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PRE-INVERSE SUBGROUPS OF ORTHODOX SEMIGROUPS

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Abstract: In this paper we consider a regular semigroup S and a subsemigroup T having the property that $T \cap A(a) \neq \emptyset$ for every a belongs to $S \setminus T$, and show that T is a maximal subgroup H_z for some idempotent z. When S is orthodox, z is medial and zSz is uniquely unit orthodox. When S is orthodox and z is a middle unit, we obtain a structure theorem for uniquely unit orthodox semigroup.

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1. Introduction

By [3] a semigroup S is called orthodox if it is regular and its idempotent form a subsemigroup. By [2] $A(a) = \{b$ belongs to S; $aba = a\}$ is the set of pre-inverses of a, and z is medial in the sense that for some idempotent z, $a = aza$ for every a belongs to $\langle E \rangle$. By [1] when S is orthodox then z is middle unit if $azb = ab$ for all a, b belongs to S. By [4] a medial idempotent z of regular semigroup is said to be normal if the band $z \langle E \rangle z$ is commutative. Let S be a regular semigroup. Consider a subsemigroup T of S having the property that $T \cap A(a) \neq \emptyset$ for every a belongs to $S \setminus T$. We define a' by $T \cap A(a) = \{a'\}$, $a'' = (a')'$ for every a

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belongs to $S \setminus T$ then $(a')'' = (a')')' = (a'')'$ which can be written as a'''. Since a' belongs to $A(a)$ we have $a'aa'$ belongs to $V(a)$ is subset of $A(a)$. If a belongs to T then $a'aa'$ belongs to $T \cap A(a) = \{a'\}$ whence $a'aa' = a'$, consequently a belongs to $T \cap A(a') = \{a''\}$ so that $a = a''$. If we write $S' = \{a'; a \text{ belongs to } S\}$ we therefore have T is a subset of S'. The converse follows from the definition of a', we thus have $T = S'$. Also observe that $a''a'a''$ belongs to $V(a')$ gives $a''a'a''$ belongs to $T \cap A(a') = \{a''\}$. Hence $a''a'a'' = a''$ and so a' belongs to $T \cap A(a'') = \{a'''\}$. Thus $a''' = a'$, from which it follows that a belongs to $T = S'$ if and only if $a = a''$. Since a'' belongs to $V(a')$ we have that S' is regular; and since b belong to $S' \cap V(a')$ gives $S' \cap A(a') = \{a''\}$. We can see that S' is an inverse semigroup with $(a')^{-1} = a''$. If we now e_1, e_2 belongs to $E(S')$ then since e_1, e_2 commute we have $e_1e_2 \cdot e_1 \cdot e_1e_2 = e_1e_2 = e_1e_2 \cdot e_2 \cdot e_1e_2$. Whence e_1, e_2 belongs to $S' \cap A(e_1e_2)$ and therefore $e_1 = (e_1e_2) = e_2$. It follows that S' is in fact a group. If we denote by z the identity element of S', then we have the following properties (for all a belongs to S) $a'z = a' = za', a'a'' = z = a''a'$. We shall call such a subgroup S' a pre-inverse subgroup of S. Some properties of pre-inverse subgroups. For every a belongs to S we define $a^* = a'aa'$ it is clear that a^* belongs to $V(a)$ and $aa^* = aa', a^*a = a'a$

2. Some Relation between a' and a^*

Theorem. *For all a belongs to S: $a'^* = a'' = a^{*\prime}$.*

Proof. Since $a'^* = a''a'a''$ belongs to $S' \cap V(a')$ is subset of $S' \cap A(a') = \{a''\}$, and also we have $a^* = a^*a^{*\prime} = a^*$ and so $a = aa^*a = aa^*a^{*\prime}a^*a = aa'a^{*\prime}a'a$. Whence $a'a^{*\prime}a'$ belongs to $S' \cap A(a)$ and therefore $a'a^{*\prime}a' = a'$. It now follows that $a^{*\prime}$ belongs to $S' \cap A(a') = \{a''\}$. □

Corollary. *For all a belongs to S: $a^{**} = zaz$.*

Proof. $a^{**} = a^{*\prime}a^*a^{*\prime} = a''a'aa'a'' = zaz$. □

Corollary. *For all a belongs to S: $a^{***} = a^*$.*

Proof. $a^{***} = a^{**\prime} = a^{**} = a^{**\prime} = a'''a^{**}a''' = a'a^{**}a' = a'aa' = a^*$. □

If we define $S^* = \{a^*; a \text{ belongs to } S\}$ we see from above results that a belongs to S^* if and only if $a = aza$ so that $S^* = zSz$. The subsemigroup S^* is regular. This also gives that $a' = za'z = (zaz)'$. As z is the identity element of S' then we have S' is subset of S^*. Every pre-inverse subgroup of S is in fact a maximal subgroup; for we have the following theorem.

Theorem. $S' = H_z$.

Proof. S' is a subset of H_z which is obvious. To obtain the converse, let a belongs to H_z. Then aa' belongs to H_z and $a'a$ belongs to H_z gives $aa' = z = a'a$ whence $a^* = a'aa' = a'z = a'$ and $a = za = a^{**}$, consequently, $a = a^{**} = a'^* = a''$ belongs to S'. □

Corollary. *S^* is uniquely unit regular with group of units H_z.*

Proof. Since S is regular and $H_z = S'$ is subset of S^*. Moreover, $H_z \cap (zaz) = S' \cap A(zaz) = \{(zaz)'\}$ and $(zaz)' = a' = za'z$ belongs to S^*. It follows that S^* is uniquely unit regular with group of unit H_z. □

Theorem. *For all a, b belongs to S: $(ab)' = (a'ab)'a' = b'(abb')'$.*

Proof. $ab \cdot (a'ab)'a' \cdot ab = a \cdot a'ab(a'ab)'a'ab = aa'ab = ab$ and so $(a'ab)'b'$ belongs to $S' \cap A(ab) = (ab)'$. Similarly we can shows the other identity.

Observe now that zaz belongs to $E(S^*)$ if and only if $zazaz = zaz$. Pre-multiplying by aa' and post multiplying by $a'a$ we see that this is equivalent to $aza = a$, that is , to $a' = z$. Thus zaz belongs to $E(S^*)$ implies $a = aa'a = aa' \cdot a'a$ belongs to $\langle E \rangle$. It follows by from these observations that we have $E(S^*)$ is contain in $z \langle E \rangle z$.

Theorem. *The following statements are equivalent,* (1) *z is medial;* (2) *(for all a, b belongs to S) $(ab)' = b'a'$; (3) $E(S^*) = z \langle E \rangle z$.*

Proof. (1) Let z is medial then $z = a'$ for every a belongs to $\langle E \rangle$. It follows by previous theorem that $(ab)' = b'(a'abb')'a' = b'za' = b'a'$.

(2) Let (for all a, b belongs to S) $(ab)' = b'a'$; then we have f' belongs to E for every f belongs to E, since S' is a group it follows that $f' = z$ for every f belongs to E. Consequently, if f_1, f_2 belongs to E then by (2) we have $(f_1 f_2)' = f_2' f_1' = zz = z$, by induction we have $a' = z$ for all a belongs to E. It follows that $a = aza$ for every a belongs to $\langle E \rangle$ whence z is medial. Let (1) is true then (3) is obvious.

(3) Let $E(S^*) = z \langle E \rangle z$. If a belongs to E then zaz is idempotent so $a' = z$ and $a = aa'a = aza$, that is, z is medial. □

Theorem. *If S is orthodox then z is medial and $E(S^*) = zEz$.*

Proof. We have b^*a^* belongs to $V(ab)$ is contain in $A(ab)$. Then $ab = abb^*a^*ab = abb'a'ab$ whence $b'a'$ belongs to $S' \cap A(ab) = (ab)'$. The result follows by previous theorem. □

Corollary. *If S is orthodox then $f' = z$ for every f belongs to E.*

Observe that if we define $E^* = \{f^*; f$ belongs to $E(S)\}$, then we have S is orthodox, we have $E^* = E(S^*)$.

Theorem. *The following statements are equivalent; (1) z is middle unit;*

(2) (for all a, b belongs to S) $(ab)^{**} = a^{**}b^{**}$.

Proof. Let (1) is true then, $(ab)^{**} = zabz = zaz \cdot zbz = a^{**}b^{**}$. Let (2) is true then for all a, b belongs to S we have $a^*aby^* = a^*zabzy^* = a^*(ab)^{**}b^* = a^*a^{**}b^{**}b^* = a^* \cdot zaz \cdot zbz \cdot b^* = a^*azbb^*$ whence $ab = azb$ and so z is a middle unit. □

Theorem. *If S is orthodox then the following statements are equuivalent; (1) z is normal medial idempotent; (2) $S^* = zSz$ is inverse.*

Proof. Let (1) is true, then by our results: $E(S^*) = zEz$ which is a semilattice. Conversley let $S^* = zSz$ is inverse then z is a medial; and by (2) it is normal. □

Example. Let B be a rectangular band and let B^1 be obtained from B by adjoining an identity element 1. Let $S = \mathbb{Z} \times B^1 \times \mathbb{Z}$ and define on S the multiplication $(m, a, p)(n, b, q) = (m_k + n, ab, p + q_k)$, where for a fixed integer $k > 1$, m_k is the greated multiple of k that is less then or equal to m. From simple investigation it is seen that S is a semigroup, also simple calculations reveal that the set of pre-inverse of (m, a, p) belongs to S is

$$A(m,a,p) = \begin{cases} \{(n,b,q);\ n_k = -m_k,\ q_k = -p_k\} \text{ if } \ a \neq 1, \\ \{(n,1,q);\ n_k = -m_k,\ q_k = -p_k\} \text{ if } a = 1, \end{cases}$$

and the set of inverses of (m, a, p) belongs to S is

$$V(m,a,p) = \begin{cases} \{(n,b,q);\ n_k = -m_k,\ y \neq 1,\ q_k = -p_k\} \text{ if } \ a \neq 1, \\ \{(n,1,q);\ n_k = -m_k,\ q_k = -p_k\} \text{ if } a = 1. \end{cases}$$

The set of idempotents of S is $E = \{(m, a, p);\, m_k = 0 = p_k\}$ and so S is orthodox. For every (m, a, p) belongs to S we define $(m, a, p)' = (-m_k, a, -p_k)$. The identity element of S' is $z = (0, 1, 0)$. By observation z is a middle unit. Now $(m, a, p)^* = (m, a, p)'(m, a, p)(m, a, p)' = (-m_k,\ a,\ -p_k)$. After simple calculations this gives $\{(m, a, p)(n, b, q)\}^* = (-m_k{-}n_k,\, ab,\, -p_k{-}q_k)$, $(n, a, q)^*(m, a, p)^* = (-m_k - n_k,\ ba,\ -p_k - q_k)$. Now $ab \neq ba$ for different a, b belongs to B, so z is not medial normal.

Let B be a band with middle unit z and let $End\ B$ be the monoid of the endomorphisms on B. Define $End_z\ B = \{f$ belongs to $End\ B$; f preserves z and $Im\ f = zBz\}$. Then $End_z\ B$ is a subsemigroup of $End\ B$. Consider the mapping $\alpha : B \longrightarrow B$ given by $\alpha(a) = zaz$ for every a belongs to B. Since z is a middle unit, we have α belongs to $End\ B$. Moreover, α clearly preserves z and $Im\ \alpha = zBz$. Hence α belongs to $End_z\ B$, α is the identity element of $End_z\ B$, then (for all a belongs to $B)g\alpha(a) = g(zaz) = zg(a)z = \alpha g(a)$, the last equality following from the fact that $\alpha\,|_{zBz} = id_{zBz}$. Hence $g\alpha = \alpha g = g$ and so $End\ B$ is a monoid.

Theorem. *Let B be a band with middle unit z and let G be a group. Let $\lambda : G \longrightarrow End_z$ described by $h \longrightarrow \lambda_h$, be a 1-preserving morphism. On the set $[B;\ G]_\lambda = \{(a, h, x)$ belongs to $Bz \times G \times zB; \lambda_h(x) = \lambda_1(a)$ defines the multification $(a, h, x)(b, h', y) = (a\lambda_h(b), hh', \lambda_{h^{-1}}(x)y)$. Then $[B; G]_\lambda$ is an orthodox semigroup with an associate subgroup of which the identity element $(z, 1, z)$ is a middle unit. Moreover, we have $E([B; G]_\lambda) \simeq B$ and $H_{(z,1,z)} \simeq G$.*

Proof. The multiplication on $[B; G]_\lambda$ is well define, for we have $a\lambda_h(b)$ belongs to $Bz \cdot zBz$ contain in Bz and $\lambda_{(h')^{-1}}xy$ belongs to $zBz \cdot zB$ contained in zB, with $\lambda_{hh'}[\lambda_{(h')}(x)\, y] = \lambda_h(x)\, \lambda_{(h')}(y)] = \lambda_1(a)\, \lambda_h[\lambda_1(b)] = \lambda_1[a\lambda_h(b)]$, by investigation shows that it is also associative. Also the semigroup $[B; G]_\lambda$ is regular that is $(a, h, x)\left(z, h^{-1}, z\right)(a, h, x) = \left(x\lambda_h(z), hh^{-1}, \lambda_h(x)\, z\right)(a, h, x) = (az, 1, \lambda_h(x)\, z)\,(a, h, x) = (a, h, x)$. The set of idempotents of $[B; G]_\lambda$ is $[B; G]_\lambda = \{(a, 1, x);\ za = xz\}$ and that the idempotent $(z, 1, z)$ is a middle unit of $[B; G]_\lambda$. If now $(a, 1, x)$ and $(b, 1, y)$ are idempotents then $(a, 1, x)\,(b, 1, y) = (a\lambda_1(b), 1, \lambda_1(x)\, y) = (azbz, 1, zxzy) = (ab, 1, xy)$, with $zab = xzb = xyz$. Hence we see that $[B; G]_\lambda$ is orthodox. Moreover, we have $E([B; G]_\lambda) \simeq B$ for, consider the mapping $\varphi : E([B; G]_\lambda) \longrightarrow B$ given by $\varphi(a, 1, x) = ax$. Now φ is surjective since for every e belongs to B we have $(ez, 1, ze)$ belongs to $E([B; G]_\lambda)$ with $\varphi(ez, 1, ze) = ez \cdot ze = e$. To see that φ is also injective, suppose that $(a, 1, x)$ and $(b, 1, y)$ are idempotent with $\varphi(a, 1, x) = \varphi(b, 1, y)$. Then $ax = by$ with $za = xz$ and $zb = yz$. It follows that $a = aza = axz = byz = bzb = b$ and similarly $x = y$; now to show that φ is morphism; for $\varphi[(a, 1, x)\,(b, 1, y)] = \varphi(ab, 1, xy) = abxy$ and $abxy = azbxzy = ayzay = ay = azayzy = axzby = axby = \varphi(a, 1, x)\varphi(b, 1, y)$. Also (b, h', y) belongs to $A(a, h, x)$ if and only if $\lambda_h(b)$ belongs to $A(a), h' = h^{-1}, \lambda_{h^{-1}}(y)$ belongs to $A(x)$. If we define by $(a, h, x)' = (z, h^{-1}, z)$, we see that $[B; G]'_\lambda$ is a pre-inverse subgroup, with identity element $(z, 1, z)$, that is isomorphic to G. It follows from second theorem that $G \simeq H_{(z,1,z)}$. ☐

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