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ON THE WEIGHTED OSTROWSKI TYPE INEQUALITY FOR $L_p(a, b)$ AND APPLICATION

Ather Qayyum
Department of Mathematics
University of Hail
P.O. Box 2440, KINGDOM OF SAUDI ARABIA
e-mail: ather@uoh.edu.sa

Abstract: The aim of this paper is to develop a weighted Ostrowski type inequality for differentiable mappings whose first derivative belongs to $L-p(a, b)$ $(p > 1)$. Applications in numerical integration are also given.

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1. Introduction

In 1938, Ostrowski [9] presented Ostrowski inequality for differentiable mappings with bounded derivatives. Since then there is a rising of obtaining sharp bounds of this inequality in terms of variety of Lebesgue spaces involving, at most, the first derivative which results in obtaining some new inequalities of Ostrowski type. During the past few years, many researchers have given considerable attention to the above inequality, in which G.V. Milovanović, and J.E. Pěcarić [10], N. Ujević [2], S.S. Dragomir and A. Sofo [8], and A.M. Fink [7] are some of them.

Ostrowski [9] proved the following interesting and useful integral inequality:

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Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and diferentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)| < \infty$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty \qquad (1)$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1997, S.S. Dragomir and S. Wang [1] considered integral inequality of Ostrowski type for $\|.\|_p$- norms $(p > 1)$ as follows:

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0 (I^0 is interior of I) and $a,b \in I^0$ with $a < b$. If $f' \in L_p(a,b)$ $\left(p > 1,\ \frac{1}{p} + \frac{1}{q} = 1\right)$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{1}{(b-a)} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right] \|f'\|_p \qquad (2)$$

for all $x \in [a,b]$, where $\|f'\|_p = \left(\int_a^b \left|f'(t)\right|^p dt \right)^{\frac{1}{p}}$ is the $\|.\|$ −norm.

In 1999, S.S. Dragomir and N.S. Barnett [1], p. 13, extended the above result (2). They proved the following inequality:

Theorem 1.3. *Let f: $[a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) and $f'' \in L_p(a,b)\,(p > 1)$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2} \right) f'(x) \right|$$
$$\le \frac{1}{2(b-a)(2q+1)^{\frac{1}{q}}} \left[(x-a)^{2q+1} + (b-x)^{2q+1} \right] \|f'\|_p \qquad (3)$$

for all $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Motivated by the result of S.S. Dragomir and N.S. Barnett [1], p. 13, in this paper, we develop weighted Ostrowski type inequality for differentiable mappings whose first derivative belongs to $L_p(a,b)\,(p > 1)$. In this way, our result is more generalized and extended as compare to the above results, because we observe that the previous results (see for example [4], [11], [7], [5], [3]) are non-weighted and our result is in the weighted form. Moreover, perturbed mid-point and trapezoid inequalities are also obtained in the form of remarks. Finally, we give applications in numerical integration.

2. Main Results

Our main result is given in the following theorem.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b), whose first derivative, i.e $f' : [a,b] \to \mathbb{R}$ belongs to $L_p(a,b)$, then*

$$\left| f(x) - \frac{1}{m(a,b)} w(x)(b-a)\left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{m(a,b)} \int_a^b f(t)w(t)dt \right|$$
$$\leq \frac{1}{m^2(a,b)} w(x) \left(\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right)$$
$$\times \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left[(x-a)^{1+\frac{1}{q}} + (b-x)^{1+\frac{1}{q}} \right] \|f''\|_{w,p} \quad (4)$$

for all $x \in [a,b]$.

Proof. The following weighted integral identity is proved in [5].

$$f(x) = \frac{1}{m(a,b)} \int_a^b P(x,t) f'(t)dt + \frac{1}{m(a,b)} \int_a^b f(t)w(t)dt, \quad (5)$$

for all $x \in [a,b]$, where $w(t)$ is a weighted function and $m(a,b) = \int_a^b w(t)dt$, (see for example [4]). Integration with weight functions are used in countless mathematical problems. Now define weighted Peano kernel, $P(.,.) : [a,b]^2 \longrightarrow \mathbb{R}$ by

$$P(x,t) = \begin{cases} \int_a^t w(u)du, & \text{if } t \in [a,x], \\ \int_b^t w(u)du, & \text{if } t \in (x,b], \end{cases}$$

for all $t \in [a,b]$.

Applying the identity (5) for $f'(.)$, as mentioned in [4] , we can state

$$f'(t) = \frac{1}{m(a,b)} \int_a^b P(t,s) f''(s) ds + \frac{1}{m(a,b)} \int_a^b f'(s) w(s) ds.$$

Substituting $f'(t)$ in the right membership of (5), we have

$$\begin{aligned} f(x) &= \frac{1}{m^2(a,b)} \int_a^b \int_a^b P(x,t) P(t,s) f''(s) ds dt + \frac{1}{m^2(a,b)} \\ &\times \int_a^b P(x,t) dt \int_a^b f'(s) w(s) ds dt + \frac{1}{m(a,b)} \int_a^b f(t) w(t) dt. \end{aligned} \tag{6}$$

Now by using, First Mean Value Theorem for Integration, we have

$$\begin{aligned} \int_a^b P(x,t) dt &= w(x)(b-a)\left(x - \frac{a+b}{2}\right), \\ \int_a^b f'(s) w(s) ds &= f'(x) m(a,b). \end{aligned}$$

From (6), by using technique [11], we have

$$\left| f(x) - \frac{1}{m(a,b)} w(x)(b-a)\left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{m(a,b)} \int_a^b f(t) w(t) dt \right|$$
$$\leq \frac{1}{m^2(a,b)} \int_a^b |P(x,t)| \, dt \int_a^b |P(x,t)| \left| f''(t) \right| dt. \tag{7}$$

Now by using Second Mean Value Theorem for Integration [4] , we have

$$\int_a^b |P(x,t)| \, dt = w(x) \left(\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right) = A.$$

Thus (7) gives

$$\int_a^b |P(x,t)| \, dt \int_a^b |P(x,t)| \left| f''(t) \right| dt$$

$$= A\left(\int_a^x |P(x,t)|\left|f''(t)\right|dt + \int_x^b |P(x,t)|\left|f''(t)\right|dt\right)$$

$$= A\left(\int_a^x |t-a|\left|w(t)f''(t)\right|dt + \int_x^b |b-t|\left|w(t)f''(t)\right|dt\right). \quad (8)$$

Now, by applying Hölder's inequality, we obtain

$$A\left(\int_a^x |t-a|\left|w(t)f''(t)\right|dt + \int_x^b |b-t|\left|w(t)f''(t)\right|dt\right)$$

$$\leq A\left[\begin{array}{c}\left(\int_a^x |t-a|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^x |w(t)f''(t)|^p\,dt\right)^{\frac{1}{p}} \\ +\left(\int_x^b |b-t|^q\,dt\right)^{\frac{1}{q}}\left(\int_x^b |w(t)f''(t)|^p\,dt\right)^{\frac{1}{p}}\end{array}\right]$$

$$= A\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left((x-a)^{q+1}\right)^{\frac{1}{q}}\left\|f''\right\|_{w,p[a,x]}\right.$$

$$\left.+\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left((b-x)^{q+1}\right)^{\frac{1}{q}}\left\|f''\right\|_{w,p[x,b]}\right]$$

$$\leq A\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[(x-a)^{1+\frac{1}{q}}+(b-x)^{1+\frac{1}{q}}\right]\left\|f''\right\|_{w,p} \quad (9)$$

Using (8) and (9) in (7), we get our required result (4). □

Remark 2.2. Putting $w(t) = 1$, in (4), we get non-weighted result which is closely related to the result obtained in [3]. For different weights one can obtain variety of results. The details are left to the interested reader.

Corollary 2.3. *Let $f\colon [a,b] \to \mathbb{R}$ be defined in Theorem* (4), *then we have perturbed mid-point inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right|$$

$$\leq w\left(\frac{a+b}{2}\right)\frac{(b-a)^{2\frac{q+1}{q}}}{2^{\frac{3q+1}{q}}(q+1)^{\frac{1}{q}}m^2(a,b)}\left\|f''\right\|_{w,p}. \quad (10)$$

Proof. This follows by the inequality (4), choosing $x = \frac{a+b}{2}$. □

Corollary 2.4. *Let $f\colon [a,b] \to \mathbb{R}$ be defined in Theorem* (4), *then we have the following perturbed trapezoid inequality*

$$\frac{f(a)+f(b)}{2} + \frac{(a-b)^2}{4m(a,b)}\left[f'(a)w(a) - f'(b)w(b)\right] - \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt$$

$$\leq \frac{(b-a)^{2\frac{q+1}{q}}}{2m^2(a,b)(q+1)^{\frac{1}{q}}}\left[\frac{w(a)+w(b)}{2}\right]\left\|f''\right\|_{w,p}. \quad (11)$$

Proof. Put in (4), $x = a$ and $x = b$, summing the above two inequalities. Using the triangle inequality and dividing by 2, we get (11). □

Corollary 2.5. *Let $f\colon [a,b] \to \mathbb{R}$ be defined in Theorem* (4) *and $f' \in L_2(a,b)$, then we have the inequality*

$$\left| f(x) - \frac{1}{m(a,b)}w(x)(b-a)\left(x - \frac{a+b}{2}\right)f'(x) - \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt \right|$$

$$\leq \frac{1}{m^2(a,b)}w(x)\left(\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right)$$

$$\times \frac{1}{\sqrt{3}}\left[(x-a)^{\frac{3}{2}} + (b-x)^{\frac{3}{2}}\right]\left\|f''\right\|_{w,2}. \quad (12)$$

Proof. Applying inequality (4) for $p = q = 2$, we get (12). □

Taking into account the fact that the mapping

$$h : [a,b] \to \mathbb{R},\ h(x) = (x-a)^{1+\frac{1}{q}} + (b-x)^{1+\frac{1}{q}}$$

has the property that

$$\inf_{x\in(a,b)} h(x) = h\left(\frac{a+b}{2}\right) = \frac{(b-a)^{1+\frac{1}{q}}}{2^{\frac{1}{q}}},$$

$$\sup_{x\in(a,b)} h(x) = h(a) = h(b) = (b-a)^{1+\frac{1}{q}},$$

we can get the best estimation from the inequality (4), only when $x = \frac{a+b}{2}$, this yields the inequality (10). It shows that the mid point estimation is better than the trapezoidal type estimation. Hence for $m(a,b) > 1$, the result given in (10) is better than the comparable results available in the literature.

3. Application in Numerical Integration

We know that Ostrowski-type integral inequalities, have enjoyed a surge in popularity. This field has developed significantly over the last few years, and has yielded many new results and powerful applications in numerical integration, probability theory, stochastics, statistics, information theory, game theory and integral operator theory. In this paper, we will apply the inequality [4] for numerical integration.

Let $I_n : a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ a sequence of intermediate points $h_i := x_{i+1} - x_i$ $(i = 0, 1, ...n-1)$. We have the following quadrature formula:

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $[a, b]$ whose first derivative $f' : (a, b) \to \mathbb{R}$ belong to $L_p(a, b)$, then we have the quadrature formula:*

$$\int_a^b w(t) f(t) dt = A(f, f', \xi, I_n) + R(f, f', \xi, I_n), \tag{13}$$

where

$$A\left(f, f', \xi, I_n\right) = \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f(\xi_i) - \sum_{i=0}^{n-1} \left[w(\xi_i) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i f'(\xi_i) \right] \tag{14}$$

and the remainder $R(f, f', \xi, I_n)$ satisfies the estimation

$$\begin{aligned} \left|R(f, f', \xi, I_n)\right| \leq\ & \frac{\|f''\|_{w,p}}{(q+1)^{\frac{1}{q}}} \sum_{i=0}^{n-1} \frac{w(\xi_i)}{m(x_i, x_{i+1})} \left\{ \frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right\} \\ & \times \left[(\xi_i - x_i)^{1+\frac{1}{q}} + (x_{i+1} - \xi_i)^{1+\frac{1}{q}} \right], \end{aligned} \tag{15}$$

for all ξ_i as above.

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$, to obtain

$$\left| f(\xi_i) m(x_i, x_{i+1}) - w(\xi_i)(h_i) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) - \int_{x_i}^{x_{i+1}} f(t) w(t) dt \right|$$

$$\leq \frac{\|f''\|_{w,p}}{m(x_i, x_{i+1})} w(\xi_i) \left(\frac{1}{4} h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right)$$

$$\times\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[(\xi_i-x_i)^{1+\frac{1}{q}}+(x_{i+1}-\xi_i)^{1+\frac{1}{q}}\right]$$

for all $\xi_i \in [x_i, x_{i+1}]$, where $h_i = x_{i+1} - x_i, (i = 0, 1, ..., n-1)$. Summing over i from 0 to $(n-1)$ and using the generalized triangular inequality, we get the desired inequality (16). □

Remark 3.2. If we choose $\xi_i = \frac{x_i+x_{i+1}}{2}$, we recapture the weighted mid-point quadrature formula

$$\int_a^b w(t) f(t)dt = A_M + R_M, \tag{16}$$

where the remainder R_M satisfies the estimation

$$|R_M| \le \sum_{i=0}^{n-1} \frac{\|f''\|_{w,p}}{m(x_i, x_{i+1})} w\left(\frac{x_i+x_{i+1}}{2}\right) \frac{1}{2^{2\left(1+\frac{1}{q}\right)}} h_i^{2\left(1+\frac{1}{q}\right)} \left(\frac{1}{q+1}\right)^{\frac{1}{q}}. \tag{17}$$

Remark 3.3. To derive the corresponding results for the Euclidean norm $\|f''\|_{w,2}$, we put $p = q = 2$, in (17).

Remark 3.4 The corresponding quadrature formulas for equidistant partitioning can be obtained by choosing $x_i = a + i\frac{b-a}{n}$ $(i = 0, 1, ..., n-1)$.

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