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GENESIS OF THE SCALAR PRODUCT AND VECTOR PRODUCT OF VECTORS

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Abstract: In handbooks of mathematics and mechanics, known to the author, there are provided ready-made definitions of the scalar product and vector product of vectors. It happens that arriving at the formula on the cosine alpha of the angle contained between two vectors, as shown in the handbook [1], and Lagrange's identity permit delivering the genesis of the both products [2]. And so, determining the formula on the cosine of the angle contained between two vectors leads to the scalar product of those two vectors, and determining the formula on the sine of the angle contained between the two vectors leads to the formulation of the vector product of those vectors.

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1. Genesis of the Scalar Product

There will be presented some reasoning which may be construed as a genesis of the scalar products of two vectors. Arriving at the formula on the cosine alpha of the angle contained between two vectors leads to the formulation of the scalar product of those two vectors.

Let us assume that in a system of co-ordinates, which has been adopted arbitrarily, there are the two specific vectors: $\vec{a}$ and $\vec{b}$ whose front ends are found

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in the beginning of the coordinate system. Those vectors may be identified by their coordinates. So the following is in place:

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}, \quad \text{and} \quad \vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k},$$

where

$$a_x = \mid \vec{a} \mid \cos\alpha, \quad a_y = \mid \vec{a} \mid \cos\beta, \quad a_z = \mid \vec{a} \mid \cos\gamma,$$

$$b_x = \mid \vec{b} \mid \cos\alpha' \quad b_y = \mid \vec{b} \mid \cos\beta' \quad b_z = \mid \vec{b} \mid \cos\gamma'.$$

Then, the versors of those vectors are determined:

$$\vec{e} = \frac{\vec{a}}{\mid \vec{a} \mid} = e_x\vec{i} + e_y\vec{j} + e_z\vec{k} = \frac{a_x}{\mid \vec{a} \mid}\vec{i} + \frac{a_y}{\mid \vec{a} \mid}\vec{j} + \frac{a_z}{\mid \vec{a} \mid}\vec{k} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k},$$

$$\vec{e}' = \frac{\vec{b}}{\mid \vec{b} \mid} = e'_x\vec{i} + e'_y\vec{j} + e'_z\vec{k} = \frac{b_x}{\mid \vec{b} \mid}\vec{i} + \frac{b_y}{\mid \vec{b} \mid}\vec{j} + \frac{b_z}{\mid \vec{b} \mid}\vec{k} = \cos\alpha'\vec{i} + \cos\beta'\vec{j} + \cos\gamma'\vec{k}.$$

Because the versor $\vec{e}$ may be expressed by its projections upon the axes x, y, z of the coordinate system, with the aid of the formula:

$$\vec{e} = \vec{e}_x + \vec{e}_y + \vec{e}_z,$$

so the coordinate e_b of the versor $\vec{e}$ to the axis as determined by the vector $\vec{b}$ will be arrived at in the known manner, moving from the vector equation to the equations of the coordinates. So the following is in place:

$$e_b = e_{xb} + e_{yb} + e_{zb},$$

where the coordinates of the projections to the axis determined by the vector $\vec{b}$ will be expressed by the formulae:

$$e_{xb} = e_x \cos\alpha' = \cos\alpha\cos\alpha',$$

$$e_{yb} = e_y \cos\beta' = \cos\beta\cos\beta',$$

$$e_{zb} = e_z \cos\gamma' = \cos\gamma\cos\gamma'.$$

On the other hand, designating the letter ϕ the angle between the vectors $\vec{a}$ and $\vec{b}$, the coordinate e_b of the versor $\vec{e}$ to the axis determined by the vector $\vec{b}$ will be determined by ma king use of the definition of the coordinate that is:

$$\begin{aligned} e_b = \mid \vec{e} \mid \cos\phi = \cos\phi &= e_{xb} + e_{yb} + e_{zb} \\ &= e_x\cos\alpha' + e_y\cos\beta' + e_z\cos\gamma' = e_x e'_x + e_y e'_y + e_z e'_z \\ &= \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' \\ &= \frac{a_x}{\mid \vec{a} \mid}\frac{b_x}{\mid \vec{b} \mid} + \frac{a_y}{\mid \vec{a} \mid}\frac{b_y}{\mid \vec{b} \mid} + \frac{a_z}{\mid \vec{a} \mid}\frac{b_z}{\mid \vec{b} \mid} = \frac{a_x b_x + a_y b_y + a_z b_z}{\mid \vec{a} \mid\mid \vec{b} \mid}. \end{aligned}$$

Let us note that the coordinate system $U\,(x, y, z)$ has been adopted completely arbitrarily. If we assume any other coordinate systems, e.g., $U'(x', y', z')$ and carry out the identical operations in them, then, they will prove that the formula on the coordinate e_b, that is the cosineϕ is expressed by the analogical formula, i.e.

$$e_b = \cos\phi = \frac{a'_x b'_x + a'_y b'_y + a'_z b'_z}{|\vec{a}||\vec{b}|}.$$

So, we may write down that

$$\cos\phi = \frac{a_x b_x + a_y b_y + a_z b_z}{|\vec{a}||\vec{b}|} = \frac{a'_x b'_x + a'_y b'_y + a'_z b'_z}{|\vec{a}||\vec{b}|} = ... = \frac{a''_x b''_x + a''_y b''_y + a''_z b''_z}{|\vec{a}||\vec{b}|},$$

hence

$$|\vec{a}||\vec{b}|\cos\phi = a_x b_x + a_y b_y + a_z b_z = a'_x b'_x + a'_y b'_y + a'_z b'_z = ...a''_x b''_x + a''_y b''_y + a''_z b''_z.$$

And that number, which does not belong with the adopted coordinate system, and which is dependent only upon the values of both vectors and cosine of the angle contained between them, will be referred to as the scalar product of two vectors. The product will be designated as follows:

$$\vec{a} \cdot \vec{b} \stackrel{df}{=} |\vec{a}||\vec{b}|\cos\phi = a_x b_x + a_y b_y + a_z b_z.$$

The above reasoning may be construed as a genesis of the scalar product of two vectors.

2. Genesis of the Vector Product

Relying on the considerations as shown in the preceding Section 1, there will be added some more reasoning, which may be construed as a genesis of the vector product of two vectors. It happens that arriving at the formula on the sine of the angle contained between two vectors leads to a formula that implies the opportunity of introducing a new vector. We will prove that this new vector will be interpreted as the product of those two vectors.

So, bearing in mind the considerations as presented in paragraph 1, we will be looking for the second power of the value of the sineϕ. Now that the value of the versors equals to one, i.e.

$$|\vec{e}| = e_x^2 + e_y^2 + e_z^2 = 1, \quad |\vec{e}'| = e_x'^2 + e_y'^2 + e_z'^2 = 1,$$

and

$$\cos\phi = e_x e'_x + e_y e'_y + e_z e'_z,$$

so, employing Lagrange's identity [1], [3], we may write:

$$\sin^2\phi = 1-\cos^2\phi = \left(e_x^2+e_y^2+e_z^2\right)\left(e_x^{,2}+e_y^{,2}+e_z^{,2}\right)-\left(e_xe_x^{,}+e_ye_y^{,}+e_ze_z^{,}\right)^2$$
$$= \left(e_ye_z^{,}-e_ze_y^{,}\right)^2+\left(e_ze_x^{,}-e_xe_z^{,}\right)^2+\left(e_xe_y^{,}-e_ye_x^{,}\right)^2$$
$$= \frac{(a_yb_z-a_zb_y)^2+(a_zb_x-a_xb_z)^2+(a_xb_y-a_yb_x)^2}{|\,\vec{a}\,|^2|\,\vec{b}\,|^2}.$$

Being experienced in computing determinants, it is easy to notice that the sum of the bracketed expressions may be rendered by means of a determinant:

$$(a_yb_z-a_zb_y)+(a_zb_x-a_xb_z)+(a_xb_y-a_yb_x)=\begin{vmatrix} 1 & 1 & 1 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

Now, if, in the first line of the determinant, we replace the ones with the versors of the axes x, y, z, that is the versors $\vec{i},\vec{j},\vec{k}$, then the determinant will represent a certain new vector designated as $\vec{c}$ and represented by the formula:

$$\vec{c}\overset{df}{=}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}=(a_yb_z-a_zb_y)\,\vec{i}+(a_zb_x-a_xb_z)\,\vec{j}+(a_xb_y-a_yb_x)\,\vec{k}.$$

The square power of the sine ϕ may be expressed as:

$$\sin^2\phi=\frac{(a_yb_z-a_zb_y)^2+(a_zb_x-a_xb_z)^2+(a_xb_y-a_yb_x)^2}{|\,\vec{a}\,|^2|\,\vec{b}\,|^2}$$
$$=\frac{\left\|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{matrix}\right\|^2}{|\,\vec{a}\,|^2|\,\vec{b}\,|^2}=\frac{|\,\vec{c}\,|^2}{|\,\vec{a}\,|^2|\,\vec{b}\,|^2},$$

whereas the value of the vector $\vec{c}$ is represented by the formula:

$$|\,\vec{c}\,|=|\,\vec{a}\,||\,\vec{b}\,|\sin\phi=\left\|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{matrix}\right\|.$$

It happens that the new vector $\vec{c}$ is perpendicular to both the vector $\vec{a}$ and vector $\vec{b}$, as the scalar product of that vector and the vectors $\vec{a}$ and $\vec{b}$ equals to naught:

$$\vec{a}\cdot\vec{c}=a_xc_x+a_yc_y+a_zc_z=0,$$
$$\vec{b}\cdot\vec{c}=b_xc_x+b_yc_y+b_zc_z=0.$$

The direction of the vector is determined by the signs of the coordinates. Because we know the coordinates of the vector $\vec{c}$, so its direction will be found by means of the formula:

$$\vec{c} = c_x\vec{i} + c_y\vec{j} + c_z\vec{k} = \vec{c}_x + \vec{c}_y + \vec{c}_z.$$

In determining the direction, we will make use of the right-hand-screw rule to the vectors $\left(\vec{a}, \vec{b}, \vec{c}\right)$. The vector product of the vectors is expressed by the formula:

$$\vec{c} \overset{df}{=} \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

The above reasoning may be construed as a genesis of the vector product of vectors.

References

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