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# ON THE RANKS OF A RATIONAL NORMAL CURVE OF $\mathbb{P}^n$ MINUS OR PLUS FINITELY MANY POINTS

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

**Abstract:** Let $X \subset \mathbb{P}^n$, be a rational normal curve. For each finite $B \subset X$ and $D \subset \mathbb{P}^n \backslash X$ we study the rank function $r_{(X \backslash B) \cup D} : \mathbb{P}^n \to \mathbb{Z}$: for any $P \in \mathbb{P}^n$ the rank $r_{(X \backslash B) \cup D}(P)$ is the minimal cardinality of a set $S \subset (X \backslash B) \cup D$ such that $P \in \langle S \rangle$.

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## 1. Introduction

Let $X \subset \mathbb{P}^n$ be an integral closed subvariety defined over the algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ the rank $r_X(P)$ of $X$ is the minimal integer $k$ such that there are $S \subset X$ with $P \in \langle S \rangle$ and $\sharp(S) = k$ (see [4], [3] and references therein). Here we introduce two related problems:

(a) Fix $B \subsetneq X$ and use only the subsets of $X \backslash B$ to compute the rank. Call $r_{X \backslash B}$ the corresponding function.

(b) Fix $D \subsetneq \mathbb{P}^n \backslash X$ and use only the subsets of $X \cup D$ to compute the rank. Call $r_{X \cup B}$ the corresponding function.

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Of course, we may also take $B \subsetneqq X$, $D \subsetneqq \mathbb{P}^n \backslash X$ and use only the subsets of $(X\backslash B) \cup D$ to compute the rank. Call $r_{(X\backslash B)\cup D}$ the corresponding function.

We study topics (a) and (b) in one of the few cases in which the function $r_X$ is known: the rational normal curve (see [2], [4], Theorem 4.1, [3], Theorem 3.8).

**Theorem 1.** *Let $X \subset \mathbb{P}^n$ be a rational normal curve. Fix a finite $B \subset X$.*

*(a) For any $P \in \mathbb{P}^n$ we have $r_{X\backslash B}(P) \le n+1$.*

*(b) For any $P \in B$ we have $r_{X\backslash B}(P) = n+1$.*

*(c) Assume $P \notin X$. Let $k$ be the first integer such that $P \in S^k(X)$, where $S^k(X) \subseteq \mathbb{P}^n$ denote the closure of the union of all $\langle S\rangle$ with $S \subset X$ and $\sharp(S) = k$. Hence $2 \le k \le \lfloor (n+2)/2 \rfloor$ (see [1], Remark 1.6). Either $r_X(P) = k$ or $r_X(P) = n+2-k$ (see [2], [4], [3]).*

*(c1) Assume $r_X(P) = k$ and $2k < n+2$. There is a unique $S_P \in X$ such that $\sharp(S_P) = k$ and $P \in \langle S_P\rangle$. If $S_P \cap B = \emptyset$, then $r_{X\backslash B}(P) = k$. If $S_P \cap B \ne \emptyset$, then $r_{X\backslash B}(P) = n+2-k$.*

*(c2) Assume $r_X(P) = n+2-k$ and $2k < n+2$. Then $r_{X\backslash B}(P) = n+2-k = r_X(P)$.*

**Theorem 2.** *Let $X \subset \mathbb{P}^n$, $n \ge 3$, be a rational normal curve.*

*(a) Assume $n = 3$. Then for any finite $D \subset \mathbb{P}^3$, the set of all $P \in \mathbb{P}^3$ such that $r_{X\cup D}(P) = 3$ is non-empty and two-dimensional.*

*(b) Assume $n = 4$. For any $Q \in \mathbb{P}^4$ there are infinitely many $P \in \mathbb{P}^4$ such that $r_{X\cup\{Q\}}(P)) = 4$. For every integer $s \ge 3$ and any general $D \subset \mathbb{P}^4$ such that $\sharp(D) = s$ we have $r_{X\cup D}(P) \le 3$ for all $P \in \mathbb{P}^4$.*

*(c) Assume $n \ge 6$ and fix any $Q \in \mathbb{P}^n$ such that $r_X(Q) = \lfloor (n+2)/2 \rfloor$ (e.g. take a general $Q \in \mathbb{P}^n$). Then $r_{X\cup\{Q\}}(P) \le n-1$ for all $P \in \mathbb{P}^n$.*

## 2. The Proofs

**Remark 1.** Let $X \subset \mathbb{P}^n$ be a rational normal curve and $Z \subset X$ be any zero-dimensional subscheme. The cohomology of line bundles on $\mathbb{P}^1$ gives $\dim(\langle Z\rangle) = \min\{n, \mathrm{length}(Z) - 1\}$.

*Proof of Theorem 1.* For any $P$ we have $r_{X\backslash B} \ge r_X(P)$. Since there is $S \subset X\backslash B$, such that $\sharp(S) = n+1$ and $\langle S\rangle = \mathbb{P}^n$, we get part (a) and the inequality $r_{X\backslash B}(P) \le n+1$ in part (b). The reverse inequality $r_{X\backslash B}(P) \ge n+1$

in part (b) is true by Remark 1. If $P \in X\backslash B$, then obviously $r_{X\backslash B}(P) = 1$. Fix any $P \in \mathbb{P}^n\backslash X$ and let $k$ be the minimal integer such that $P \in \mathrm{Sec}^k(X)$. It is well-known that $k$ is the first integer such that there is a zero-dimensional scheme $Z \subset X$ such length$(Z) = k$ and $P \in \langle Z\rangle$ (see, e.g., the proofs of [3], theorem 3.8 or 3.11). If $Z$ is reduced, then the minimality of $k$ gives $r_X(P) = k$. Obviously we get $r_{X\backslash B}(P) = t$ if $Z$ is reduced and $Z \cap B = \emptyset$. If $Z$ is not reduced and $2k \le n+1$, then $r_X(P) = n+2-k$ (see [2], [4], Theorem 4.1, [3], Theorem 3.8). Now assume $2k < n+2$. In this case the scheme $Z$ is uniquely determined by $P$ (use Remark 1). Hence we proved part (c1). Now assume that either $Z$ is not reduced or $Z \cap B \neq \emptyset$. Hence we cannot use $Z$ to compute $r_{X\backslash B}(P)$. The proof of [3], Theorem 3.8 (or use Remark 1 and linear algebra) gives $r_{X\backslash B}(P) \ge n+2-k$. Hence to prove part (c2) it is sufficient to prove the reverse inequality.

(i) Here we assume $Z$ unreduced. Hence there is $O \in Z_{red}$ such that $2O \subseteq Z$ (we allow the case $O \in B$). In this case the proof is just a photocopy of the proof of [3], Theorem 3.8. First assume $k = 2$. In this case the result is true (in characteristic 0), because the set $\Gamma$ of all hyperplanes of $\mathbb{P}^n$ containing $P$ has no base points. Hence for a general such hyperplane $H$ the scheme $H \cap X$ is formed by $n$ distinct points. Since $\Gamma$ has no base points, $B \cap H = \emptyset$. Hence the set $H \cap X$ ($H$ general in $\Gamma$) gives $r_{X\backslash B}(P) \le n$. Now assume $k > 2$, i.e. $2O \neq Z$. Hence $Z'' := Z - 2O$ is a degree $k-2 > 0$ effective divisor on $X$. Since $P \notin \langle Z''\rangle$ and $\dim(\langle Z''\rangle) = k-3$, we have $\dim(\langle Z'' \cup \{P\}\rangle) = k-2$. Let $\Psi$ be the linear series on $X \cong \mathbb{P}^1$ cut out by the hyperplanes containing $\langle Z'' \cup \{P\}\rangle$. Since $Z$ is the minimal subscheme of $X$ such that $P \in \langle Z\rangle$ and $\dim(\langle Z\rangle) > \dim(\langle Z'' \cup \{P\}\rangle)$ , we have $\langle Z'' \cup \{P\}\rangle \cap Z = Z''$. Hence $\Psi$ has no base points. Hence (characteristic zero) a general divisor $E \in \Psi$ is reduced and $E \cap B = \emptyset$. The divisor $E$ gives $r_{X\backslash B}(P) \le n+2-k$, concluding the proof in this case.

(ii) Here we consider the only remain the case, i.e. here $Z$ is reduced and $Z \cap B \neq \emptyset$. Let $\Phi$ be the complete degree $n-k+2$ linear system on $X \cong \mathbb{P}^1$. Let $Z_1 \subset Z$ be any subset such that $\sharp(Z_1) = k-2$. Remark 1 implies that $\mathcal{I}_{Z_1}(1)$ is spanned and that $\Phi$ is induced the set of all hyperplanes containing $Z_1$. For any $A \in \Phi$ we have $\langle A| \cap \langle Z\rangle \neq \emptyset$. Hence the linear subseries $\Phi_P$ of the effective divisors $A$ such that $P \in |A|$ is a linear subseries of dimension at least $n+2-2k > 0$.

**Claim.** *$\Phi_P$ has no base point.*

*Proof of the Claim.* Since $\langle Z_1 \cup \{P\}\rangle$ has dimension $k-1$ by the minimality of $k$, either $\langle Z_1 \cup \{P\}\rangle \cap X = Z_1$ (scheme-theoretically) or $\langle Z_1 \cup \{P\}\rangle \cap X$ is a degree $k-1$ divisor (Remark 1). The latter case contradicts the minimality of $k$. Hence $\langle Z_1 \cup \{P\}\rangle \cap X = Z_1$. Thus $\Phi_P$ has no base points, proving the claim.

The claim implies (in characteristic zero) that a general $E \in \Phi_P$ is reduced and contain no point of $B$. Since $\sharp(E) = n+2-k$, we get $r_{X\setminus B}(P) \le n+2-k$, concluding the proof of part (c2). □

*Proof of Theorem 2.* (i) First assume $n = 3$ and fix a finite $D \subset \mathbb{P}^3$. We may assume $D \cap X = \emptyset$. The set of all $P \in \mathbb{P}^3$ such that $r_{X\cup D}(P) \le 2$ is the union of the following 4 sets $S_0$, $S_1$, $S_2$ and $S_3$. $S_0 := X \cup D$. $S_1$ is the set of all $P \in \mathbb{P}^3$ such that $r_X(P) \le 2$, i.e. (see [2], [4], Theorem 4.1, [3], Theorem 3.8) $S_1 = \mathbb{P}^3\setminus TX$. $S_2$ is the union of all cones with vertex any $Q \in D$ and $X$ as a basis. If $\sharp(D) = 1$, then $S_3 = \emptyset$. If $\sharp(D) \ge 2$, then $S_3$ is the union of all lines spanned by two points of $D$. Obviously $TX\setminus TX \cap (S_0 \cup S_1 \cup S_2 \cup S_3)$ has dimension 2.

(ii) Assume $n = 4$. For any $P \in \mathbb{P}^4$ we have $r_{X\cup D}(P) \le r_X(P) \le 4$ and $r_X(P) = 4$ if and only if $P \in TX\setminus X$. Hence it is sufficient to compute the ranks with respect to $X \cup D$ of the points of $TX\setminus X$. We may assume $D \cap X = \emptyset$. For any $O \in \mathbb{P}^4$, let $\ell_O : \mathbb{P}^4\setminus\{O\} \to \mathbb{P}^3$ be the linear projection from $O$. Let $X_O$ denote the closure of $\ell_O(X\setminus\{O\})$ in $\mathbb{P}^3$.

(ii.1) Here we assume $\sharp(D) = 1$, say $D = \{Q\}$. Fix a hyperplane $H \subset \mathbb{P}^4$ such that $Q \notin H$ and see $H$ as the target of $\ell_Q$. Since $Q \notin X$, $X_Q$ is either a cuspidal degree 4 curve with arithmetic genus 1 (case $Q \in TX\setminus X$) or a nodal degree 4 curve with arithmetic genus 1 (case $r_X(Q) = 2$) or a smooth and rational curve of degree 4 (case $Q$ not in the secant variety of $X$). In the latter case the curve $X_Q$ has a cuspidal projection in $\mathbb{P}^2$ in the sense of [5] (see [5], example 1 of page 109), i.e. there is $U \in \mathbb{P}^3$ such that $r_{X_Q}(U) = 3$. Obviously every point $P$ of the line $\langle\{Q, U\}\rangle$ has $r_{X\cup\{Q\}}(P) = 4$.

(ii.1.1) Here we assume that $X_Q$ is singular and prove the existence of a one-dimensional family of points $U \in H$, such that $r_{X_Q}(U) = 3$ (each of them lying on $TX_Q$). Hence even in this case we will find infinitely many points $P \in \mathbb{P}^4$ such that $r_{X\cup\{Q\}}(P) = 4$. Let $\nu : \mathbb{P}^1 \to X_Q$ denote the normalization map. First assume the existence of $O \in (X_Q)_{reg}$ such that $P \in T_OX_Q$. Since $X_Q$ is the complete intersection of 2 quadrics, the divisor $2O$ is the scheme-theoretic intersection of $X_Q$ and $T_OX_Q$, i.e. the rational map $\psi$ from $X$ onto $\mathbb{P}^1$ induced by the linear projection from the line $T_OX_Q$ is induced by a sub-series of the complete linear system associated to the line bundle $\mathcal{O}_{X_Q}(1)(-2O)$. Since $h^0(X, \mathcal{O}_X(1)(-2O)) = 2$, $\psi$ is induced by the complete linear system $|\mathcal{O}_{X_Q}(1)(-2O)|$. Since $\mathcal{O}_X(1)(-2O)$ is spanned (even at $Q$) by Riemann-Roch,

$\psi$ is a degree 2 morphism $X \to \mathbb{P}^1$. By the Riemann-Hurwitz formula the degree 2 morphism $\psi \circ \nu : \mathbb{P}^1 \to \mathbb{P}^1$ has 2 ramification points (since $\mathrm{char}(\mathbb{K}) \neq 2$). Hence all points of $T_OX_Q$, except at most 2 of them, have $X_Q$-rank 2. If $X$ is a cuspidal curve, then at least one of the ramification points of $\psi \circ \nu$ is over a smooth point of $X$. Hence if $X$ is a cuspidal curve, then each $T_OX_Q$ contains one or two points with $X_Q$-rank 3. Hence $\mathbb{P}^3$ has a one-dimensional family of points with $X_Q$-rank 3. Now assume that $X$ is an ordinary node. $T_OX_Q$ contains at least one point with rank 3 if and only if $X$ is not obtained from $\mathbb{P}^1$ gluing together the two ramification points of the pencil $\psi \circ \nu : \mathbb{P}^1 \to \mathbb{P}^1$. Any degree 2 morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is induced by a hyperplane of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$, i.e. by the linear projection of a smooth conic $C$ from a point of $\mathbb{P}^2 \backslash C$. Since $\mathrm{char}(\mathbb{K}) \neq 2$, the projection point is the intersection of the two tangent lines of $C$ at the two ramification points. Thus this pencil is uniquely determined by the two ramification points. Since $\mathrm{Pic}^1(X) \cong (X_Q)_{reg}$, we get the existence of a unique $O \in (X_Q)_{reg}$ such that $T_OX_Q$ contains no rank 3 point. Now fix any $E \in H$ such that $r_{X_Q}(E) \geq 3$, but there is no $O \in X_{reg}$ such that $E \in T_OX_Q$. Since $S^2(X_Q) = \mathbb{P}^3$ and the Grassmannian of lines in $\mathbb{P}^3$ is complete, there are a line $V$ and a length 2 subscheme $Z \subset X_Q$ such that $E \in V$, $V = \langle Z \rangle$ and $V \cap X_Q = Z$. Since $r_{X_Q}(E) > 2$, $Z$ is not reduced. Since $P \notin T_OX_Q$ for some $O \in X_{reg}$, $Z_{red} = \{Q\}$. Let $M$ be a general plane containing $V$. Since $V \cap X$ has length 2, $(M \backslash V) \cap X$ has length 2, i.e. the linear projection from $V$ induces a degree 2 rational map from $X_Q$ onto $\mathbb{P}^1$. Since $\mathrm{char}(\mathbb{K}) \neq 2$, this rational map is separable. Thus $(M \backslash V) \cap X_Q$ is formed by two distinct points. Hence a general element of $V$ has rank 2. In summary, the set of all points with rank 3 is non-empty, one-dimensional and contained in $TX_Q$.

(ii.2) Now assume $\sharp(D) \geq 3$. For any $Q \in \mathbb{P}^4$ let $\mathbb{B}_Q$ denote the set of all $P \in \mathbb{P}^4$ such that $r_{X \cup \{Q\}}(P) \geq 4$. To conclude the proof of the case $n = 4$ it is sufficient to notice that $\mathbb{B}_{Q_1} \cap \mathbb{B}_{Q_2} \cap \mathbb{B}_{Q_3} = \emptyset$ for a general $(Q_1, Q_2, Q_3) \in \mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^4$.

(iii) Assume $n \geq 6$ and fix $Q$ as in part (c). Since $r_{X \cup \{Q\}}(P) \leq r_X(P)$ for all $P \in \mathbb{P}^n$, it is sufficient to prove that $r_{X \cup \{Q\}}(P) \leq n - 1$ for all $P \in TX \backslash X$ (see [2], [4], Theorem 4.1, [3], Theorem 3.8). Fix $P \in TX \backslash X$ and call $O \in X$ the unique point of $X$ such that $P \in T_OX$ (the uniqueness of $O$ follows from Remark 1). We will prove the inequality $r_{X \cup \{Q\}} \leq 2 + \lfloor (n+2)/2 \rfloor$. Since $n \geq 6$, this would be sufficient to prove part (c). Since $r_X(Q) = \lfloor (n+2)/2 \rfloor$, $Q \notin T_OX$. Hence $\langle \{Q\} \cup T_AX \rangle$ is a plane. Since the set of all $Q' \in \mathbb{P}^n$ such that $r_X(Q') = \lfloor (n+2)/2 \rfloor$ is open, there is $Q' \in \langle \{Q\} \cup T_AX \rangle \backslash \langle \{O, Q\} \rangle$ such that $r_X(Q') = \lfloor (n+2)/2 \rfloor$. Fix $S \subset X$ such that $\sharp(S) = \lfloor (n+2)/2 \rfloor$. Our choice of $Q'$ gives $P \in \langle \{Q, O\} \cup S \rangle$. □

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