International Journal of Pure and Applied Mathematics
Volume 61 No. 1 2010, 7-10

RANKS OF SUBVARIETIES OF $\mathbb{P}^n$ OVER NON-ALGEBRICALLY CLOSED FIELDS

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Abstract: Here we define the symmetric tensor rank over a non-algebraically closed field with respect to a subvariety $X \subset \mathbb{P}^n$. We compute it when X is a rational normal curve.

AMS Subject Classification: 14N05
Key Words: rank, rational normal curve, symmetric tensor rank

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Fix a field K and an algebraically closure $\overline{K}$ of K. Let $X \subset \mathbb{P}^n$ be a geometrically integral closed subvariety defined over K. For any $P \in \mathbb{P}^n(\overline{K})$ the rank $r_X(P)$ of X is the minimal integer k such that there is $S \subset X(\overline{K})$ with $P \in \langle S \rangle$ and $\sharp(S) = k$ (see [4], [3] and references therein). Since $P \in \mathbb{P}^n(\overline{K})$, there is a finite extension L of K such that $P \in \mathbb{P}^n(L)$. We call L-rank $r_{X,L}(P)$ of P the minimal integer k such that there is $S \subset X(L)$ with $P \in \langle S \rangle$ and $\sharp(S) = k$; we require that each $Q \in S$ is defined over L. Fix L and $P \in \mathbb{P}^n(L)$. We say that a finite extension M of L computes the rank $r_X(P)$ of P if there is $S \subset X(M)$ such that $P \in \langle S \rangle$ and $\sharp(S) = r_X(Q)$. Our guess is that finite extensions M as above should not exists, except for trivial cases like the real closed fields ($\mathbb{C}$ always work if $K = \mathbb{R}$) (see Remark 2). One of our aims (for fixed X and L) is to find M computing the ranks of all $P \in \mathbb{P}^n(L)$ or prove that no such finite

Received: December 9, 2009

extension exists. The X-L-rank (or just the L-rank) $r_{X,L}(P)$ of $P \in \mathbb{P}^n(L)$ is the minimal integer $\sharp(S)$, where $S \subset X(L)$ and $P \in \langle S\rangle$; to get the finiteness of the function $r_{L,X}$ we need to assume that $X(L)$ spans $\mathbb{P}^n$. Here we compute the function $r_{X,L}$ in one of the few cases in which the function r_X is known: the rational normal curve. In this case we need to assume $\mathrm{char}(\mathbb{K}) = 0$, because only under this assumption a complete description of the X-rank function is known (see [2], [4], Theorem 4.1, [3], Theorem 3.8)

Theorem 1. *Fix an integer $n \geq 2$ and a field K such that $\mathrm{char}(K) = 0$. Let $X \subset \mathbb{P}^n$ be the rational normal curve over K, i.e. we assume that X is the completion of the affine curve $t \mapsto (t, t^2, \dots, t^n) \in \mathbb{A}^n$ defined over K. Fix $P \in \mathbb{P}^n(K)$ and let k be the minimal integer such that $P \in S^k(X)$, where $S^k(X) \subseteq \mathbb{P}^n$ denote the closure of the union of all $\langle S\rangle$ with $S \subset X(\overline{K})$ and $\sharp(S) = k$. Hence $1 \leq k \leq \lfloor (n+2)/2 \rfloor$ (see [1], Remark 1.6). If $k = 1$, then $r_X(P) = r_{X,K}(P) = 1$. Assume $k \geq 2$ and $2k \neq n+2$. Either $r_X(P) = k$ or $r_X(P) = n+2-k$ (see [2], [4], Theorem 4.1, [3], Theorem 3.8).*

(a) There is an extension M of K such that $r_{X,M}(P) = r_X(P)$ and $\deg([M : K]) \leq n+2-k$.

(b) Assume $r_X(P) = k$. Then either $r_{X,K}(P) = k$ or $r_{X,K}(P) \geq n+2-k$. The latter case occurs for some P with $r_X(P) = k$ if and only if K has a Galois extension of degree x with $2 \leq x \leq k$.

Remark 1. Let $X \subset \mathbb{P}^n$ be a rational normal curve and $Z \subset X$ be any zero-dimensional subscheme. The cohomology of line bundles on $\mathbb{P}^1$ gives $\dim(\langle Z\rangle) = \min\{n, \mathrm{length}(Z) - 1\}$.

Proof of Theorem 1. Since $r_{X,K}(P) = r_X(P) = 1$ for all $P \in X(K)$, we may assume $k \geq 2$. The integer k is the minimal integer such that there is a zero-dimensional scheme $Z \subset X(\overline{K})$ such that $P \in \langle Z\rangle$. Since $2k < n+2$, Z is unique (Remark 1). Hence Z is defined over K (but its connected components are not necessarily defined over K). Z is reduced if and only if $r_X(P) = k$ (see [2], [4], Theorem 4.1, [3], Theorem 3.8).

(i) Assume $r_X(P) = k$, i.e. assume that Z is reduced. There is a finite extension M of K such that each point of Z is defined over M. Hence $r_{X,M}(P) = k$ (if Z is reduced). Since Z is unique, $r_{X,K}(P) = k$ if and only if each $Q \in Z$ is defined over K. For any k we may find at least one such P, because $X(K)$ is Zariski dense in $X(\overline{K})$. For each k we may find $P \in \mathbb{P}^n(K)$ such that $r_X(P) = k$ and $r_{X,K} > k$ if and only if there is an integer $x \in \{2, \dots, k\}$ and a degree x Galois extension M of K: take as Z the union of x distinct points of $X(M) = \mathbb{P}^1(M)$ which are conjugate over K and $k - x$ general points of $X(K)$.

(ii) Now assume $r_{X,K}(P) \neq k$. Remark 1 implies $r_X(P) \geq n+2-k$. Hence to conclude the proof it is sufficient to prove the reverse inequality for an extension M of K of degree at most $n+2-k$. Let V be the complete linear system on $X \cong \mathbb{P}^1$ formed by the degree $n+2-k$ effective divisors. Let V_P the linear subsystem of V formed by the divisors $D \in V$ such that $P \in \langle V \rangle$. V_P has codimension $k-1$ in V. Since $P \in \mathbb{P}^n$, both V and V_P are defined over K. The inequality $r_{X,K}(P) \leq n+2-k$ is true if and only if there is a reduced $D \subset V_P$ such that each point of D is defined over K.

Claim. *The linear system V_P is base point free.*

Proof of the Claim. The claim only concerns $\overline{K}$. Take a subscheme Z' of Z (defined over $\overline{K}$) such that length$(Z') = k-2$. It exists because $k \geq 2$ and $\overline{K}$ is algebraically closed. Let W be the linear system on X induced by the set of all hyperplanes containing Z'. Let W_P be the subsystem of W formed by the hyperplanes containing P. Since $P \notin \langle Z' \rangle$, W_P has codimension 1 in W. Remark 1 gives that W is the complete linear system of all degree $n+2-k$ divisors on X. Since $P \notin S^{k-1}(X)$, for each $Q \in X \backslash Z'$, we have $P \neq \langle Z' \cup \{Q\} \rangle$, i.e. (use Remark 1) $Q \notin \langle \{P\} \cup Z' \rangle$, i.e. Q is not a base point of W_P. For the same reason for each Z'' such that $Z' \subsetneqq Z'' \subsetneqq Z$ and $(Z'')_{red} \neq (Z')_{red}$, the point $(Z')_{red} \backslash (Z'')_{red}$ is not a base point of W_P. Hence $W_P = V_P$ is base point free. The claim also follows from the proof of [4], Theorem 4.1, second paragraph: the case given in the case $[\phi] \in \sigma_{r-1}(v_d(\mathbb{P}^1))$ in the case $[\phi] \in \sigma_r(v_d(\mathbb{P}^1)) \backslash \sigma_{r-1}(v_d(\mathbb{P}^1))$ gives $f = 0$ base points.

By the claim and our characteristic zero assumption there is a non-empty open subset Ω of $V_P(\overline{K})$ parametrizing the the set of all reduced divisors. The maximal such Ω is defined over K. Since V_P is a projective space and K is infinite, $V_P(K)$ is Zariski dense in $V_P(\overline{K})$. Hence $\Omega(K) \neq \emptyset$. Any degre $n-k+2$ divisor $D \in \Omega(K)$ is reduced and all its connected component are defined over an extension M of K such that $\deg([M:K]) \leq n+2-k$. □

Remark 2. Take $K = \mathbb{Q}$ and look at step (i) of the proof of Theorem 1. This proof shows that no finite extension M of $\mathbb{Q}$ computes the X-$\overline{\mathbb{Q}}$-ranks of all $P \in \mathbb{P}^n(\mathbb{Q})$.

Acknowledgments

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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