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EIGENVALUE INTERVALS FOR $3n$-TH ORDER THREE-POINT BOUNDARY VALUE PROBLEMS

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Abstract: We study the existence of eigenvalue intervals of λ, for which there exist a positive solution with respect to a cone, of the $3n$-th order three-point boundary value problem

$$(-1)^n y^{(3n)} = \lambda f(t, y(t)), \quad t \in [t_1, t_3],$$

subject to general three-point boundary conditions

$$\alpha_{3i-2,1} y^{(3i-3)}(t_1) + \alpha_{3i-2,2} y^{(3i-2)}(t_1) + \alpha_{3i-2,3} y^{(3i-1)}(t_1) = 0,$$
$$\alpha_{3i-1,1} y^{(3i-3)}(t_2) + \alpha_{3i-1,2} y^{(3i-2)}(t_2) + \alpha_{3i-1,3} y^{(3i-1)}(t_2) = 0,$$
$$\alpha_{3i,1} y^{(3i-3)}(t_3) + \alpha_{3i,2} y^{(3i-2)}(t_3) + \alpha_{3i,3} y^{(3i-1)}(t_3) = 0,$$

for $1 \leq i \leq n$, where $n \geq 1$, $t_1 < t_2 < t_3$ and $f : [t_1, t_3] \times \mathbb{R} \to \mathbb{R}$ is continuous, λ is a parameter.

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1. Introduction

The study of the existence of positive solutions of boundary value problem on optimal intervals for higher order differential equations has gained prominence and it became a rapidly growing field due to its importance of applications in different fields. The existence of positive solutions for higher order boundary value problems have been studied by many authors, Eloe and Henderson [6], Anderson [2], Davis, Henderson, Prasad and Yin [4], Anderson and Davis [3], Li [16], Guo, Sun and Zhao [13], Shahed [20].

Recently, Prasad, Kameswararao and Murali [18] studied the eigenvalue intervals for which there exist positive solutions of the even order nonlinear differential equation

$$(-1)^n y^{(2n)} = \lambda f(t, y(t), y''(t), ..., y^{(2(n-1))}), \quad t \in [a, c],$$

subject to the three-point boundary conditions

$$\alpha_{i+1} y^{(2i)}(a) + \beta_{i+1} y^{(2i+1)}(a) = y^{(2i)}(a),$$
$$\gamma_{i+1} y^{(2i)}(b) = y^{(2i)}(c),$$

for $1 \le i \le n$.

In this paper, we consider the $3n$-th order nonlinear differential equation

$$(-1)^n y^{(3n)} = \lambda f(t, y(t)), \quad t \in [t_1, t_3], \tag{1.1}$$

satisfying the general three-point boundary conditions

$$\begin{aligned} \alpha_{3i-2,1} y^{(3i-3)}(t_1) + \alpha_{3i-2,2} y^{(3i-2)}(t_1) + \alpha_{3i-2,3} y^{(3i-1)}(t_1) = 0, \\ \alpha_{3i-1,1} y^{(3i-3)}(t_2) + \alpha_{3i-1,2} y^{(3i-2)}(t_2) + \alpha_{3i-1,3} y^{(3i-1)}(t_2) = 0, \\ \alpha_{3i,1} y^{(3i-3)}(t_3) + \alpha_{3i,2} y^{(3i-2}(t_3) + \alpha_{3i,3} y^{(3i-1)}(t_3) = 0, \end{aligned} \tag{1.2}$$

for $n \ge 1$, and $1 \le i \le n$, $t_1 < t_2 < t_3$. We determine the eigenvalue intervals for the boundary value problem (1.1)-(1.2) to posses at least one positive solution.

For convenience we adopt the following notation:

$$\beta_j = \alpha_{3i-3+j,1} t_j + \alpha_{3i-3+j,2}, \quad \gamma_j = \alpha_{3i-3+j,1} t_j^2 + 2\alpha_{3i-3+j,2} t_j + 2\alpha_{3i-3+j,3},$$
$$l_j = \alpha_{3i-3+j,1} s^2 - 2\beta_j s + \gamma_j,$$

and define

$$m_{kj} = \frac{\alpha_{3i-3+k,1}\gamma_j - \alpha_{3i-3+j,1}\gamma_k}{2(\alpha_{3i-3+k,1}\beta_j - \alpha_{3i-3+j,1}\beta_k)},$$
$$M_{kj} = \frac{\beta_{3i-3+k,1}\gamma_j - \beta_j\gamma_k}{(\alpha_{3i-3+k,1}\beta_j - \alpha_{3i-3+j,1}\beta_k)}$$

for $k, j = 1, 2, 3$ and also let $m = \max\{m_{12}, m_{13}, m_{23}\}$

$$M = \min\left\{m_{23} + \sqrt{m_{23}^2 - M_{23}}, \quad m_{13} + \sqrt{m_{13}^2 - M_{13}}\right\}$$

and

$$d_i = [\alpha_{3i-2,1}(\beta_2\gamma_3 - \beta_3\gamma_2) - \beta_1(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) + \gamma_1(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)].$$

We assume the following conditions throughout this paper:

(A1) $f : [t_1, t_3] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(A2) $\alpha_{3i-2,1} > 0$, $\alpha_{3i-1,1} > 0$ and $\alpha_{3i,1} > 0$ for $1 \le i \le n$ are real constants, such that $\frac{\alpha_{3i-2,2}}{\alpha_{3i-2,1}} < \frac{\alpha_{3i-1,2}}{\alpha_{3i-1,1}} < \frac{\alpha_{3i,2}}{\alpha_{3i,1}}$.

(A3) $m \le t_1 \le t_2 \le t_3 \le M$, $2\alpha_{3i-1,3}\alpha_{3i-1,1} > \alpha_{3i-1,2}^2$, $2\alpha_{3i-2,3}\alpha_{3i-2,1} < \alpha_{3i-2,2}^2$, $2\alpha_{3i,3}\alpha_{3i,3} > \alpha_{3i,2}^2$.

(A4) $m_{23}^2 > M_{23}$, $m_{12}^2 < M_{12}$, $m_{13}^2 > M_{13}$ and $d_i > 0$.

Define the nonnegative extended real numbers f_0, f^0, f_∞, f^∞ by

$$f_0 = \lim_{y\to 0^+} \min_{t\in[t_1,t_3]} \frac{f(t,y)}{y},$$

$$f^0 = \lim_{y\to 0^+} \max_{t\in[t_1,t_3]} \frac{f(t,y)}{y},$$

$$f_\infty = \lim_{y\to \infty} \min_{t\in[t_1,t_3]} \frac{f(t,y)}{y},$$

$$f^\infty = \lim_{y\to \infty} \max_{t\in[t_1,t_3]} \frac{f(t,y)}{y},$$

and assume that they will exist.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous boundary value problem corresponding to (1.1)-(1.2) and estimate the bounds for the Green's function. In Section 3, we present a lemma which is needed in the main result and establish a criteria to determine the eigenvalue intervals for which the boundary value problem (1.1)-(1.2) has at least one positive solution by using the Krasnosel'skii fixed point theorem. Later our results are illustrated with an example.

2. The Green's Function and Bounds

In this section, we estimate the bounds of the Greens function for the homogeneous three point boundary value problem corresponding to (1.1)-(1.2) and we prove certain lemmas which are needed to prove our main results.

Let $G_i(t,s)$ be the Green's function for the homogeneous problem

$$-y''' = 0, \quad t \in [t_1, t_3], \tag{2.1}$$

satisfying the general three point boundary conditions

$$\sum_{k=1}^{3} \alpha_{3i-3+j,k} y^{(k-1)}(t_j) = 0 \tag{2.2}$$

for $j = 1, 2, 3$ and $1 \le i \le n$. First we need few results on the related third order homogeneous boundary value problem (2.1)-(2.2).

Lemma 2.1. *The homogeneous boundary value problem* (2.1)-(2.2) *has only the trivial solution if and only if* $d_i = [\alpha_{3i-2,1}(\beta_2\gamma_3 - \beta_3\gamma_2) - \beta_1(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) + \gamma_1(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)] \neq 0$ *for* $1 \le i \le n$.

Proof. On application of boundary conditions (2.2) to the general solution of (2.1), it can be established. □

Lemma 2.2. *For* $1 \le i \le n$, *the Green's function* $G_i(t,s)$ *for the homogeneous boundary value problem* (2.1)-(2.2) *is given by*

$$G_i(t,s) = \begin{cases} G_{i_1}(t,s), & t_1 < s < t \le t_2 < t_3, \\ G_{i_2}(t,s), & t_1 \le t < s < t_2 < t_3, \\ G_{i_3}(t,s), & t_1 \le t < t_2 < s < t_3, \\ G_{i_4}(t,s), & t_1 < t_2 < s < t \le t_3, \\ G_{i_5}(t,s), & t_1 < t_2 \le t < s < t_3, \\ G_{i_6}(t,s), & t_1 \le s < t_2 < t < t_3, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} G_{i_1}(t,s) =& \frac{1}{2d_i}[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) \\ &- t^2(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)] \times l_1, \\ G_{i_2}(t,s) =& \frac{1}{2d_i}[-(\beta_1\gamma_3 - \beta_3\gamma_1) + t(\alpha_{3i-2,1}\gamma_3 - \alpha_{3i,1}\gamma_1) \\ &- t^2(\alpha_{3i-2,1}\beta_3 - \alpha_{3i,1}\beta_1)] \times l_2 \\ &+ \frac{1}{2d_i}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{3i-2,1}\gamma_2 - \alpha_{3i-1,1}\gamma_1) \\ &+ t^2(\alpha_{3i-2,1}\beta_2 - \alpha_{3i-1,1}\beta_1] \times l_3, \\ G_{i_3}(t,s) =& \frac{1}{2d_i}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{3i-2,1}\gamma_2 - \alpha_{3i-1,1}\gamma_1) \\ &+ t^2(\alpha_{3i-2,1}\beta_2 - \alpha_{3i-1,1}\beta_1)] \times l_3, \end{aligned}$$

$$\begin{aligned} G_{i_4}(t,s) =& \frac{1}{2d_i}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{3i-1,2,1}\gamma_3-\alpha_{3i,1}\gamma_2) \\ &-t^2(\alpha_{3i-1,1}\beta_3-\alpha_{3i,1}\beta_2)]\times l_1 \\ &+\frac{1}{2d_i}[(\beta_1\gamma_3-\beta_3\gamma_1)-t(\alpha_{3i-2,1}\gamma_3-\alpha_{3i,1}\gamma_1) \\ &+t^2(\alpha_{3i-2,1}\beta_3-\alpha_{3i,1}\beta_1]\times l_2, \\ G_{i_5}(t,s) =& \frac{1}{2d_i}[(\beta_1\gamma_2-\beta_2\gamma_1)-t(\alpha_{3i-2,1}\gamma_2-\alpha_{3i-1,1}\gamma_1) \\ &+t^2(\alpha_{3i-2,1}\beta_2-\alpha_{3i-1,1}\beta_1)]\times l_3, \\ G_{i_6}(t,s) =& \frac{1}{2d_i}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{3i-1,1}\gamma_3-\alpha_{3i,1}\gamma_2) \\ &-t^2(\alpha_{3i-1,1}\beta_3-\alpha_{3i,1}\beta_2)]\times l_1. \end{aligned}$$

Proof. $G_i(t,s)$ is constructed by using standard methods [19]. □

Lemma 2.3. *Assume the conditions* $(A1)$-$(A4)$ *are satisfied. Then, for* $1 \le i \le n$*, the Green's function* $G_i(t,s)$ *of the boundary value problem* (2.1)-(2.2) *satisfies* $G_i(t,s) > 0$*, for* $(t,s) \in [t_1,t_3]\times[t_1,t_3]$.

Proof. For $(t,s) \in [t_1,t_3]\times[t_1,t_3]$, $G_i(t,s)$ stated as in (2.3), if we consider sequentially, from $(A2)$-$(A4)$,

$$G_i(t,s) > 0, \quad \text{for} \quad (t,s) \in [t_1,t_3]\times[t_1,t_3]. \tag{2.4}$$

□

Lemma 2.4. *Assume the conditions* $(A1)$-$(A4)$ *are satisfied. Then, for* $1 \le i \le n$*, the Green's function* $G_i(t,s)$ *given by* (2.3) *satisfies that*

$$G_i(t,s) \le \max\left\{G_i(t_1,s), G_i(s,s), G_i(t_3,s)\right\}.$$

Proof. This can be proved by proceeding sequentially with the branches of $G_i(t,s)$ in (2.3).

Case 1. For $t_1 < s < t < t_2 < t_3$

$$\begin{aligned} G_i(t,s) = G_{i_1}(t,s) = &\frac{1}{2d_i}[-(\beta_2\gamma_3-\beta_3\gamma_2)+t(\alpha_{3i-1,1}\gamma_3-\alpha_{3i,1}\gamma_2) \\ &-t^2(\alpha_{3i-1,1}\beta_3-\alpha_{3i,1}\beta_2]\times l_1 \end{aligned}$$

which is decreasing in t from $(A2)$-$(A4)$. Therefore $G_{i_1}(t,s) \le G_{i_1}(s,s) \le G_{i_1}(t_1,s)$. Hence $G_i(t,s) \le G_i(t_1,s)$.

Case 2. For $t_1 \leq t < t_2 < s < t_3$

$$G_i(t,s) = G_{i_3}(t,s) = \frac{1}{2d_i}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{3i-2,1}\gamma_2 - \alpha_{3i-1,1}\gamma_1) + t^2(\alpha_{3i-2,1}\beta_2 - \alpha_{3i-1,1}\beta_1)] \times l_3$$

which is increasing in t from $(A2)$-$(A4)$. Therefore $G_{i_3}(t,s) \leq G_{i_3}(s,s) \leq G_{i_3}(t_3,s)$. Hence $G_i(t,s) \leq G_i(t_3,s)$.

Case 3. For $t_1 \leq t < s < t_2 < t_3$

$$\begin{aligned} G_i(t,s) &= G_{i_2}(t,s) \\ &= \frac{1}{2d_i}[-(\beta_1\gamma_3 - \beta_3\gamma_1) + t(\alpha_{3i-2,1}\gamma_3 - \alpha_{3i,1}\gamma_1) - t^2(\alpha_{3i-2,1}\beta_3 - \alpha_{3i,1}\beta_1)] \times l_2 \\ &+ \frac{1}{2d_i}[(\beta_1\gamma_2 - \beta_2\gamma_1) - t(\alpha_{3i-2,1}\gamma_2 - \alpha_{3i-1,1}\gamma_1) + t^2(\alpha_{3i-2,1}\beta_2 - \alpha_{3i-1,1}\beta_1] \times l_3 \end{aligned}$$

which is increasing in t by $(A2)$-$(A4)$ and Case 2. Therefore $G_{i_2}(t,s) \leq G_{i_2}(s,s)$. Hence $G_i(t,s) \leq G_i(s,s)$.

Case 4. For $t_1 < t < t_2 < s < t < t_3$.

$$\begin{aligned} G_i(t,s) &= G_{i_4}(t,s) \\ &= \frac{1}{2d_i}[-(\beta_2\gamma_3 - \beta_3\gamma_2) + t(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) - t^2(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)] \times l_1 \\ &+ \frac{1}{2d_i}[(\beta_1\gamma_3 - \beta_3\gamma_1) - t(\alpha_{3i-2,1}\gamma_3 - \alpha_{3i,1}\gamma_1) + t^2(\alpha_{3i-2,1}\beta_3 - \alpha_{3i,1}\beta_1)] \times l_2 \end{aligned}$$

which is decreasing in t from Case 1 and Case 2. Therefore $G_{i_4}(t,s) \leq G_{i_4}(s,s)$. Hence $G_i(t,s) \leq G_i(s,s)$.

Similarly we can prove when the Green's function $G_i(t,s) = G_{i_5}(t,s)$ and $G_i(t,s) = G_{i_6}(t,s)$ as in Case 2 and Case 1 respectively, where $G_{i_5}(t,s)$, $G_{i_6}(t,s)$ are given as in (2.3). From all above cases

$$G_i(t,s) \leq \max\{G_i(t_1,s), G_i(s,s), G_i(t_3,s)\}. \qquad \square$$

Lemma 2.5. *Assume that the conditions $(A1)$-$(A4)$ hold. For $1 \leq i \leq n$, and fixed $s \in [t_1,t_3]$, the Green's function $G_i(t,s)$ in (2.3) satisfies*

$$\min_{t \in [t_2,t_3]} G_i(t,s) \geq m_i \parallel G_i(.,s) \parallel,$$

where

$$m_i = \min\left\{\frac{G_{i_1}(t_3, s)}{G_{i_1}(t_1, s)}, \frac{G_{i_4}(t_3, s)}{G_{i_4}(t_1, s)}, \frac{G_{i_5}(t_1, s)}{G_{i_5}(t_3, s)}\right\}$$

and $\|.\|$ *is defined by* $\|x\| = \max\{x(t) : t \in [t_1, t_3]\}$.

Proof. For $s \in [t_1, t_2]$, $G_i(t, s) = G_{i_1}(t, s)$ which is decreasing in t by $(A2)$-$(A4)$. Therefore

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i_1}(t, s)}{G_{i_1}(s, s)} \geq \frac{G_{i_1}(t_3, s)}{G_{i_1}(t_1, s)}.$$

For $s \in [t_2, t_3]$ and $t_1 < t_2 \leq t < s < t_3$. $G_i(t, s) = G_{i_5}(t, s)$ which is increasing in t on $[t_1, t_3]$ by $(A2)$-$(A4)$. Therefore

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i_5}(t, s)}{G_{i_5}(s, s)} \leq \frac{G_{i_5}(t_1, s)}{G_{i_5}(t_3, s)}.$$

For $s \in [t_2, t_3]$ and $t_1 < t_2 < s < t < t_3 . G_i(t, s) = G_{i_4}(t, s)$ which is decreasing in t on $[t_1, t_3]$ by $(A2)$-$(A4)$. Therefore

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i_4}(t, s)}{G_{i_4}(s, s)} \geq \frac{G_{i_4}(t, s)}{G_{i_4}(t_1, s)} \geq \frac{G_{i_4}(t_3, s)}{G_{i_4}(t_1, s)}.$$

Therefore from Lemma 2.4 and by all the above cases we have

$$\min_{t \in [t_2, t_3]} G_i(t, s) \geq m_i \parallel G(., s) \parallel,$$

where

$$m_i = \min\left\{\frac{G_{i_1}(t_3, s)}{G_{i_1}(t_1, s)}, \frac{G_{i_4}(t_3, s)}{G_{i_4}(t_1, s)}, \frac{G_{i_5}(t_1, s)}{G_{i_5}(t_3, s)}\right\}. \qquad \square$$

Lemma 2.6. *Assume the conditions* $(A1)$-$(A4)$ *are satisfied and* $G_i(t, s)$ *as in* (2.3). *Let us define* $H_1(t, s) = G_1(t, s)$ *and recursively define*

$$H_j(t, s) = \int_{t_1}^{t_3} H_{j-1}(t, r) G_j(r, s) dr$$

for $2 \leq j \leq n$, *then* $H_n(t, s)$ *is the Green's function for the homogeneous problem corresponding to* (1.1)-(1.2).

Lemma 2.7. *Assume the conditions* $(A1)$-$(A4)$ *holds. If we define*

$$K = \prod_{j=1}^{n} K_j, \quad L = \prod_{j=1}^{n} m_j L_j,$$

then the Green's function $H_n(t, s)$ *in Lemma 2.6 satisfies*

$$0 \leq H_n(t, s) \leq K \parallel G_n(., s) \parallel, \quad (t, s) \in [t_1, t_3] \times [t_1, t_3], \tag{2.5}$$

and

$$H_n(t,s) \geq m_n L \parallel G_n(.,s) \parallel, \quad (t,s) \in [t_2,t_3] \times [t_1,t_3] \tag{2.6}$$

where m_n *is given as in Lemma 2.5,*

$$K_j = \int_{t_1}^{t_3} \parallel G_j(.,s) \parallel ds > 0, \quad \textit{for } 1 \leq j \leq n,$$

and

$$L_j = \int_{t_2}^{t_3} \parallel G_j(.,s) \parallel ds > 0, \quad \textit{for } l \leq j \leq n.$$

Proof. By using Lemma 2.5 and induction on n, we can easily establish the proof. □

3. Existence of Positive Solutions

In this section, we present the lemma which is needed in the main result and establish a criteria to determine the eigenvalue intervals for which the boundary value problem (1.1)-(1.2) has at least one positive solution.

Let $y(t)$ be the solution of the boundary value problem (1.1)-(1.2) and is given by

$$y(t) = \lambda \int_{t_1}^{t_3} H_n(t,s) f(s,(y(s))ds, \quad \text{for all } t \in [t_1,t_3]. \tag{3.1}$$

Define

$$X = \{u : u \in C[t_1,t_3]\}$$

with norm

$$\parallel u \parallel = \max_{t \in [t_1,t_3]} |u(t)|.$$

Then $(X, \parallel . \parallel)$ is a Banach space. Define a set κ by

$$\kappa = \{u \in X : u(t) \geq 0 \text{ on } [t_1,t_3] \text{ and } \min_{t \in [t_2,t_3]} u(t) \geq \frac{m_n L}{K} \parallel u \parallel\}. \tag{3.2}$$

Then κ is a cone in X.

Define the operator $T : \kappa \to X$ by

$$(Ty)(t) = \lambda \int_{t_1}^{t_3} H_n(t,s) f(s,(y(s))ds, \quad \text{for all } t \in [t_1,t_3]. \tag{3.3}$$

If $y \in \kappa$ is a fixed point of T, then y satisfies (3.1) and hence y is a positive solution of the boundary value problem (1.1)-(1.2). We seek a fixed point of the operator T in the cone κ.

Lemma 3.1. *The operator T defined in* (3.3) *is a self map on κ.*

Proof. Let $y \in \kappa$. From Lemma 2.7, we have $(Ty)(t) \geq 0$ on $[t_1, t_3]$ and

$$(Ty)(t) = \lambda \int_{t_1}^{t_3} H_n(t,s) f(s, y(s)) ds$$
$$\leq K\lambda \int_{t_1}^{t_3} \parallel G_n(.,s) \parallel f(s, y(s)) ds,$$

so that

$$\parallel Ty \parallel \leq K\lambda \int_{t_1}^{t_3} \parallel G_n(.,s) \parallel f(s, y(s)) ds.$$

Next, if $y \in \kappa$, then we have

$$(Ty)(t) = \lambda \int_{t_1}^{t_3} H_n(t,s) f(s, y(s)) ds$$
$$\geq m_n L\lambda \int_{t_1}^{t_3} \parallel G_n(.,s) \parallel f(s, y(s)) ds$$
$$\geq \frac{m_n L}{K} \parallel Ty \parallel, \quad t \in [t_1, t_3].$$

Hence $Ty \in \kappa$ and so $T : \kappa \to \kappa$. □

The operator T is completely continuous by an application of the Ascoli-Arzela Theorem.

To establish the eigenvalue intervals we will employ the following fixed point theorem due to Krasnosel'skii.

Theorem 3.1. *Let X be a Banach space, $\kappa \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \kappa$ is completely continuous operator such that either*

(i) $\parallel Tu \parallel \leq \parallel u \parallel, \quad u \in \kappa \cap \partial\Omega_1$ *and* $\parallel Tu \parallel \geq \parallel u \parallel, \quad u \in \kappa \cap \partial\Omega_2$*, or*

(ii) $\parallel Tu \parallel \geq \parallel u \parallel, \quad u \in \kappa \cap \partial\Omega_1$ *and* $\parallel Tu \parallel \leq \parallel u \parallel, \quad u \in \kappa \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.2. *Assume that the conditions* $(A1)$-$(A4)$ *are satisfied. Then, for each λ satisfying*

$$\frac{K}{m_n^2 L^2 [\int_{t_2}^{t_3} \parallel G_n(.,s) \parallel ds] f_\infty} < \lambda < \frac{1}{K[\int_{t_1}^{t_3} \parallel G_n(.,s) \parallel ds] f^0}, \tag{3.4}$$

there exists at least one positive solution of the boundary value problem (1.1)-(1.2) *in* κ.

Proof. Let λ be given as in (3.4). Now $\epsilon > 0$ be chosen such that

$$\frac{K}{m_n^2 L^2 [\int_{t_2}^{t_3} \| G_n(.,s) \| ds](f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{K[\int_{t_1}^{t_3} \| G_n(.,s) \| ds](f^0 + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined as in (3.3). By the definition of f^0, there exits $r_1 > 0$ such that

$$\max_{t \in [t_1, t_3]} \frac{f(t,y)}{y} \leq (f^0 + \epsilon), \quad \text{for} \quad 0 < y \leq r_1.$$

It follows that

$$f(t,y) \leq (f^0 + \epsilon) y, \quad \text{for} \ \ 0 < y \leq r_1.$$

So choosing $y \in \kappa$ with $\| y \| = r_1$. Then we have

$$\begin{aligned} (Ty)(t) &= \lambda \int_{t_1}^{t_3} H_n(t,s) f(s, y(s)) ds \\ &\leq \lambda \int_{t_1}^{t_3} K \| G_n(.,s) \| f(s, y(s)) ds \\ &\leq K\lambda (f^0 + \epsilon) \int_{t_1}^{t_3} \| G_n(.,s) \| y(s) ds \\ &\leq \lambda K (f^0 + \epsilon) \int_{t_1}^{t_3} \| G_n(.,s) \| \| y \| ds \\ &\leq \| y \| . \end{aligned}$$

Consequently $\| Ty \| \leq \| y \|$. So if we define

$$\Omega_1 = \{ y \in X : \| y \| < r_1 \},$$

then

$$\| Ty \| \leq \| y \|, \quad \text{for} \ \ y \in \kappa \cap \partial \Omega_1. \tag{3.5}$$

By the definition of f_∞, there exits $\overline{r}_2 > 0$ such that

$$\min_{t \in [t_1, t_3]} \frac{f(t,y)}{y} \geq (f_\infty - \epsilon) y, \quad \text{for} \ \ y \geq \overline{r}_2.$$

It follows that,

$$f(t,y) \geq (f_\infty - \epsilon) y, \quad \text{for} \ \ y \geq \overline{r}_2.$$

Let

$$r_2 = \max\{2r_1, \frac{K}{m_n L} \overline{r}_2\}$$

and let

$$\Omega_2 = \{y \in X : \| y \| < r_2\}.$$

Now choose $y \in \kappa \cap \partial\Omega_2$ with $\| y \| = r_2$, so that

$$\min_{t\in[t_2,t_3]} y(t) \geq \frac{m_n L}{K} \| y \| = \overline{r}_2.$$

Consider,

$$\begin{aligned}(Ty)(t) &= \lambda \int_{t_1}^{t_3} H_n(t,s) f(s, y(s)) ds \\ &\geq \lambda \int_{t_2}^{t_3} m_n L \| G_n(.,s) \| f(s, y(s)) ds \\ &\geq \lambda m_n L \int_{t_2}^{t_3} \| G_n(.,s) \| (f_\infty - \epsilon) y(s) ds \\ &\geq \frac{\lambda m_n^2 L^2}{K} (f_\infty - \epsilon) \int_{t_2}^{t_3} \| G_n(.,s) \| \| y \| ds \\ &\geq \| y \| .\end{aligned}$$

Thus,

$$\| Ty \| \geq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{3.6}$$

Applying part (i) of Theorem 3.1 to (3.5) and (3.6) yields that T has a fixed point $y(t) \in \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is a positive solution of the boundary value problem (1.1)-(1.2) for the given λ. □

Theorem 3.3. *Assume that the conditions (A1)-(A4) satisfied. Then, for each λ satisfying*

$$\frac{K}{m_n^2 L^2 [\int_{t_2}^{t_3} \| G_n(.,s) \| ds] f_0} < \lambda < \frac{1}{K[\int_{t_1}^{t_3} \| G_n(.,s) \| ds] f^\infty}, \tag{3.7}$$

there exists at least one positive solution of the boundary value problem (1.1)-(1.2) in κ.

Proof. Let λ be given as in (3.7). Now $\epsilon > 0$ be chosen such that

$$\frac{K}{m_n^2 L^2 [\int_{t_2}^{t_3} \| G_n(.,s) \| ds](f_0 - \epsilon)} \leq \lambda \leq \frac{1}{K[\int_{t_1}^{t_3} \| G_n(.,s) \| ds](f^\infty + \epsilon)}.$$

Let T be cone preserving completely continuous operator that was defined by (3.3). By the definition of f_0, there exits an $J_1 > 0$ such that

$$\min_{t\in[t_1,t_3]} \frac{f(t,y)}{y} \geq (f_0 - \epsilon), \quad \text{for} \quad 0 < y \leq J_1.$$

It follows that,

$$f(t,y) \geq (f_0 - \epsilon)y, \quad \text{for} \quad 0 < y \leq J_1.$$

So choose $y \in \kappa$ with $\| y \| = J_1$, then

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{t_3} H_n(t,s) f(s, y(s)) ds \\
&\geq \lambda \int_{t_2}^{t_3} m_n L \| G_n(.,s) \| f(s,y(s)) ds \\
&\geq \lambda m_n L (f_0 - \epsilon) \int_{t_2}^{t_3} \| G_n(.,s) \| y(s) ds \\
&\geq \lambda \frac{m_n^2 L^2}{K} (f_0 - \epsilon) \int_{t_2}^{t_3} \| G_n(.,s) \| \| y \| ds \\
&\geq \| y \| .
\end{aligned} \tag{3.8}$$

Consequently, $\| Ty \| \geq \| y \|$. So if we define

$$\Omega_1 = \{ y \in X : \| y \| < J_1 \},$$

then

$$\| Ty \| \geq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1.$$

It remains for us to consider f^∞. By the definition of f^∞, there exist $\overline{J}_2 > 0$ such that

$$\max_{t \in [t_1, t_3]} \frac{f(t,y)}{y} \leq (f^\infty + \epsilon), \quad \text{for} \quad y \geq \overline{J}_2.$$

It follows that,

$$f(t,y) \leq (f^\infty + \epsilon)y, \quad \text{for} \quad y \geq \overline{J}_2.$$

There are two cases.

Case 1. Suppose f is bounded, then there is $L > 0$ such that $f(t,y) \leq L$, for $0 < y < \infty$. Let

$$J_2 = \max\{2J_1, KL\lambda \int_{t_1}^{t_3} \| G_n(.,s) \| ds\}.$$

Then, for $y \in \kappa$ with $\| y \| = J_2$, we have

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{t_3} H_n(t,s) f(s,y(s)) ds \\
&\leq \lambda \int_{t_1}^{t_3} K \| G_n(.,s) \| f(s,y(s)) ds \leq \lambda LK \int_{t_1}^{t_3} \| G_n(.,s) \| ds \leq \| y \|,
\end{aligned}$$

so that $\| Ty \| \leq \| y \|$. So, if we define

$$\Omega_2 = \{y \in X : \|y\| < J_2\},$$

then

$$\| Ty \| \leq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{3.9}$$

Case 2. Suppose f is unbounded. Let $J_2 > \max\{2J_1, \overline{J}_2\}$ be such that $f(t, y) \leq f(t, J_2)$, for $0 < y \leq J_2$. Let $y \in \kappa$ with $\| y \| = J_2$, we have

$$\begin{aligned}
(Ty)(t) &= \lambda \int_{t_1}^{t_3} H_n(t, s) f(s, y(s)) ds \\
&\leq \lambda K \int_{t_1}^{t_3} \| G_n(., s) \| f(s, y(s)) ds \\
&\leq \lambda K \int_{t_1}^{t_3} \| G_n(., s) \| f(s, J_2) ds \\
&\leq \lambda K \int_{t_1}^{t_3} \| G_n(., s) \| (f^\infty + \epsilon) J_2 ds \leq J_2 = \| y. \|
\end{aligned}$$

Thus, $\| Ty \| \leq \| y \|$. For this case, if we define

$$\Omega_2 = \{y \in X : \|y\| < J_2\},$$

then

$$\| Ty \| \leq \| y \|, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_2. \tag{3.10}$$

Thus, in either of the cases, an application of part (ii) of Theorem 3.1 to (3.8), (3.9) and (3.10) yields that T has fixed point $y(t) \in \kappa \cap (\Omega_2 \backslash \Omega_1)$. This fixed point is a solution of the boundary value problem (1.1)-(1.2) for the given λ. □

Example. Consider the following eigenvalue problem

$$y^{(6)} + \lambda y(200 - 199.5e^{-7y}) = 0, \quad t \in [0, 1], \tag{3.11}$$

subject to the boundary conditions

$$\begin{aligned}
y(0) + 4y'(0) + \frac{5}{4} y''(0) &= 0, \\
y\Big(\frac{1}{2}\Big) + \frac{1}{2} y'\Big(\frac{1}{2}\Big) + y''\Big(\frac{1}{2}\Big) &= 0, \\
y(1) + \frac{1}{2} y'(1) + \frac{1}{2} y''(1) &= 0, \\
y'''(0) + 4y^{(iv)}(0) + \frac{5}{2} y^{(v)}(0) &= 0, \\
y'''\Big(\frac{1}{2}\Big) + \frac{1}{4} y^{(iv)}\Big(\frac{1}{2}\Big) + \frac{1}{2} y^{(v)}\Big(\frac{1}{2}\Big) &= 0, \\
y'''(1) + \frac{1}{4} y^{(iv)}(1) + \frac{1}{4} y^{(v)}(1) &= 0.
\end{aligned} \tag{3.12}$$

The Green's function $H_2(t,s) = \int_0^1 H_1(t,r)G_2(r,s)dr$ for the boundary value problem (3.11)-(3.12) can be constructed, after computation of $G_1(t,s)$ and $G_2(t,s)$, where

$$G_1(t,s) = \frac{4}{3}\begin{cases} [\frac{9+2t-4t^2}{8}][s^2-8s+5], & 0<s<t\le\frac{1}{2}<1 \\ [\frac{5t^2-4t-9}{2}][\frac{4s^2-8s+11}{4}]+ & \\ \quad [\frac{24+9t-12t^2}{4}][\frac{2s^2-6s+6}{2}], & 0\le t<s<\frac{1}{2} \\ [\frac{24+9t-12t^2}{4}][\frac{2s^2-6s+6}{2}], & 0\le t<\frac{1}{2}<s<1 \\ [\frac{9+2t-4t^2}{8}][s^2-8s+5]+ & \\ \quad [\frac{-5t^2+4t+9}{2}][\frac{4s^2-8s+11}{4}], & 0<\frac{1}{2}<s<t\le 1 \\ [\frac{24+9t-12t^2}{4}][\frac{2s^2-6s+6}{2}], & 0<\frac{1}{2}\le t<s<1 \\ [\frac{-4t^2+2t+9}{8}][s^2-8s+5], & 0\le s<\frac{1}{2}<t<1. \end{cases}$$

$$G_2(t,s) = \frac{4}{9}\begin{cases} [\frac{3+4t-4t^2}{8}][s^2-8s+5], & 0<s<t\le\frac{1}{2}<1 \\ [\frac{11t^2-12t-7}{4}][\frac{2s^2-3s+3}{2}]+ & \\ \quad [\frac{9+14t-13t^2}{4}][\frac{2s^2-5s+4}{2}], & 0\le t<s<\frac{1}{2} \\ [\frac{9+14t-13t^2}{4}][\frac{2s^2-5s+4}{2}], & 0\le t<\frac{1}{2}<s<1 \\ [\frac{3+4t-4t^2}{8}][s^2-8s+5]+ & \\ \quad [\frac{-11t^2+12t+7}{4}][\frac{2s^2-3s+3}{2}], & 0<\frac{1}{2}<s<t\le 1 \\ [\frac{9+14t-13t^2}{4}][\frac{2s^2-5s+4}{2}], & 0<\frac{1}{2}\le t<s<1 \\ [\frac{-4t^2+4t+3}{8}][s^2-8s+5], & 0\le s<\frac{1}{2}<t<1. \end{cases}$$

From Lemma 2.4 and (2.3)-(2.5), we get

$$m_1 = 0.7777, \quad K_1 = 16.3458, \quad L_1 = 13.990,$$

$$m_2 = 0.8, \quad K_2 = 0.750591, \quad L_2 = 0.44527.$$

Therefore $L = 3.875772$, $K = 12.26901097$. We found that $f_\infty = 200$, and $f^0 = 0.5$. Employing Theorem 3.2, we get the optimal eigenvalue interval $0.0009652 < \lambda < 0.01165$, for which (3.11)-(3.12) has a positive solution.

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