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THE CHARACTERIZATION FOR THE ROTATION-INVARIANT SEGAL-BARGMANN SPACE

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Abstract: The Segal-Bargmann space is the set of holomorphic functions on $\mathbb{C}^d$ that are square-integrable with respect to the complex Gaussian measure $\mu_t(z)dz = (\pi t)^{-d} e^{-z^2/t} dz$, and is denoted by $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$.

In this work, we consider the rotation-invariant subspace of the Segal-Bargmann space. The complex rotation-invariant function F is determined by its values on $(z, 0, \ldots, 0) \cong \mathbb{C}^1$ and it is a complex even function. Conversely, any even holomorphic function on $\mathbb{C}^1$ has an extension to a complex rotation-invariant holomorphic function on $\mathbb{C}^d$. Thus the space of complex rotation-invariant functions in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$ with respect to some non-Gaussian measure. This non-Gaussian measure is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$. We give a characterization for a complex function to be in the rotation-invariant subspace of the Segal-Bargmann space.

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1. Introduction

The Segal-Bargmann space is the set of holomorphic functions on $\mathbb{C}^d$ that are square-integrable with respect to the complex Gaussian measure $\mu_t(z)dz = (\pi t)^{-d}e^{-z^2/t}dz$, and is denoted by $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$. Here we use notation $z^2 = z_1^2 + z_2^2 + \cdots + z_d^2$ for $z = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$. The Segal-Bargmann space is unitarily equivalent to the Hilbert space $L^2(\mathbb{R}^d, \rho_t)$ where $\rho_t(x) = (2\pi t)^{-d/2}e^{-x^2/2t}dz$ by the unitary map B_t given by the formula

$$(B_t f)(z) = \int_{\mathbb{R}^d} f(x) \frac{e^{-(z-x)^2/2t}}{(2\pi t)^{d/2}}\, dx$$

for all $f \in L^2(\mathbb{R}^d, \rho_t)$ and $z \in \mathbb{C}^d$.

In this work, we consider the rotation-invariant subspace of the Segal-Bargmann space. The complex rotation-invariant function F is determined by its values on $(z, 0, \ldots, 0) \cong \mathbb{C}^1$ and it is a complex even function. Conversely, any even holomorphic function on $\mathbb{C}^1$ has an extension to a complex rotation-invariant holomorphic function on $\mathbb{C}^d$. Thus the space of complex rotation-invariant functions in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$ with respect to some non-Gaussian measure. This non-Gaussian measure is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$. We give a characterization for a complex function to be in the rotation-invariant subspace of the Segal-Bargmann space.

2. Main Results

Denote by $SO(d)$ the set of $d \times d$ real orthogonal matrices with determinant one and by $SO(d, \mathbb{C})$ the set of $d \times d$ complex orthogonal matrices with determinant one. Define a bilinear form $(\cdot, \cdot)$ on $\mathbb{F}^d$ by

$$(x, y) = x_1y_1 + x_2y_2 + \cdots + x_dy_d$$

for all $x, y \in \mathbb{F}^d$ where $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$. Then the elements of $SO(d)$ and $SO(d, \mathbb{C})$ preserve the bilinear form on $\mathbb{R}^d$ and $\mathbb{C}^d$, respectively.

Definition 1. Let F be a function on $\mathbb{F}^d$ and let G be a group of $d \times d$ matrices. We say that F is *G-invariant* if

$$F(Ax) = F(x) \quad \text{for all } A \in G \text{ and all } x \in \mathbb{F}^d.$$

Notice that if F is an $SO(d)$-invariant holomorphic function on $\mathbb{C}^d$, then by analytic continuation it is $SO(d, \mathbb{C})$-invariant.

Denote by $\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ the set of all $SO(d,\mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, and $\mathcal{H}(\mathbb{C})^e$ the set of all holomorphic even functions on $\mathbb{C}$.

For any $d \geq 2$, the map $\psi\colon \mathcal{H}(\mathbb{C})^e \to \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ defined by

$$\psi(f)(z) = f\left(\sqrt{(z,z)}\right)$$

for all $f \in \mathcal{H}(\mathbb{C})^e$ and all $z = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$ is a linear isomorphism whose inverse is given by

$$\phi(F)(w) = F(w, 0, \ldots, 0)$$

for all $F \in \mathcal{H}(\mathbb{C})^{SO(d,\mathbb{C})}$ and all $w \in \mathbb{C}$.

Denote by $\mathcal{B}(\mathbb{C}^d)$ the Borel σ-algebra in $\mathbb{C}^d$ and by $\mathcal{B}(\mathbb{C})$ the Borel σ-algebra in $\mathbb{C}$ and define $\Phi_i\colon (\mathbb{C}^d, \mathcal{B}(\mathbb{C}^d), \mu_t) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$, $i = 1, 2$, to be the branch of $\sqrt{(z,z)}$ with a smaller and larger argument respectively and for each $E \in \mathcal{B}(\mathbb{C})$ define

$$\lambda_i(E) = \mu_t(\Phi_i^{-1}(E)),$$

and let $\lambda = (\lambda_1 + \lambda_2)/2$. In [10] we show that λ is absolutely continuous with respect to Lebesgue measure with a non-Gaussian density.

Define

$$\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})} = \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \cap L^2(\mathbb{C}^d, \mu_t)$$

and

$$\mathcal{H}L^2(\mathbb{C}, \lambda)^e = \mathcal{H}(\mathbb{C})^e \cap L^2(\mathbb{C}, \lambda).$$

Then they are Hilbert spaces and the map $\psi\colon \mathcal{H}L^2(\mathbb{C}, \lambda)^e \to \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ is unitary.

Lemma 2. *The set $\{w^{2n}\}_{n=0}^{\infty}$ is an orthogonal basis for the Hilbert space $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$. In particular, the set $\{(z,z)^{2n}\}_{n=0}^{\infty}$ is an orthogonal basis for the Hilbert space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$. Moreover, for any $n \in \mathbb{N} \cup \{0\}$*

$$\|w^{2n}\|^2_{L^2(\mathbb{C},\lambda)} = \frac{a_0}{\Gamma(d/2)} 2^{2n+d} t^{2n} \Gamma(n+1)\Gamma(n+d/2),$$

where a_0 is a constant independent of n and d. In particular

$$\|w^{2(n+1)}\|^2_{L^2(\mathbb{C},\lambda)} = 4t^2(n+1)(n+d/2)\|w^{2n}\|^2_{L^2(\mathbb{C},\lambda)}. \tag{1}$$

Proof. The proof of this lemma uses the same technique as in [1], [4] and [6] and will be omitted. □

Define the operators A and D on $\mathcal{H}(\mathbb{C})^e$ as follows

$$(Af)(w) = \frac{w^2}{t^2} f(w),$$

$$(Df)(w) = \frac{d^2}{dw^2} f(w) + \frac{d-1}{w}\frac{d}{dw} f(w).$$

For each $n \in \mathbb{N}$, we define

$$\mathcal{D}(A^n) = \{f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e \mid A^n f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e\}$$

and

$$\mathcal{D}(D^n) = \{f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e \mid D^n f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e\}.$$

Note that A^n and D^n are densely defined on $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ because their domains contain the set of complex even polynomials, which is dense in $\mathcal{H}L^2(\mathbb{C},\lambda)^e$.

Theorem 3. *For any $n \in \mathbb{N}$, $\mathcal{D}(A^n) = \mathcal{D}(D^n)$.*

Proof. Let $f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e$ and write $f(w) = \sum_{n=0}^{\infty} a_n w^{2n}$ for all $w \in \mathbb{C}$. It follows from (1) that

$$\begin{aligned}\|Af\|^2_{L^2(\mathbb{C},\lambda)} &= \frac{1}{t^4}\sum_{n=0}^{\infty} |a_n|^2 \|w^{2n+2}\|^2_{L^2(\mathbb{C},\lambda)} \\ &= \frac{1}{t^2}\sum_{n=0}^{\infty} |a_n|^2 4(n+1)(n+d/2)\|w^{2n}\|^2_{L^2(\mathbb{C},\lambda)} \qquad (2)\end{aligned}$$

and

$$\begin{aligned}\|Df\|^2_{L^2(\mathbb{C},\lambda)} &= \sum_{n=0}^{\infty} |a_n|^2 \|Dw^{2n}\|^2_{L^2(\mathbb{C},\lambda)} \\ &= \sum_{n=0}^{\infty} |a_n|^2 4n^2(2n+d-2)^2 \|w^{2n-2}\|^2_{L^2(\mathbb{C},\lambda)} \\ &= \frac{1}{t^2}\sum_{n=0}^{\infty} |a_n|^2 4n(n-1+d/2)\|w^{2n}\|^2_{L^2(\mathbb{C},\lambda)}. \qquad (3)\end{aligned}$$

Since $4(n+1)(n+d/2) = 4n(n-1+d/2) + 8n + 2d$, it follows that

$$\|Af\|^2_{L^2(\mathbb{C},\lambda)} = \|Df\|^2_{L^2(\mathbb{C},\lambda)} + \frac{8}{t^2}\sum_{n=0}^{\infty} |a_n|^2 n \|w^{2n}\|^2_{L^2(\mathbb{C},\lambda)} + \frac{2d}{t^2}\|f\|^2_{L^2(\mathbb{C},\lambda)}.$$

Thus $\|Af\|^2_{L^2(\mathbb{C},\lambda)} < \infty$ implies that $\|Df\|^2_{L^2(\mathbb{C},\lambda)} < \infty$. On the other hand, if $\|Df\|^2_{L^2(\mathbb{C},\lambda)} < \infty$, then by (3) $\frac{8}{t^2}\sum_{n=0}^{\infty} |a_n|^2 n \|w^{2n}\|^2_{L^2(\mathbb{C},\lambda)} \leq \|Df\|^2_{L^2(\mathbb{C},\lambda)}$. Hence $\|Af\|^2_{L^2(\mathbb{C},\lambda)} < \infty$, so we have $\mathcal{D}(A) = \mathcal{D}(D)$. Now assume that $\mathcal{D}(A^k) = \mathcal{D}(D^k)$. Thus we have that

$$f \in \mathcal{D}(D^{k+1}) \Longleftrightarrow D^{k+1} f \in L^2(\mathbb{C},\lambda)$$

$$\begin{aligned} &\iff Df \in \mathcal{D}(D^k) \\ &\iff Df \in \mathcal{D}(A^k) \\ &\iff w^{2k} Df \in L^2(\mathbb{C}, \lambda). \end{aligned}$$

Next we will show that $w^{2k}Df \in L^2(\mathbb{C},\lambda)$ if and only if $D(w^{2k}f) \in L^2(\mathbb{C},\lambda)$. Since $f(w) = \sum_{n=0}^{\infty} a_n w^{2n}$ for all $w \in \mathbb{C}$,

$$\|w^{2k}Df\|^2_{L^2(\mathbb{C},\lambda)} = \sum_{n=0}^{\infty} |a_n|^2 4n^2(2n+d-2)^2 \|w^{2(n+k-1)}\|^2$$

and

$$\|D(w^{2k}f)\|^2_{L^2(\mathbb{C},\lambda)} = \sum_{n=0}^{\infty} |a_n|^2 4(n+k)^2(2n+2k+d-2)^2 \|w^{2(n+k-1)}\|^2.$$

Thus $\|D(w^{2k}f)\|^2_{L^2(\mathbb{C},\lambda)} < \infty$ implies that $\|w^{2k}(Df)\|^2_{L^2(\mathbb{C},\lambda)} < \infty$. Since

$$(n+k)^2(2n+2k+d-2)^2 \leq Mn^2(2n+d-2)^2,$$

where M is a constant depending on K, $\|w^{2k}(Df)\|^2_{L^2(\mathbb{C},\lambda)} < \infty$ implies that $\|D(w^{2k}f)\|^2_{L^2(\mathbb{C},\lambda)} < \infty$. If follows that

$$\begin{aligned} f \in \mathcal{D}(D^{k+1}) &\iff D(w^{2k}f) \in L^2(\mathbb{C},\lambda) \\ &\iff w^{2k}f \in \mathcal{D}(D) \\ &\iff w^{2k}f \in \mathcal{D}(A) \\ &\iff w^{2k+2}f \in L^2(\mathbb{C},\lambda) \\ &\iff f \in \mathcal{D}(A^{k+1}). \end{aligned}$$

Hence, $\mathcal{D}(A^k) = \mathcal{D}(D^k)$ for all $k \in \mathbb{N}$. □

Theorem 4. *The operators A and D are densely defined on $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ with $A^* = D$.*

Proof. Let $g \in \mathcal{D}(A^*)$. Then there exists $h \in \mathcal{H}L^2(\mathbb{C},\lambda)^e$ such that $\langle Af, g\rangle = \langle f, h\rangle$ for all $f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e$. Write

$$f(w) = \sum_{n=0}^{\infty} a_n w^{2n}, \quad g(w) = \sum_{n=0}^{\infty} b_n w^{2n} \quad \text{and} \quad h(w) = \sum_{n=0}^{\infty} c_n w^{2n},$$

for all $w \in \mathbb{C}$. Then $\langle Af, g\rangle = \langle f, h\rangle$ implies

$$\frac{1}{t^2}\sum_{n=0}^{\infty} a_n \bar{b}_{n+1} \|w^{2n+2}\|^2_{L^2((\mathbb{C},\lambda)} = \sum_{n=0}^{\infty} a_n \bar{c}_n \|w^{2n}\|^2_{L^2(\mathbb{C},\lambda)}.$$

Thus by using (1) we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} c_n &= \frac{b_{n+1}}{t^2}\frac{\|w^{2n+2}\|^2}{\|w^{2n}\|^2} \\ &= \frac{4b_{n+1}t^2(n+1)(n+d/2)\|w^{2n}\|^2}{t^2\|w^{2n}\|^2} \\ &= b_{n+1}(2n+2)(2n+d). \end{aligned}$$

Hence $h(w) = \displaystyle\sum_{n=0}^{\infty} b_{n+1}D(w^{2(n+1)}) = (Dg)(w)$ for all $w \in \mathbb{C}$ and therefore $\mathcal{D}(A^*) \subseteq \mathcal{D}(D)$. Let $f \in \mathcal{D}(A)$ and $g \in \mathcal{D}(D)$. If $f(w) = \sum_{n=0}^{\infty} a_n w^{2n}$ and $g(w) = \sum_{n=0}^{\infty} b_n w^{2n}$ for all $w \in \mathbb{C}$, then

$$\langle Af, g\rangle = \frac{1}{t^2}\sum_{n=0}^{\infty} a_n\overline{b}_{n+1}\|w^{2n+2}\|^2 = \frac{1}{t^2}\sum_{n=0}^{\infty} a_n\overline{b}_{n+1}4t^2(n+1)(n+\frac{d}{2})\|w^{2n}\|^2$$

and

$$\langle f, Dg\rangle = \sum_{n=0}^{\infty} a_n\overline{b}_{n+1}2(n+1)(2n+d)\|w^{2n}\|^2 = \langle Af, g\rangle\,.$$

Thus $D(g) = A^*(g)$ and $g \in \mathcal{D}(A^*)$. Hence $\mathcal{D}(D) = \mathcal{D}(A^*)$ and $A^* = D$. □

Proposition 5. *For any $f \in \mathcal{H}(\mathbb{C})$, if $w^2 f \in L^2(\mathbb{C}, \lambda)$, then $f \in L^2(\mathbb{C}, \lambda)$. In particular, if $Af \in \mathcal{H}L^2(\mathbb{C}, \lambda)^e$ then $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)^e$.*

Proof. Let $\mathbb{D} = \{w \in \mathbb{C} \colon |w| \le 1\}$. Then $\|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \le \|w^2 f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} < \infty$. Next, we will show that there is a constant $C > 0$ such that

$$\|f\|_{L^2(\mathbb{C},\lambda)} \le C\|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

Let $w \in \mathbb{D}$. Denote by $\mathcal{C}(w)$ the annulus $\{v \in \mathbb{C} \colon 2 \le |v - w| \le 3\}$. For any $v \in \mathcal{C}(w)$, we use the polar coordinates with the origin at w so that $v - w = re^{i\theta}$. Since $f \in \mathcal{H}(\mathbb{C})$, we expand f as a power series around $v = w$:

$$f(v) = f(w) + \sum_{n=1}^{\infty} a_n(v-w)^n.$$

Hence,

$$\int_{\mathcal{C}(w)} f(v)dv = 5\pi f(w) + \sum_{n=1}^{\infty}\int_2^3\int_0^{2\pi} a_n r^{n+1}e^{i\theta}d\theta dr = 5\pi f(w).$$

Thus

$$f(w) = \frac{1}{5\pi}\int_{\mathcal{C}(w)} f(v)dv$$

$$= \frac{1}{5\pi} \int_{\mathbb{C}-\mathbb{D}} \chi_{\mathcal{C}(w)}(v) \frac{1}{\Lambda(v)} f(v)\Lambda(v)dv$$
$$= \frac{1}{5\pi} \left\langle \chi_{\mathcal{C}(w)} \frac{1}{\Lambda}, f \right\rangle_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

By Cauchy-Schwarz inequality we have that

$$|f(w)| \leq \frac{1}{5\pi} \|\chi_{\mathcal{C}(w)} \frac{1}{\Lambda}\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

Since Λ is strictly positive and continuous on $\mathcal{C}(w)$, $\frac{1}{\Lambda}$ is bounded on $\mathcal{C}(w)$. Thus the first L^2-norm is finite and

$$\frac{1}{5\pi} \|\chi_{\mathcal{C}(w)} \frac{1}{\Lambda}\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \leq \frac{1}{5\pi} \|\chi_{\mathcal{C}^*} \frac{1}{\Lambda}\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} < \infty,$$

where $\mathcal{C}^* = \{v \in \mathbb{C} \colon 1 < |z| < 4\}$ which $\mathcal{C}(w) \subset \mathcal{C}^*$ for all $w \in \mathbb{D}$. It follows that there exists a constant c such that

$$|f(w)| \leq c\|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

Hence

$$\int_{\mathbb{C}} |f(w)|^2 d\lambda(w) = \int_{\mathbb{D}} |f(w)|^2 d\lambda(w) + \int_{\mathbb{C}-\mathbb{D}} |f(w)|^2 d\lambda(w)$$
$$\leq c^2 \|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \lambda(\mathbb{D}) + \|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)}$$
$$\leq C\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)},$$

so the proposition is proved. □

Theorem 6. *$f \in \cap_{n=1}^{\infty} \mathcal{D}(D^n)$ if and only if for all $n \in \mathbb{N}$ there is $C > 0$ such that*

$$|f(w)|^2 \leq \frac{Ce^{|w|^2/t}}{(1+|w|^{d-1})(1+|w|^{2n})^2} \qquad \textit{for all } w \in \mathbb{C}. \tag{4}$$

Proof. ($\Rightarrow$) Note that

$$f \in \cap_{n=1}^{\infty} \mathcal{D}(D^n) \Longleftrightarrow f \in \mathcal{D}(A^n) \text{ for all } n \in \mathbb{N},$$
$$\Longleftrightarrow w^{2n} f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e \text{ for all } n \in \mathbb{N}.$$

It follows from Theorem 10 in [10] that, for all $f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e$,

$$|f(w)| \leq \frac{e^{|w|^2/2t}\|w^{2n}f\|_{L^2(\mathbb{C},\lambda)}}{(1+|w|^{d-1})^{1/2}}.$$

Then for all $n \in \mathbb{N}$,

$$w^{2n} f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e \Longrightarrow |w^{2n} f(w)| \leq \frac{e^{|w|^2/2t}\|w^{2n}f\|_{L^2(\mathbb{C},\lambda)}}{(1+|w|^{d-1})^{1/2}}$$

$$\implies |f(w)| \leq \frac{e^{|w|^2/2t}(\|f\|_{L^2(\mathbb{C},\lambda)} + \|w^{2n}f\|_{L^2(\mathbb{C},\lambda)})}{(1+|w|^{d-1})^{1/2}(1+|w|^{2n})}$$
$$\implies |f(w)|^2 \leq \frac{Ce^{|w|^2/t}}{(1+|w|^{d-1})(1+|w|^{2n})^2}.$$

($\Leftarrow$) Assume that (4) holds. Following Corollary 8 of [10] we know that the L^2-norm with respact to λ is equivalent to the L^2-norm with respect to the measure $\beta(w)dw$ where

$$\beta(w) = \frac{e^{-|w|^2/t}}{\pi t}|w|^{d-1}.$$

Thus

$$\begin{aligned}\|w^{2n}f\|^2_{L^2(\mathbb{C},\beta)} &= \frac{1}{\pi t}\int_{\mathbb{C}} |w^{2n}f(w)|^2 e^{-|w|^2/t}|w|^{d-1}dw \\ &\leq \frac{C}{\pi t}\int_{\mathbb{C}} |w|^{4n}\frac{e^{|w|^2/t}}{(1+|w|^{d-1})(1+|w|^{2n+2})^2}e^{-|w|^2/t}|w|^{d-1}dw \\ &= \frac{C}{\pi t}\int_{\mathbb{C}} \frac{|w|^{4n+d-1}}{(1+|w|^{d-1})(1+|w|^{2n+2})^2}dw < \infty.\end{aligned}$$

Hence $w^{2n}f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e$ for all $n \in \mathbb{N}$. ∎

Define the operator $\mathcal{T}\colon \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \to \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ by

$$(\mathcal{T}F)(z) = (z,z)F(z)$$

for all $z \in \mathbb{C}^d$ where $(z,z) = z_1^2 + \cdots + z_d^2$.

Lemma 7. *For any $F \in \mathcal{H}(C^d)^{SO(d,\mathbb{C})}$ and any $n \in \mathbb{N}$,*

$$\|\mathcal{T}^n(F)\|_{L^2(\mathbb{C}^d,\mu_t)} = t^{2n}\|A^n(\phi(F))\|_{L^2(\mathbb{C},\lambda)}.$$

Proof. Let $F \in \mathcal{H}(C^d)^{SO(d,\mathbb{C})}$ and $f = \phi(F)$. Then for all $w \in \mathbb{C}$, $f(w) = F(w,0,\dots,0)$ and for all $n \in \mathbb{N}$

$$\begin{aligned}\|\mathcal{T}^n(F)\|^2_{L^2(\mathbb{C}^d,\mu_t)} &= \int_{\mathbb{C}^d} |(\mathcal{T}^nF)(z)|^2\mu_t(z)dz \\ &= \int_{\mathbb{C}} |\phi(\mathcal{T}^nF)(w)|^2\lambda(w)dw \\ &= \int_{\mathbb{C}} |(\mathcal{T}^nF)(w,0,\dots,0)|^2\lambda(w)dw \\ &= \int_{\mathbb{C}} |w^{2n}F(w,0,\dots,0)|^2\lambda(w)dw \\ &= \int_{\mathbb{C}} |w^{2n}f(w)|^2\lambda(w)dw\end{aligned}$$

$$= t^{2n}\|A^n(f)\|^2_{L^2(\mathbb{C},\lambda)},$$

so the lemma is proved. □

Theorem 8. *Let $F \in \mathcal{H}(C^d)^{SO(d,\mathbb{C})}$. Then $\mathcal{T}^n(F) \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ for all $n \in \mathbb{N}$ if and only if for all $n \in \mathbb{N}$ there is $C > 0$ such that*

$$|F(z)|^2 \le \frac{Ce^{|(z,z)|/t}}{\Big(1+|(z,z)|^n\Big)^2\Big(1+|(z,z)|^{(d-1)/2}\Big)} \qquad \text{for all } z \in \mathbb{C}^d. \qquad (5)$$

Proof. Let $F \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and $f = \phi(F)$.

($\Rightarrow$) Suppose $\mathcal{T}^n(F) \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ for all $n \in \mathbb{N}$. Then $A^n(f) \in \mathcal{H}L^2(\mathbb{C}, \lambda)^e$ for all $n \in \mathbb{N}$. Thus by Theorem 6, for all $n \in \mathbb{N}$ there is a constant C such that

$$|f(w)|^2 \le \frac{Ce^{|w|^2/t}}{(1+|w|^{2n})^2(1+|w|^{d-1})} \qquad \text{for all } z \in \mathbb{C}^d.$$

This implies that

$$|F(z)|^2 \le \frac{Ce^{|(z,z)|/t}}{\Big(1+|(z,z)^n|\Big)^2\Big(1+|(z,z)|^{(d-1)/2}\Big)}$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{C}^d$.

($\Leftarrow$) Assume that (5) holds. Then

$$|f(w)|^2 \le \frac{Ce^{|w|^2/t}}{(1+|w|^{2n})^2(1+|w|^{d-1})} \qquad \text{for all } w \in \mathbb{C}.$$

By Theorem 6, it follows that $A^n(f) \in \mathcal{H}L^2(\mathbb{C}, \lambda)^e$ for all $n \in \mathbb{N}$, and that $\mathcal{T}^n(F) \in \mathcal{H}L^2(\mathbb{C}^d, \lambda)^{SO(d,\mathbb{C})}$. □

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