-fibre of a set $X$ is the class of all $A$-structures on $X$.

2. $A$ is called **properly fibred over $\text{Set}$** providing that the following conditions hold:

   (i) (**Fibre-smallness**) For each set $X$, the $A$-fibre of $X$ is a set.

   (ii) (**Terminal separator property**) For each singleton set $X$, the $A$-fibre of $X$ has precisely one element.

   (iii) If $\xi$ and $\eta$ are $A$-structures on a set $X$ such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are $A$-morphisms, then $\xi = \eta$.

**Definition 2.6.** (see [20]) A category $A$ is called a **topological universe over $\text{Set}$** providing that the following conditions hold:

1. $A$ is well-structured over $\text{Set}$, i.e.: (i) $A$ is a concrete category; (ii) $A$ has the fibre-smallness condition; (iii) $A$ has the terminal separator property.

2. $A$ is cotopological over $\text{Set}$.

3. Final episinks in $A$ are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any $A$-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of $f$ and $g_\lambda$ for each $\lambda$, is again a final episink.

**Definition 2.7.** (see [22]) A category $A$ is called a **topos** providing that the following conditions hold:

1. There is a terminal object $U$ in $A$, i.e., for each $A$-object $A$, there exists one and only one $A$-morphism from $A$ to $U$.

2. $A$ has equalizers i.e., for any $A$-objects $A$ and $B$ and $A$-morphisms $A \xrightarrow{f} B$,

   ![Diagram](https://example.com/diagram.png)

   there exist an $A$-object $C$ and an $A$-morphism $h : C \to A$ such that:

   (a) $f \circ h = g \circ h$,

   (b) for each $A$-object $C'$ and $A$-morphism $h' : C' \to A$ with $f \circ h' = g \circ h'$, there exists a unique $A$-morphism $\overline{f'} : C' \to C$ such that $h' = h \circ \overline{f'}$, i.e., the diagram
commutes.

(3) \( \mathbf{A} \) is Cartesian closed.

(4) There is a subobject classifier in \( \mathbf{A} \), i.e., there is an \( \mathbf{A} \)-object \( \Omega \) and \( \mathbf{A} \)-morphism \( v : U \rightarrow \Omega \) such that for each \( \mathbf{A} \)-monomorphism \( m : A' \rightarrow A \), there exists a unique \( \mathbf{A} \)-morphism \( \phi_m : A \rightarrow \Omega \) such that the following diagram is a pullback:

\[
\begin{array}{ccc}
A' & \xrightarrow{m} & A \\
\downarrow & & \downarrow v \\
A & \xrightarrow{\phi_m} & \Omega.
\end{array}
\]

**Remark 2.8.** Let \( \mathbf{A} \) be any category with a subobject classifier. If \( f \) is any bimorphism in \( \mathbf{A} \), then \( f \) is an isomorphism in \( \mathbf{A} \) (cf. [4]).

**Definition 2.9.** (see [5], [22]) A lattice \( H \) is called a **complete Heyting algebra**, if \( H \) satisfies the following conditions hold:

1. \( H \) is a complete lattice.
2. For any \( a, b \in H \), the set \( \{ x \in H : x \land a \leq b \} \) has a greatest element denoted by \( a \rightarrow b \) (called **pseudo-complement** of \( a \) and \( b \)), i.e., \( x \land a \leq b \) if and only if \( x \leq (a \rightarrow b) \).

In particular, for each \( a \in H \), \( N(a) = a \rightarrow 0 \) is called the **negation** or the **pseudocomplement** of \( a \).

**Result 2.C.** (see [5, Example 6 on p. 46]) Let \( H \) be a complete Heyting algebra and let \( a, b \in H \). Then:

1. If \( a \leq b \), then \( N(b) \leq N(a) \), i.e., \( N : H \rightarrow H \) is an involutive order reversing operation in \((H, \leq)\).
2. \( a \leq NN(a) \).
3. \( N(a) = NNN(a) \).
4. \( N(a \lor b) = N(a) \land N(b) \) and \( N(a \land b) = N(a) \land N(b) \).
Throughout this paper, we use $H$ as a complete Heyting algebra with the least element $0$ and the largest element $1$.

3. The Category $\text{IVRel}(H)$

We introduce the category $\text{IVRel}(H)$ consisting of interval-valued $H$-fuzzy relational spaces and relation preserving mappings between them, and show that it has similar structures as those of $\text{IVRel}(H)$.

Let $D(H)$ be the set of all closed subintervals of $H$. The elements of $D(H)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M_L, M_U]$, where $M_L$ and $M_U$ are the lower and the upper end points respectively. Especially, we denote $0 = [0,0], 1 = [1,1]$, and $a = [a,a]$ for every $a \in H$. We also note that

(i) $(\forall M, N \in D(H))(M = N \iff M_L = N_L, M_U = N_U)$.

(ii) $(\forall M, N \in D(H))(M \leq N \iff M_L \leq N_L, M_U \leq N_U)$.

For every $M \in D(H)$, the complement of $M$, denoted by $M^c$, is defined by $M^c = N(M) = [N(M_U), N(M_L)]$.

Definition 3.1. Let $X$ be a nonempty set. Then a mapping $A = [A_L, A_U] : X \rightarrow D(H)$ is called an interval-valued $H$-fuzzy set (in short, $\text{IVHFS}$) in $X$, where $A_L$ and $A_U$ are $H$-fuzzy sets in $X$ satisfying $A_L(x) \leq A_U(x)$ for each $x \in X$.

We will denote the set of all IVHFSs in $X$ as $D(H)^X$.

Definition 3.2. Let $X$ be a nonempty set. Then a mapping $R = [R_L, R_U] : X \times X \rightarrow D(H)$ is called an interval-valued $H$-fuzzy relation (in short, $\text{IVHFR}$) on $X$, where $R_L$ and $R_U$ are $H$-fuzzy relations on $X$ satisfying $R_L(x, y) \leq R_U(x, y)$ for each $(x, y) \in X \times X$. The pair $(X, R)$ is called an interval-valued $H$-fuzzy relational space (in short, $\text{IVFRS}$).

Definition 3.3. Let $(X, R_X)$ and $(Y, R_Y)$ be an IVFRSs. A mapping $f : X \rightarrow Y$ is called a relation preserving mapping if $R^L_X \leq R^L_Y \circ f^2$ and $R^U_X \leq R^U_Y \circ f^2$, where $f^2 = f \times f$.

The following is the immediate result of Definition 3.3.

Proposition 3.4. Let $(X, R_X), (Y, R_Y)$ and $(Z, R_Z)$ be IVFRSs.

(1) The identity mapping $id_X : (X, R_X) \rightarrow (X, R_X)$ is a relation preserving mapping.

(2) If $f : (X, R_X) \rightarrow (Y, R_Y)$ and $g : (Y, R_Y) \rightarrow (Z, R_Z)$ are relation
Then, by the definition of $R \in \text{IVHFR}$, denote by $f$ and be an IVHFR and let $f$ be an IVHFR and let $f$ be a set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IVHFRSs and let $f$ be a set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IVHFRSs and relation preserving mappings between them. Every IVRel($H$)-morphism will be called an IVRel($H$)-morphism.

**Theorem 3.5.** IVRel($H$) is topological over Set.

**Proof.** Let $X$ be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IVHFRSs indexed by a class $\Gamma$. Let $(f_\alpha : X \to X_\alpha)_{\Gamma}$ be any source of mappings. We define the mapping $R = [R^L, R^U] : X \times X \to D(H)$ as follows: For each $(x, y) \in X \times X$,

$$R^L(x, y) = \bigwedge_{\alpha \in \Gamma} R^L_\alpha(f(x), f(y)) \quad \text{and} \quad R^U(x, y) = \bigwedge_{\alpha \in \Gamma} R^U_\alpha(f(x), f(y)).$$

Then, by the definition of $R$, $R^L \leq R^U$. Thus $(X, R) \in \text{IVRel}(H)$. Moreover, $f_\alpha : (X, R) \to (X_\alpha, R_\alpha)$ is an IVRel($H$)-mapping for each $\alpha \in \Gamma$.

For any $(Y, R_Y) \in \text{IVRel}(H)$, let $g : Y \to X$ be any mapping for which $f_\alpha \circ g : (Y, R_Y) \to (X_\alpha, R_\alpha)$ is an IVRel($H$)-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (Y, R_Y) \to (X, R)$ is an IVRel($H$)-mapping. Hence $R$ is the initial structure on $X$ with respect to $(X, (f_\alpha), ((X_\alpha, R_\alpha)))$. This completes the proof. \hfill \Box

**Example 3.5.** (1) Inverse image of an IVHFR. Let $X$ be a set, let $(Y, R_Y)$ be an IVHFRS and let $f : X \to Y$ be any mapping. Then there exists the initial IVHFR $R$ on $X$ for which $f : (X, R) \to (Y, R_Y)$ is an IVRel($H$)-mapping. In this case, $R$ is called the inverse image of $R_Y$ under $f$. In particular, if $X \subset Y$ and $f : X \to Y$ is the canonical mapping, then $(X, R)$ is called an interval-valued $H$-fuzzy relational subspace of $(Y, R_Y)$, where $R = [R^L, R^U]$ is the inverse image of $R_Y$ under $f$. In fact, $R^L = R_Y^L|_{X \times X}$ and $R^U = R_Y^U|_{X \times X}$.

(2) Interval-valued fuzzy product structure. Let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IVHFRSs and let $X = \prod X_\alpha$ be the product set of $(X_\alpha)_\Gamma$. Then there exists the initial IVHFR $R$ on $X$ for which each projection $\pi_\alpha : (X, R) \to (X_\alpha, R_\alpha)$ is an IVRel($H$)-mapping. In this case, $R$ is called the product of $(R_\alpha)_\Gamma$ and denoted by $R = \prod R_\alpha$ and $((X_\alpha, R_\alpha))_\Gamma$ is called the interval-valued $H$-fuzzy product relational space of $((X_\alpha, R_\alpha))_\Gamma$. In fact, $(\text{IR})^L = \bigwedge_{\Gamma} R^L_\alpha \circ \pi^2_\alpha$ and $R^L_\alpha = \bigcap_{\Gamma} R^U_\alpha \circ \pi^2_\alpha$.

In particular, if $H = \{1, 2\}$, then

$$(R_1 \times R_2)^L((x_1, y_1), (x_2, y_2)) = R^L_1(x_1, x_2) \wedge R^L_2(y_1, y_2)$$
and
\[(R_1 \times R_2)^U((x_1, y_1), (x_2, y_2)) = R_1^U(x_1, x_2) \land R_2^U(y_1, y_2)\]
for any
\[(x_1, y_1), (x_2, y_2) \in X_1 \times X_2.\]

**Corollary 3.5.** \(\text{IVRel}(H)\) is complete and cocomplete. Moreover, by definition, it is easy to show that \(\text{IVRel}(H)\) is well-powered and co-well-powered.

From Result 2.A and Theorem 3.5, it is clear that \(\text{IVRel}(H)\) is cotopological. However, we show directly that \(\text{IVRel}(H)\) is cotopological.

**Theorem 3.6.** \(\text{IVRel}(H)\) is cotopological over \(\text{Set}\).

**Proof.** Let \(X\) be any set and let \(((X_\alpha, R_\alpha))_\Gamma\) be any family of IVHFRS indexed by a class \(\Gamma\). Let \((f_\alpha : X_\alpha \to X)_\Gamma\) be any sink of mappings. We define the mapping \(R = [R^L, R^U] : D(H) \to D(H)\) as follows: For each \((x, y) \in X \times X\),
\[R^L(x, y) = \bigvee_{f_\alpha^{-1} \circ f_\alpha^{-1}(x, y)} R^L_\alpha(x_\alpha, y_\alpha)\]
and
\[R^U(x, y) = \bigvee_{f_\alpha^{-1} \circ f_\alpha^{-1}(x, y)} R^U_\alpha(x_\alpha, y_\alpha),\]
where \(f_\alpha^{-1} = f_\alpha^{-1} \times f_\alpha^{-1}\). Then clearly \((X, R) \in \text{IVRel}(H)\). Moreover, \(f_\alpha : (X_\alpha, R_\alpha) \to (X, R)\) is an \(\text{IVRel}(H)\)-mapping for each \(\alpha \in \Gamma\).

For any \((Y, R_Y) \in \text{IVRel}(H)\), let \(g : X \to Y\) be any mapping for which \(g \circ f_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y)\) is an \(\text{IVRel}(H)\)-mapping for each \(\alpha \in \Gamma\). Then we can easily check that \(g : (X, R) \to (Y, R_Y)\) is an \(\text{IVRel}(H)\)-mapping. Hence \(R\) is the final structure on \(X\) with respect to \(((X_\alpha, R_\alpha)), (f_\alpha), X)\). This completes the proof.  

**Example 3.6.** (1) *Interval-valued H-fuzzy quotient relation.* Let \((X, R) \in \text{IVRel}(H)\), let \(\sim\) be an equivalence relation on \(X\) and let \(\varphi : X \to X/\sim\) be the canonical mapping. Then there exists the final interval-valued H-fuzzy relation \(R_{X/\sim} = [R^L_{X/\sim}, R^U_{X/\sim}]\) on \(X/\sim\) for which \(\varphi : (X, R) \to (X/\sim, R_{X/\sim})\) is an \(\text{IVRel}(H)\)-mapping. In this case, \(R_{X/\sim}\) is called the interval-valued H-fuzzy quotient relation of \(X\) by \(R\).

(2) *Sum of interval-valued H-fuzzy relations.* Let \(((X_\alpha, R_\alpha))_\Gamma\) be a family of IVHFRSs, let \(X\) be the sum of \((X_\alpha)_\Gamma\) and let \(j_\alpha : X_\alpha \to X\) be the canonical (injection) mapping for each \(\alpha \in \Gamma\). Then there exists the final IVHFR \(R\) on \(X\). In fact, for each \(((x_\alpha, \alpha), (y_\beta, \beta)) \in X \times X\), \(R^L((x_\alpha, \alpha), (y_\beta, \beta)) = \bigvee_{\Gamma} \mu_{R_\alpha}(x, y)\) and \(R^U((x_\alpha, \alpha), (y_\beta, \beta)) = \bigvee_{\Gamma} R^U(x, y)\). In this case, \(R\) is called the sum of
(Rα)Γ and (X, R) is called the sum of ((Xα, Rα))Γ.

**Theorem 3.7.** Final episinks in IVRel(H) are preserved by pullbacks.

**Proof.** Let (gα : (Xα, Rα) → (Y, RY))Γ be any final episink in IVRel(H) and let f : (W, RW) → (Y, RY) be any IVRel(H)-mapping. For each α ∈ Γ, let Uα = {(w, xα) ∈ W × Xα : f(w) = gα(xα)} and let us define the mapping RUα = [RLUα, RUα : Uα × Uα → D(H)] as follows: For each ((w, xα), (w', xα')) ∈ Uα × Uα,

\[RUα((w, xα), (w', xα')) = RLW(w, w') \land RUα(xα, xα')\]

and

\[RUα((w, xα), (w', xα')) = RUW(w, w') \land RUα(xα, xα')\].

Let eα : Uα → W and pα : Uα → Xα denote the usual projections of Uα. Then clearly (Uα, RUα) ∈ IVRel(H) for each α ∈ Γ. Moreover, eα : (Uα, RUα) → (W, RW) and pα : (Uα, RUα) → (Xα, Rα) are IVRel(H)-mappings for each α ∈ Γ. And the following diagram is a pullback square in IVRel(H):

\[
\begin{array}{ccc}
(Uα, Rα) & \xrightarrow{pα} & (Xα, Rα) \\
\downarrow{eα} & & \downarrow{gα} \\
(W, RW) & \xrightarrow{f} & (Y, RY).
\end{array}
\]

By the process of the proof of Theorem 3.5 in [11], we can see that (eα : (Uα, RUα) → (W, RW))Γ is a final episink in IVRel(H) and RW is the unique final IVHFR on W with respect to (eα)Γ. This completes the proof. □

For any singleton set \{a\}, since the IVHFR R on \{a\} is not unique, the category IVRel(H) is not properly fibred over Set. Hence, by Theorems 3.6 and 3.7, we obtain the following result.

**Theorem 3.8.** IVRel(H) satisfies all the conditions of a topological universe over Set except the terminal separator property.

**Theorem 3.9.** IVRel(H) is Cartesian closed over Set.

**Proof.** It is clear that IVRel(H) has products by Corollary 3.5. We will show that IVRel(H) has exponential objects.

For any IVHFRs X = (X, RX) and Y = (Y, RY), let YX be the set of all mappings from X into Y. We define the mapping R = [RL, RU] : YX × YX →
$D(H)$ as follows: For each $(f, g) \in Y^X \times Y^X,$

$$R_L(f, g) = \bigwedge \{h \in H : R^f_X(x, y) \land h \leq R^g_Y(f(x), g(y))\}$$

for each $(x, y) \in X \times X$ and

$$R_U(f, g) = \bigwedge \{h \in H : R^f_X(x, y) \land h \leq R^g_Y(f(x), g(y))\}$$

for each $(x, y) \in X \times X$.

Then clearly $(Y^X, R) \in \text{IVRel}(H).$ Let $Y^X = (Y^X, R).$ Then, by the definition of $R,$

$$R^f_X(x, y) \land R^g_Y(f(x), g(y)) \leq R^h_Y(f(x), g(y))$$

and

$$R^f_X(x, y) \land R^g_Y(f(x), g(y)) \leq R^h_Y(f(x), g(y))$$

for each $(f, g) \in Y^X$ and $(x, y) \in X \times X.$

Define $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(x, f) = f(x)$ for each $(x, f) \in X \times Y^X.$ Let $((x, f), (y, g)) \in (X \times Y^X) \times (X \times Y^X).$ Then, by the process of the proof of Theorem 2.7 in [11], $e_{X,Y} : X \times Y^X \to Y$ is an $\text{IVRel}(H)$-mapping.

For any $Z = (Z, R_Z) \in \text{IVRel}(H),$ let $h : X \times Z \to Y$ be an $\text{IVRel}(H)$-mapping. We define $\overline{h} : Z \to Y^X$ by $\overline{h}(z)(x) = h(x, z)$ for each $z \in Z$ and each $x \in X.$ Let $z, z' \in Z$ and let $x, x' \in X.$ Then, by the process of the proof of Theorem 2.7 in [11], $\overline{h} : Z \to Y^X$ is an $\text{IVRel}(H)$-mapping. Moreover, $\overline{h}$ is the unique $\text{IVRel}(H)$-mapping such that $e_{X,Y} \circ (1_X \times \overline{h}) = h.$ This completes the proof.

\[\Box\]

**Remark 3.10.** $\text{IVRel}(H)$ has no subobject classifier. Hence $\text{IVRel}(H)$ is not topos.

**Example 3.11.** Let $H = \{0, 1\}$ be the two points chain and let $X = \{a\}.$ Let $R_1$ and $R_2$ be the IVHFRs on $X$ given by $R_1(a, a) = 0$ and $R_2(a, a) = 1.$ Let $1_X : (X, R_1) \to (X, R_2)$ be the identity mapping. Then clearly, $1_X$ is both a monomorphism and an epimorphism in $\text{IVRel}(H).$ But, $1_X$ is not an isomorphism in $\text{IVRel}(H).$ Hence $\text{IVRel}(H)$ has no subobject classifier (See [4]).

### 4. The Relations between Rel$(H)$, IRel$(H)$ and IVRel$(H)$

**Definition 4.1.** (see [11]) The concrete category $\text{Rel}(H)$ is defined by: Objects are $(X, R),$ called $H$-fuzzy relational space on $X,$ where $X$ is any set and $R$ is a mapping from $X \times X$ to $H.$ A morphism $f : (x, R_X) \to (Y, R_Y)$ is
a mapping from $X$ to $Y$ satisfying $R_X(x, y) \leq R_Y(f(x), f(y))$, i.e., $R_X(x, y) \leq (R_Y \circ f^2)(x, y)$ for each $(x, y) \in X \times X$. Every $\text{Rel}(H)$-morphism is called a $\text{Rel}(H)$-mapping.

**Definition 4.2.** (see [13]) The concrete category $\text{IRel}(H)$ is defined by: Objects are $(X, R) = (X, \mu_R, \nu_R)$, called an intuitionistic $H$-fuzzy relational space on $X$, where $X$ is any set and $\mu_R, \nu_R \in H^{X \times X}$ satisfying $\mu_R(x, y) \leq N(\nu_R(x, y))$ for each $(x, y) \in X \times X$. A morphism $f : (X, R_X) \rightarrow (Y, R_Y)$ is a mapping satisfying $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$ and $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$, where $f^2 = f \times f$. Every $\text{IRel}(H)$-morphism is called an $\text{IRel}(H)$-mapping.

**Lemma 4.3.** Define $G_1, G_2 : \text{IRel}(H) \rightarrow \text{Rel}(H)$ by

$G_1(X, R) = (X, R^L)$, $G_2(X, R) = (X, R^U)$ and $G_1(f) = G_2(f) = f$.

Then $G_1$ and $G_2$ are functors.

**Proof.** Clearly $G_1(X, R) = (X, R^L) \in \text{Rel}(H)$ for each $(X, R) \in \text{IVRel}(H)$. Let $(X, R_X), (Y, R_Y) \in \text{IVRel}(H)$ and let $f : (X, R_X) \rightarrow (Y, R_Y)$ be an $\text{IVRel}(H)$-mapping. Then $R_X^L \leq R_Y^L \circ f^2$ and $R_X^U \leq R_Y^U \circ f^2$. Thus $G_1(f) = f : (X, R_X^L) \rightarrow (Y, R_Y)$ and $G_2(f) : (X, R_X^U) \rightarrow (Y, R_Y^U)$ are $\text{Rel}(H)$-mappings. Hence $G_1$ and $G_2$ are functors. □

**Lemma 4.4.** Define $F : \text{Rel}(H) \rightarrow \text{IVRel}(H)$ by $F_1(X, R) = (X, [R, R])$ and $F(f) = f$. Then $F$ is a functor.

**Proof.** It is obvious. □

**Theorem 4.5.** The functor $F : \text{Rel}(H) \rightarrow \text{IVRel}(H)$ is a left adjoint of the functors $G_1$ and $G_2$.

**Proof.** For each $(X, R) \in \text{Rel}(H)$, $1_X : (X, R) \rightarrow G_1F(X, R) = (X, R)$ is a $\text{Rel}(H)$-mapping. Let $(Y, R_Y) \in \text{IVRel}(H)$ and let $f : (X, R) \rightarrow G_1(Y, R_Y)$ be an $\text{IVRel}(H)$-mapping. We will show that $f : F(X, R) = (X, [R, R]) \rightarrow (Y, R_Y)$ is an $\text{IVRel}(H)$-mapping. Since $f : (X, R) = G_1(Y, R_Y) \rightarrow (Y, R_Y^L)$ is a $\text{Rel}(H)$-mapping, $R \leq R_Y^L \circ f^2$. Since $R_X^L \leq R_Y^L$, $R \leq R_Y^U \circ f^2$. So $f : F(X, R) \rightarrow (Y, R_Y)$ is an $\text{IRel}(H)$-mapping. Hence $1_X$ is a $G_1$-universal map for $(X, R)$ in $\text{Rel}(H)$. Similarly, we can see that $1_X$ is a $G_2$-universal map for $(X, R)$ in $\text{Rel}(H)$. This completes the proof. □

For each $(X, R) \in \text{Rel}(H)$, $F(X, R) = (X, [R, R])$ is called an interval-valued $H$-fuzzy relation in $X$ induced by $(X, R)$. Let us denote the category consisting of all induced interval-valued $H$-fuzzy relations and $\text{IVRel}(H)$-mappings as $\text{IVRel}^*(H)$. Then it is clear $\text{IVRel}^*(H)$ is a full subcategory of $\text{IVRel}(H)$.

**Theorem 4.6.** Two categories $\text{Rel}(H)$ and $\text{IVRel}^*(H)$ are isomorphic.
Proof. It is clear that \( F : \text{Rel}(H) \rightarrow \text{IVRel}'(H) \) is a functor by Lemma 4.4. Consider the restriction \( G_1 : \text{IRel}'(H) \rightarrow \text{Rel}(H) \) of the functor \( G_1 \) in Lemma 4.3. Let \((X, R) \in \text{Rel}(H)\). Then, by Lemma 4.4, \( F(X, R) = (X, [R, R])\). Thus \( G_1F(X, R) = G_1(X, [R, R]) = (X, R)\). So \( G_1 \circ F = 1_{\text{Rel}(H)} \). Now let \((X, [R, R]) \in \text{IRel}'(H)\). Then, by Lemma 4.3, \( G_1(X, [R, R]) = (X, R)\). Thus \( FG_1(X, [R, R]) = (X, [R, R])\). So \( F \circ G_1 = 1_{\text{IRel}'(H)}\). Hence \( F : \text{Rel}(H) \rightarrow \text{IRel}'(H) \) is an isomorphism. This completes the proof.

**Lemma 4.7.** We define \( G : \text{IVRel}(H) \rightarrow \text{IRel}(H) \) as follows:

\[
G(X, R) = (X, R^L, N(R^U)) \quad \forall (X, R) \in \text{Ob(IVRel}(H))
\]

and

\[
G(f) = f \quad \forall f \in \text{Mor(IVRel}(H)).
\]

Then \( G \) is a functor.

Proof. Let \((X, R) \in \text{Ob(IVRel}(H))\). Then, by Result 2.C(2),

\[
R^L \leq R^U \leq NN(R^U).
\]

Thus \( G(X, R) = (X, R, N(R^U)) \in \text{IRel}(H) \). Let \( f : (X, R_X) \rightarrow (Y, R_Y) \) be any \( \text{IVRel}(H) \)-morphism. Then

\[
R^L_X \leq f^1 \circ f^2 \quad \text{and} \quad R^U_X \leq R^U_Y \circ f^2.
\]

Let \((x, y) \in X \times X\). Then \( R^U_X(x, y) \leq R^U_Y(f(x), f(y)) \). Thus \( N(R^U_X(x, y)) \geq N(R^U_Y(f(x), f(y))) \). So \( N(f^1) \geq N(f^2) \). Hence \( f = f \circ G(X, R_X) \rightarrow G(Y, R_Y) \) is an \( \text{IRel}(H) \)-morphism. \( G \) is a functor.

**Lemma 4.8.** We define \( K : \text{IRel}(H) \rightarrow \text{IVRel}(H) \) as follows:

\[
K(X, \mu_R, \nu_R) = (X, [\mu_R, N(\nu_R)]) \quad \forall (X, \mu_R, \nu_R) \in \text{Ob(IRel}(H))
\]

and

\[
K(f) = f \quad \forall f \circ \text{Mor(IRel}(H)).
\]

Then \( K \) is a functor.

Proof. Let \((X, \mu_R, \nu_R) \in \text{Ob(IRel}(H))\). Then clearly \( \mu_R \leq N(\nu_R) \). Thus \((X, \mu_R, \nu_R) \in \text{IRel}(H) \). Let \( f : (X, \mu_R, \nu_R) \rightarrow (Y, \mu_R, \nu_R) \) be an \( \text{IRel}(H) \)-morphism. Then

\[
\mu_{RX} \leq \mu_{RY} \circ f^2 \quad \text{and} \quad \nu_{RX} \geq \nu_{RY} \circ f^2.
\]

We can easily see that \( N(\nu_{RX}) \leq N(\nu_{RY}) \circ f^2 \). Thus

\[
K(f) : K(X, \mu_{RX}, \nu_{RX}) \rightarrow (Y, \mu_{RY}, \nu_{RY})
\]

is an \( \text{IVRel}(H) \)-morphism. Hence \( K \) is a functor.

**Theorem 4.9.** Two categories \( \text{IRel}(H) \) and \( \text{IVRel}(H) \) are isomorphic.
Proof. By Lemmas 4.7 and 4.8, we can easily show that
\[ G \circ K = 1_{\text{IRel}(H)} \quad \text{and} \quad K \circ G = 1_{\text{IVRel}(H)}. \]
□

References


