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FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS AND FUZZY MAPPINGS

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Abstract: In this paper, a common fixed point theorem for a pair of multi-valued mappings which satisfy a generalized ϕ-weakly contractive condition in a complete metric space setting is proved. A fixed point theorem is also proved when the mapping is a non-self mapping with domain a nonempty closed subset of a complete and convex metric space. A coincidence point theorem concerning a single-valued mapping f and a pair of ϕ-weakly contractive multi-valued mappings with respect to f is also proved. Next two theorems deal with fuzzy mappings in a metric linear space setting: the first one is a common fixed point theorem for a pair of fuzzy mappings, a fuzzy version of the result obtained for a pair of multi-valued mappings, and the second one is a coincidence point theorem concerning a ϕ-weakly contractive fuzzy mapping with respect to mapping f. These results extend the work of several authors.

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1. Introduction

In this paper, we consider some fixed point theorems for ϕ-weakly contractive multi-valued mappings and generalized ϕ-weakly contractive multi-valued mappings. Also fuzzy version of some of these results are presented. Al-

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ber and Guerre-Delabriere (cf. [1]) first introduced the ϕ-weakly contractive single-valued mappings. Zhang and Song (cf. [21]) introduced the generalized ϕ-weakly contractive single-valued mappings. Bae (cf. [6]) proved fixed point theorems for weakly contractive multi-valued mappings with inwardness or weakly inwardness condition, but our work in this paper is in another direction. Also the definition of weakly contractive mappings that appeared in [7], more precisely called (θ, δ)-weak contraction, is different from the type considered in this paper. In [2], Azam and Beg considered a fixed point theorem of such weakly contractive fuzzy mappings. We have discussed this type of mappings separately in [9]. Also in Section 2, we mention the different conditions imposed on ϕ by different researchers. We have used stronger condition on ϕ than needed, but the proof go through when ϕ is just lower semi-continuous. In Theorem 3.4, we proved a common fixed point theorem for a pair of generalized ϕ-weakly contractive multi-valued mappings in a complete metric space setting. Theorem 3.6 deals with the existence of a fixed point of such a non-self mapping in a complete and convex metric space setting. Both the theorems extend the work of [21]. Theorem 3.7 concerns the existence of a point of coincidence of a single-valued mapping f and a pair of ϕ-weakly contractive multi-valued mappings with respect to f. This theorem extend the work of Beg and Abbas (cf. [8, Theorem 2.1]), Azam and Shakeel (cf. [3, Theorem 2.3]). In Theorem 4.1, we established a fixed point theorem for a pair of fuzzy mappings satisfying a generalized ϕ-weakly contractive condition, which extend the work of Azam and Beg (cf. [2, Theorem 4.2]). In Theorem 4.2, a coincidence point theorem has been established for a ϕ-weakly contractive fuzzy mapping with respect to a mapping f. A different type of generalization of ϕ-weakly contractive mappings like that of Dutta and Choudhury (cf. [13]) was also considered by us and the results are under publication (cf. [10]).

2. Basic Definitions and Lemmas

In this section first we give the following basic definitions and lemmas for multi-valued mappings, and then that for the fuzzy mappings. (X, d) always represents a metric space, H represents the Haudorff distance induced by the metric d, $K(X)$ the family of nonempty compact subsets of X. Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X, and let $T : X \to \mathcal{P}(X)$ be a multi-valued mapping. An element $x \in X$ such that $x \in T(x)$ is called a fixed point of T. We denote by $Fix(T)$ the set of all fixed points of T, i.e.,

$$Fix(T) = \{x \in X : x \in T(x)\}.$$

Note that, x is a fixed point of a multi-valued mapping T if and only if $d(x, T(x)) = 0$, whenever $T(x)$ is a closed subset of X.

Lemma 2.1. (cf. [17]) *Let A and B be nonempty compact subsets of a metric space (X, d). If $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Lemma 2.2. *$d(x, A) \leq d(x, y) + d(y, A)$ for each $x, y \in X$ and $A \in K(X)$.*

Definition 2.3. Let (X, d) be a complete metric space and $f : X \to X$ be a self-mapping. Then a mapping $T : X \to K(X)$ is said to be ϕ-weakly contractive with respect to f if for all $x, y \in X$

$$H(T(x), T(y)) \leq d(f(x), f(y)) - \phi(d(f(x), f(y))),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$. If f is the identity mapping on X, then the mapping $T : X \to K(X)$ satisfying the above inequality is said to be ϕ-weakly contractive.

Definition 2.4. Let (X, d) be a complete metric space and $f : X \to X$ be a self-mapping. Then a pair of multi-valued mappings $T, S : X \to K(X)$ is said to be ϕ-weakly contractive with respect to f if for all $x, y \in X$

$$H(T(x), S(y)) \leq d(f(x), f(y)) - \phi(d(f(x), f(y))),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$. If f is the identity mapping on X, then the pair $T, S : X \to K(X)$ satisfying the above inequality is said to be ϕ-weakly contractive.

Definition 2.5. Let (X, d) be a complete metric space and $f : X \to X$ be a self-mapping. Then a pair of multi-valued mappings $T, S : X \to K(X)$ is said to be generalized ϕ-weakly contractive with respect to f if for all $x, y \in X$

$$H(T(x), S(y)) \leq M(f(x), f(y)) - \phi(M(f(x), f(y))),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$,

$$M(x, y) = \max\{d(x, y), d(T(x), x), d(S(y), y), \frac{1}{2}[d(y, T(x)) + d(x, S(y))]\}.$$

If f is the identity mapping on X, then the pair $T, S : X \to K(X)$ satisfying the above inequality is said to be generalized ϕ-weakly contractive.

Definition 2.6. Let $f : X \to X$ be a self-mapping and $T : X \to CB(X)$ be a multi-valued mapping. Then a point $u \in X$ is said to be a coincidence point of f and T if $f(u) \in T(u)$.

A real linear space X with a metric d is called a metric linear space if $d(x+z, y+z) = d(x, y)$ and $\alpha_n \to \alpha$, $x_n \to x \implies \alpha_n x_n \to \alpha x$. Let (X, d) be a metric linear space. A fuzzy set A in a metric linear space X is a function from X into $[0, 1]$. If $x \in X$, the function value $A(x)$ is called the grade of membership of x in A. The α-level set of A, denoted by A_α, is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1], \quad A_0 = \overline{\{x : A(x) > 0\}}.$$

Here $\bar{B}$ denotes the closure of the (non-fuzzy) set B.

Definition 2.7. A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup_{x \in X} A(x) = 1$.

When A is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, A is identified with an approximation of x_0. For $x \in X$, let $\{x\} \in W(X)$ with membership function equal to the characteristic function χ_x of the set $\{x\}$.

Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in X and $W(X)$ be a sub-collection of all approximate quantities.

Definition 2.8. Let $A, B \in W(X)$, $\alpha \in [0, 1]$. Then we define

$$p_\alpha(A, B) = \inf_{x \in A_\alpha, \, y \in B_\alpha} d(x, y), \quad p(A, B) = \sup_\alpha p_\alpha(A, B),$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha), \quad D(A, B) = \sup_\alpha D_\alpha(A, B).$$

where $H(A_\alpha, B_\alpha) = \max\{\sup\{d(u, B_\alpha) : u \in A_\alpha\}, \sup\{d(u, A_\alpha) : u \in B_\alpha\}\}$ is the Hausdorff distance induced by the metric d.

The function $D_\alpha(A, B)$ is called an α-distance between $A, B \in W(X)$, and D a metric on $W(X)$. We note that p_α is a non-decreasing function of α and thus $p(A, B) = p_1(A, B)$. In particular if $A = \{x\}$, then $p(\{x\}, B) = p_1(x, B) = d(x, B_1)$. Next we define an order on the family $W(X)$, which characterizes the accuracy of a given quantity.

Definition 2.9. Let $A, B \in W(X)$. Then A is said to be more accurate than B, denoted by $A \subset B$ (or B includes A), if and only if $A(x) \leq B(x)$ for each $x \in X$. The relation $\subset$ induces a partial order on the family $W(X)$.

Definition 2.10. Let X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping if and only if F is a mapping from the set X into $W(Y)$.

Definition 2.11. For $F : X \to W(X)$, we say that $u \in X$ is a fixed point of F if $\{u\} \subset F(u)$, i.e. if $u \in F(u)_1$.

Lemma 2.12. (cf. [16]) *Let $x \in X$ and $A \in W(X)$. Then $\{x\} \subset A$ if and* *only if* $p_\alpha(x, A) = 0$ *for each* $\alpha \in [0, 1]$.

Remark 2.13. Note that from the above lemma it follows that for $A \in W(X)$, $\{x\} \subset A$ if and only if $p(\{x\}, A) = 0$. If no confusion arises instead of $p(\{x\}, A)$ we will write $p(x, A)$.

Lemma 2.14. (cf. [16]) $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ *for each* $x, y \in X$ *and* $A \in W(X)$.

Lemma 2.15. (cf. [16]) *If* $\{x_0\} \subset A$*, then* $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ *for each* $B \in W(X)$.

Lemma 2.16. (cf. [15]) *Let* (X, d) *be a complete metric linear space,* $F : X \to W(X)$ *be a fuzzy mapping and* $x_0 \in X$*. Then there exists* $x_1 \in X$ *such that* $\{x_1\} \subset F(x_0)$.

Remark 2.17. Let $f : X \to X$ be a self map and $T : X \to W(X)$ be a fuzzy mapping such that $\cup\{T(X)\}_\alpha \subseteq f(X)$ for $\alpha \in [0, 1]$. Then from Lemma 2.16, it follows that for any chosen point $x_0 \in X$ there exist points $x_1, y_1 \in X$ such that $y_1 = f(x_1)$ and $\{y_1\} \subset T(x_0)$. Here $T(x)_\alpha = \{y \in X : T(x)(y) \geq \alpha\}$.

Definition 2.18. Let (X, d) be a complete metric linear space and $f : X \to X$ be a self-mapping. Then a fuzzy mapping $T : X \to W(X)$ is said to be ϕ-weakly contractive with respect to f if for all $x, y \in X$

$$D(T(x), T(y)) \leq d(f(x), f(y)) - \phi(d(f(x), f(y))),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$. If f is the identity mapping on X, then the mapping $T : X \to W(X)$ satisfying the above inequality is said to be ϕ-weakly contractive.

Definition 2.19. Let $f : X \to X$ be a self-mapping and $T : X \to W(X)$ be a fuzzy mapping. Then a point $u \in X$ is said to be a coincidence point of f and T if $\{f(u)\} \subset T(u)$, i.e. if $f(u) \in T(u)_1$.

Definition 2.20. Let (X, d) be a complete metric linear space and $f : X \to X$ be a self-mapping. Then a pair of fuzzy mappings $T, S : X \to W(X)$ is said to be ϕ-weakly contractive with respect to f if for all $x, y \in X$

$$D(T(x), S(y)) \leq d(f(x), f(y)) - \phi(d(f(x), f(y))),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$. If f is the identity mapping on X, then the mappings $T, S : X \to W(X)$ satisfying the above inequality is said to be ϕ-weakly contractive.

Definition 2.21. Let (X, d) be a complete metric linear space and $f : X \to X$ be a self-mapping. Then a pair of fuzzy mappings $T, S : X \to W(X)$ is said to be generalized ϕ-weakly contractive with respect to f if for all $x, y \in X$

$$D(T(x), S(y)) \leq M^*(f(x), f(y)) - \phi(M^*(f(x), f(y))),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$,

$$M^*(x, y) = \max\{d(x, y), p(T(x), x), p(S(y), y), \frac{1}{2}[p(y, T(x)) + p(x, S(y))]\}.$$

If f is the identity mapping on X, then the mappings $T, S : X \to W(X)$ satisfying the above inequality is said to be generalized ϕ-weakly contractive.

3. Fixed Point Theorems for Multi-Valued Mappings

We begin this section with the following definition and lemma.

Definition 3.1. (cf. [5]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X$, $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 3.2. (cf. [5]) *If K is nonempty closed subset of a complete and metrically convex metric space (X, d), then for any $x \in K$, $y \notin K$, there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

In this section, we consider multi-valued mappings satisfying Definitions 2.3, 2.4 and 2.5. In this connection the following remark is pertinent concerning different conditions imposed on ϕ by many authors.

Remark 3.3. Different researchers have used different conditions on ϕ while discussing ϕ-weakly contractive mappings.

(*i*) Instead of continuity, lower semi-continuity was used by many, for example: Zhang and Song, Dragan Doric, etc.

(*ii*) The non-decreasing condition has been dropped by many as it is not needed in the proof.

(*iii*) The condition $\lim_{t\to\infty} \phi(t) = \infty$ has been used by Alber and Guerre-Delabriere, Bae, and Azam and Beg.

We note that Azam and Beg never used it in their work. Also Rhoades, Dutta and Choudhury, Zhang and Song did not include this condition on ϕ in their work. Dutta and Choudhury used the following example:

$$\phi(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \le t \le 1, \\ \frac{1}{2} & \text{if } t > 1. \end{cases}$$

This also does not satisfy the above condition and the condition imposed by Bae, i.e.,

$$\limsup_{t\to 0^+} \frac{t}{\phi(t)} < \infty.$$

However, if $\phi(t) = kt$ where k is a positive constant, then it satisfies both the conditions: $\lim_{t\to\infty} \phi(t) = \infty$ and $\limsup_{t\to 0^+} \frac{t}{\phi(t)} < \infty$. For our work any of these conditions is not needed.

The following two theorems extend the main theorem of Zhang and Song (cf. [21, Theorem 2.1]) to a pair of multi-valued mappings $S, T : X \to K(X)$.

Theorem 3.4. *Let (X, d) be a complete metric space and $T, S : X \to K(X)$ be a pair of multi-valued mappings such that for all $x, y \in X$*

$$H(Tx, Sy) \le M(x, y) - \phi(M(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$,

$$M(x, y) = \max\{d(x, y), d(T(x), x), d(S(y), y), \frac{1}{2}[d(y, T(x)) + d(x, S(y))]\}.$$

Then there there exists a point $u \in X$ such that $u \in T(u)$ and $u \in S(u)$.

Proof. Clearly $M(x, y) = 0$ if and only if $x = y$ is a common fixed point of T and S. Let $\{x_k\}$ be a sequence in X such that $x_{2k+1} \in T(x_{2k})$ and $x_{2k+2} \in S(x_{2k+1})$ for all $k \ge 0$, and

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\le H(T(x_{2k}), S(x_{2k+1})) \\ &\le M(x_{2k}, x_{2k+1}) - \phi(M(x_{2k}, x_{2k+1})) \le M(x_{2k}, x_{2k+1}) \\ &\le \max\{d(x_{2k}, x_{2k+1}), d(T(x_{2k}), x_{2k}), d(S(x_{2k+1}), x_{2k+1}), \\ &\qquad \frac{1}{2}[d(x_{2k+1}, T(x_{2k})) + d(x_{2k}, S(x_{2k+1}))]\} \\ &\le \max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k}) + d(T(x_{2k}), x_{2k+1}), d(x_{2k}, x_{2k+1}) \\ &\quad + d(S(x_{2k+1}), x_{2k})\}[\text{because } d(x_{2k+1}, T(x_{2k})) = 0 = d(x_{2k}, S(x_{2k+1}))] \\ &= \max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k}), d(x_{2k}, x_{2k+1})\} = d(x_{2k}, x_{2k+1}) \\ &\implies d(x_{2k+1}, x_{2k+2}) \le M(x_{2k}, x_{2k+1}) \le d(x_{2k}, x_{2k+1}). \quad (1) \end{aligned}$$

Similarly $d(x_{2k+2}, x_{2k+3}) \le M(x_{2k+1}, x_{2k+2}) \le d(x_{2k+1}, x_{2k+2})$. Thus for $n =$

$0, 1, 2, \cdots$ we have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, which shows that $\{d(x_n, x_{n+1})\}$ is a non-increasing bounded sequence of positive real numbers and therefore, tends to a limit $\ell \geq 0$. If possible let $\ell > 0$. For any $n \geq 1$ we have,

$$d(x_{n+1}, x_{n+2}) \leq M(x_n, x_{n+1}) \leq d(x_n, x_{n+1}),$$

and so, $\lim_{n\to\infty} M(x_n, x_{n+1}) = \ell$. Thus taking $n \to \infty$ from the inequality (1), we have $\ell \leq \ell - \phi(\ell)$, and so $\phi(\ell) \leq 0$, which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore, $\ell = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. If possible let $\{x_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ such that $d(x_m, x_n) > \epsilon$ for infinitely many values of m and n with $m < n$. This assures that there exists two sequences (m_k) and (n_k) of natural numbers with $m_k < n_k$ such that $d(x_{2m_k}, x_{2n_k+1}) > \epsilon$ for all $k \geq 1$. Let n_k be the least positive integer exceeding m_k satisfying $d(x_{2m_k}, x_{2n_k+1}) > \epsilon$. Then

$$d(x_{2m_k}, x_{2n_k-1}) \leq \epsilon \quad \text{and} \quad d(x_{2m_k}, x_{2n_k+1}) > \epsilon,$$

and so

$$\begin{aligned}\epsilon < d(x_{2m_k}, x_{2n_k+1}) &\leq d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1}) \\ &\leq \epsilon + d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1}).\end{aligned}$$

Hence taking $k \to \infty$ we have

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon.$$

Now for any $\ell_1, \ell_2 \in \mathbb{N}$, we show that $\lim_{k\to\infty} d(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) = \epsilon$. Note that,

$$\begin{aligned}&d(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) \\ &\leq d(x_{2m_k+\ell_1}, x_{2m_k+\ell_1-1}) + d(x_{2m_k+\ell_1-1}, x_{2m_k+\ell_1-2}) + \cdots + d(x_{2m_k+1}, x_{2m_k}) \\ &\quad + d(x_{2m_k}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k+2}) + \cdots + d(x_{2n_k+\ell_2-1}, x_{2n_k+\ell_2}).\end{aligned}$$

Hence taking $k \to \infty$ we have

$$\lim_{k\to\infty} d(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) \leq \epsilon. \tag{2}$$

Again,

$$\begin{aligned}&d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2m_k+2}) + \cdots + d(x_{2m_k+\ell_1-1}, x_{2m_k+\ell_1}) \\ &\quad + d(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) + d(x_{2n_k+\ell_2}, x_{2n_k+\ell_2-1}) + \cdots \\ &\quad + d(x_{2n_k+2}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k}).\end{aligned}$$

Hence taking $k \to \infty$ we have

$$\epsilon \leq \lim_{k\to\infty} d(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}). \tag{3}$$

(2) and (3) yields $\lim_{k\to\infty} d(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) = \epsilon$ for any $\ell_1, \ell_2 \in \mathbb{N}$.

Choose ℓ_1 and ℓ_2 so that $2m_k + \ell_1$ is even, $2n_k + \ell_2$ is odd and $(2n_k + \ell_2) - (2m_k + \ell_1) = 1$. Hence by (1),

$$d(x_{2m_k+\ell_1+1}, x_{2n_k+\ell_2+1}) \le M(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) \le d(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}),$$

and so taking $k \to \infty$ we have, $\lim_{k\to\infty} M(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) = \epsilon$. Now by (1) we have

$$d(x_{2m_k+\ell_1+1}, x_{2n_k+\ell_2+1}) \le M(x_{2m_k+\ell_1}, x_{2n_k+\ell_2}) - \phi(M(x_{2m_k+\ell_1}, x_{2n_k+\ell_2})),$$

and hence taking $k \to \infty$ we have $\epsilon \le \epsilon - \phi(\epsilon) \implies \phi(\epsilon) \le 0$, which is a contradiction as $\epsilon > 0$ and $\phi(t) > 0$ for t positive. Hence, the sequence $\{x_n\}$ is a Cauchy sequence. From the completeness of X it now follows that $x_n \to u$ for some $u \in X$. Now we show that $u \in T(u)$ and $u \in S(u)$. Note that

$$\begin{aligned} d(S(u), u) \le M(x_{2k}, u) &= \max\{d(x_{2k}, u), d(T(x_{2k}), x_{2k}), d(S(u), u), \\ &\qquad \frac{1}{2}[d(u, T(x_{2k})) + d(x_{2k}, S(u))]\} \\ &\le \max\{d(x_{2k}, u), d(x_{2k+1}, x_{2k}) + d(T(x_{2k}), x_{2k+1}), d(S(u), u), \\ &\qquad \frac{1}{2}[d(u, x_{2k+1}) + d(x_{2k+1}, T(x_{2k})) + d(x_{2k}, u) + d(u, S(u))]\} \end{aligned}$$

$$= \max\{d(x_{2k}, u), d(x_{2k+1}, x_{2k}), d(S(u), u), \frac{1}{2}[d(u, x_{2k+1}) + d(x_{2k}, u) + d(u, S(u))]\}.$$

Hence, taking $k \to \infty$ we have

$$\begin{aligned} d(u, S(u)) &\le \lim_{k\to\infty} M(x_{2k}, u) \\ &\le \max\{0, 0, d(S(u), u), \frac{1}{2}[0 + 0 + d(u, S(u))]\} = d(S(u), u), \end{aligned}$$

and so $\lim_{k\to\infty} M(x_{2k}, u) = d(S(u), u)$. We know

$$\begin{aligned} d(u, S(u)) &\le d(u, x_{2k+1}) + d(x_{2k+1}, S(u)) \\ &\le d(u, x_{2k+1}) + H(T(x_{2k}), S(u)) \\ &\le d(u, x_{2k+1}) + M(x_{2k}, u) - \phi(M(x_{2k}, u)). \end{aligned}$$

Now taking $k \to \infty$ as ϕ is continuous, we have

$d(u, S(u)) \le 0 + d(S(u), u) - \phi(d(S(u), u)) \implies \phi(d(S(u), u)) \le 0 \implies \phi(d(S(u), u)) = 0 \implies d(S(u), u) = 0 \implies u \in S(u)$. Similarly, we can show $u \in T(u)$. □

Remark 3.5. The proof holds if ϕ is lower semi-continuous only, i.e., continuity condition can be relaxed and nondecreasing condition can be dropped. But for simplicity we have assumed these conditions. For details see Remark 3.3.

Taking S and T as the self-mapping of X, we get the theorem of Zhang and Song (cf. [21, Theorem 2.1]) as the following corollary.

Corollary 3.5.1. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings such that for all $x, y \in X$*

$$d(Tx, Sy) \leq M(x, y) - \phi(M(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$,

$$M(x, y) = \max\{d(x, y), d(T(x), x), d(S(y), y), \frac{1}{2}[d(y, T(x)) + d(x, S(y))]\}.$$

Then there there exists a point $u \in X$ such that $u = T(u) = S(u)$.

Example. Let $X = [0, 5]$ and $d(x, y) = |x - y|$ for $x, y \in X$. Define $T : X \to X$ as follows:

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq 4, \\ -2x + 10, & \text{if } 4 < x \leq 5. \end{cases}$$

Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{1}{36}t^2$. If $x = 4$, $y = 5$ then

$$d(T(x), T(y)) = |2 - 0| = 2, \quad d(x, y) = 1,$$

and so $d(T(x), T(y)) \leq d(x, y) - \phi(d(x, y))$ is not satisfied, i.e., T is not ϕ-weakly contractive. If $x, y \in [0, 4]$, then

$$d(Tx, Ty) = \frac{1}{2}|x - y|, \ M(x, y) = \max\{|x - y|, \frac{x}{2}, \frac{y}{2}, \frac{1}{2}[|x - \frac{y}{2}| + |y - \frac{x}{2}|]\}.$$

If $x, y \in [4, 5]$, then

$$d(Tx, Ty) = |-2x + 10 + 2y - 10| = 2|x - y|,$$

$$M(x, y) = \max\{|x - y|, |x + 2x - 10|, |y + 2y - 10|, \frac{1}{2}[|x + 2y - 10| + |y + 2x - 10|]\}.$$

If $x \in [0, 4]$ and $y \in [4, 5]$, then

$$d(Tx, Ty) = |\frac{x}{2} + 2y - 10|,$$

$$M(x, y) = \max\{|x - y|, \frac{x}{2}, |y + 2y - 10|, \frac{1}{2}[|\frac{x}{2} + 2y - 10| + |y - \frac{x}{2}|]\}.$$

Hence, T is generalized ϕ-weakly contractive and zero is a fixed point.

Next theorem gives a fixed point for a multi-valued mapping defined on a closed subset of a complete and convex metric space satisfying a generalized ϕ-weakly contractive condition.

Theorem 3.6. *Let K be a nonempty closed subset of a complete and* *convex metric space* (X, d) *and* $T : K \to K(X)$ *be a mapping such that for all* $x, y \in K$

$$H(Tx, Ty) \leq M(x, y) - \phi(M(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ *is a continuous and non-decreasing function such that* $\phi(t) > 0$ *for* $t \in (0, \infty)$ *and* $\phi(0) = 0$,

$$M(x, y) = \max\{d(x, y), d(T(x), x), d(T(y), y), \frac{1}{2}[d(y, T(x)) + d(x, T(y))]\}.$$

Suppose that $T(x) \subset K$ *for each* $x \in \partial K$ *(the boundary of* K*). Then there exists a point* $u \in K$ *such that* $u \in T(u)$.

Proof. Clearly $M(x, y) = 0$ if and only if $x = y$ is a fixed point of T. We select a sequence $\{x_n\}$ in the following way. Let $x_0 \in K$ and $x'_1 \in T(x_0)$. If $x'_1 \in K$ let $x_1 = x'_1$; otherwise select a point $x_1 \in \partial K$ such that $d(x_0, x_1) + d(x_1, x'_1) = d(x_0, x'_1)$. Thus $x_1 \in K$ and by Lemma 2.1 we can choose a point $x'_2 \in T(x_1)$ so that $d(x'_1, x'_2) \leq H(T(x_0), T(x_1))$. Now put $x'_2 = x_2$ if $x'_2 \in K$, otherwise let x_2 be a point of ∂K such that $d(x_1, x_2) + d(x_2, x'_2) = d(x_1, x'_2)$. By induction we can obtain a sequence $\{x_n\}$, $\{x'_n\}$ such that for $n = 1, 2, 3, \cdots$

(i) $x'_{n+1} \in T(x_n)$,

(ii) $d(x'_n, x'_{n+1}) \leq H(T(x_{n-1}), T(x_n))$,

where

(iii) $x'_{n+1} = x_{n+1}$ if $x'_{n+1} \in K$, or

(iv) $d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$ if $x'_{n+1} \notin K$.

Now let

$$P = \{x_i \in \{x_n\} : x_i = x'_i, i = 1, 2, \cdots\},$$
$$Q = \{x_i \in \{x_n\} : x_i \neq x'_i, i = 1, 2, \cdots\}.$$

Observe that if $x_n \in Q$ for some n, then $x_{n+1} \in P$. Now for $n \geq 2$ we estimate the distance $d(x_n, x_{n+1})$. There arises three cases.

Case 1. The case that $x_n \in P$ and $x_{n+1} \in P$. In this case we have,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x'_n, x'_{n+1}) \leq H(T(x_{n-1}), T(x_n)) \\ &\leq M(x_{n-1}, x_n) - \phi(M(x_{n-1}, x_n)) \leq M(x_{n-1}, x_n) \\ &\leq \max\{d(x_{n-1}, x_n), d(T(x_{n-1}), x_{n-1}), d(T(x_n), x_n), \\ &\qquad \frac{1}{2}[d(x_n, T(x_{n-1})) + d(x_{n-1}, T(x_n))]\} \end{aligned}$$
$$\leq \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(T(x_{n-1}), x_n), d(x_n, x_{n+1}) + d(T(x_n), x_{n+1}),$$
$$\frac{1}{2}[d(x_n, T(x_{n-1})) + d(x_{n-1}, x_{n+1}) + d(x_{n+1}, T(x_n))]\}$$

$$\leq \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})]\}$$
$$[\text{because } d(x_{n+1},T(x_n)) = 0 = d(x_n,T(x_{n-1}))] \leq d(x_{n-1},x_n),$$

as the inequality $d(x_n,x_{n+1}) > d(x_{n-1},x_n)$ implies $M(x_{n-1},x_n) = d(x_n,x_{n+1})$, and hence we have

$$\begin{aligned} &d(x_n,x_{n+1}) \leq M(x_{n-1},x_n) - \phi(M(x_{n-1},x_n)) \\ &\implies d(x_n,x_{n+1}) \leq d(x_n,x_{n+1}) - \phi(d(x_n,x_{n+1})) \implies \phi(d(x_n,x_{n+1})) \leq 0 \\ &\implies \phi(d(x_n,x_{n+1})) = 0 \implies d(x_n,x_{n+1}) = 0 \end{aligned}$$

and so $M(x_{n-1},x_n) = 0$ which implies x_n is a fixed point and then we are done.

Case 2. The case that $x_n \in P$ and $x_{n+1} \in Q$. In this case we use (iv) and proceeding in the same way as Case 1 we obtain,

$$d(x_n,x_{n+1}) \leq d(x_n,x'_{n+1}) \leq d(x'_n,x'_{n+1}) \leq H(T(x_{n-1}),T(x_n)) \leq d(x_{n-1},x_n).$$

Case 3. The case that $x_n \in Q$ and $x_{n+1} \in P$. From the construction of the sequence $\{x_n\}$ it is clear that two consecutive terms of $\{x_n\}$ can not be in Q, and hence $x_{n-1} \in P$ and $x'_{n-1} = x_{n-1}$. Using this below we obtain,

$$\begin{aligned} d(x_n,x_{n+1}) &\leq d(x_n,x'_n) + d(x'_n,x_{n+1}) \\ &= d(x_n,x'_n) + d(x'_n,x'_{n+1}) \\ &= d(x_n,x'_n) + H(T(x_{n-1}),T(x_n)) \\ &= d(x_n,x'_n) + d(x_{n-1},x_n) \qquad (\text{ as Case 1}) \\ &= d(x_{n-1},x'_n) \\ &= d(x'_{n-1},x'_n) \\ &\leq H(T(x_{n-2}),T(x_{n-1})) \\ &\leq d(x_{n-2},x_{n-1}). \end{aligned}$$

The only other possibility, $x_n \in Q$, $x_{n+1} \in Q$ cannot occur. Thus for $n \geq 2$ we have

$$d(x_n,x_{n+1}) \leq \begin{cases} d(x_{n-1},x_n), \\ d(x_{n-2},x_{n-1}). \end{cases}$$

If we write $d_{n+1} = d(x_n,x_{n+1})$, then either $d_{n+1} \leq d_n$ or $d_{n+1} \leq d_{n-1}$ for all $n \geq 1$, which shows that $\{d_n\}$ is a non-increasing bounded sequence of positive real numbers and therefore tends to a limit $\ell \geq 0$. If possible let $\ell > 0$. For any $n \geq 1$ we have

$$\begin{aligned} &d(x_{n+1},x_{n+2}) \leq M(x_n,x_{n+1}) \leq d(x_n,x_{n+1}) \text{ or } d(x_{n+1},x_{n+2}) \\ &\qquad \leq M(x_{n-1},x_n) \leq d(x_{n-1},x_n). \end{aligned}$$

Hence in any case, $\lim_{n\to\infty} M(x_n,x_{n+1}) = \ell = \lim_{n\to\infty} M(x_{n-1},x_n)$. Again we have either $d(x_{n+1}, x_{n+2}) \leq M(x_n, x_{n+1}) - \phi(M(x_n, x_{n+1}))$ or, $d(x_{n+1}, x_{n+2}) \leq M(x_{n-1}, x_n) - \phi(M(x_{n-1}, x_n))$. In any case, taking $n \to \infty$ we have $\ell \leq \ell - \phi(\ell) \implies \phi(\ell) \leq 0$, which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore, $d_n \to 0$ as $n \to \infty$. In the same way as in Theorem 3.4, it can be shown that $\{x_n\}$ is a Cauchy sequence. Since X is complete and K is closed, $\{x_n\}$ converges to a point in K. Let $u = \lim_{n\to\infty} x_n$. Also observe that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, each of whose terms is in the set P (i.e., $x_{n_k} = x'_{n_k}, k = 1, 2, \cdots$). Thus by (i), $x'_{n_k} \in T(x_{n_k-1})$. Now

$$\begin{aligned} d(u, T(u)) \leq M(x_{n_k-1}, u) &= \max\{d(x_{n_k-1}, u), d(T(x_{n_k-1}), x_{n_k-1}), d(u, T(u)), \\ &\qquad \frac{1}{2}[d(u, T(x_{n_k-1})) + d(x_{n_k-1}, T(u))]\} \\ &\leq \max\{d(x_{n_k-1}, u), d(x_{n_k-1}, x'_{n_k}) + d(T(x_{n_k-1}), x'_{n_k}), d(u, T(u)), \\ &\qquad \frac{1}{2}[d(u, x'_{n_k}) + d(x'_{n_k}, T(x_{n_k-1})) + d(x_{n_k-1}, u) + d(u, T(u))]\}. \end{aligned}$$

Taking $k \to \infty$ and noting that $x'_{n_k} \in T(x_{n_k-1})$ we have

$$\begin{aligned} d(u, T(u)) &\leq \lim_{k\to\infty} M(x_{n_k-1}, u) \\ &\leq \max\{0, 0, d(u, T(u)), \frac{1}{2}[d(u, T(u))]\} \leq d(u, T(u)) \end{aligned}$$

and so $\lim_{k\to\infty} M(x_{n_k-1}, u) = d(u, T(u))$. We know,

$$\begin{aligned} d(u, T(u)) &\leq d(u, x_{n'_k}) + d(x_{n'_k}, T(u)) \\ &\leq d(u, x'_{n_k}) + H(T(x_{n_k-1}, T(u)) \\ &\leq d(u, x'_{n_k}) + M(x_{n_k-1}, u) - \phi(M(x_{n_k-1}, u)). \end{aligned}$$

Now taking $k \to \infty$ as ϕ is continuous we have

$d(u, T(u)) \leq 0 + d(u, T(u)) - \phi(d(u, T(u))) \implies \phi(d(u, T(u))) \leq 0 \implies \phi(d(u, T(u))) = 0 \implies d(u, T(u)) = 0 \implies u \in T(u)$ and hence is the theorem. ☐

Next theorem deals with the existence of a coincidence point of a single-valued mapping and a pair of multi-valued mappings.

Theorem 3.7. *Let (X, d) be a metric space, $f : X \to X$ be a self-mapping and $T, S : X \to K(X)$ be a pair of multi-valued mappings such that for each $x, y \in X$,*

$$H(T(x), S(y)) \leq d(f(x), f(y)) - \phi(d(f(x), f(y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$. If $\cup_{x\in X} T(x) \subset f(X)$, $\cup_{x\in X} S(x) \subset f(X)$ and $f(X)$ is a complete subspace of X, then there exists a point $p \in X$ such *that* $f(p) \in T(p)$ *and* $f(p) \in S(p)$.

Proof. Let x_0 be a point in X. Choose a point $x_1 \in X$ such that $f(x_1) \in T(x_0)$. This can be done since $\cup_{x \in X} T(x) \subset f(X)$. Similarly, choose $x_2 \in X$ such that $f(x_2) \in S(x_1)$ and such that $d(f(x_1), f(x_2)) \leq H(T(x_0), S(x_1))$. In general, having chosen $x_k \in X$, we obtain $x_{k+1} \in X$ such that $f(x_{2k+1}) \in T(x_{2k})$, $f(x_{2k+2}) \in S(x_{2k+1})$, and such that

$$d(f(x_{2k-1}), f(x_{2k})) \leq H(T(x_{2k-2}), S(x_{2k-1}))$$

for any positive integer k. Hence, by the given hypothesis

$$\begin{aligned} d(f(x_{2k-1}), f(x_{2k})) &\leq H(T(x_{2k-2}), S(x_{2k-1})) \\ &\leq d(f(x_{2k-2}), f(x_{2k-1})) - \phi(d(f(x_{2k-2}), f(x_{2k-1}))) \\ &\leq d(f(x_{2k-2}), f(x_{2k-1})), \end{aligned} \tag{4}$$

and similarly,

$$d(f(x_{2k}), f(x_{2k+1})) \leq H(S(x_{2k-1}), T(x_{2k})) \leq d(f(x_{2k-1}), f(x_{2k})), \tag{5}$$

which show that $\{d(f(x_n), f(x_{n+1}))\}$ is a non-increasing bounded sequence of positive real numbers and therefore, tends to a limit $\ell \geq 0$. If possible, let $\ell > 0$, then taking $n \to \infty$ by (4) and (5), we have $\ell \leq \ell - \phi(\ell)$, and so $\phi(\ell) \leq 0$, which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore $\ell = 0$, i.e., $\lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0$. Next we show that $\{f(x_n)\}$ is a Cauchy sequence. Let us denote $f(x_n)$ by y_n. If possible let $\{y_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$ such that $d(y_m, y_n) > \epsilon$ for infinitely many values of m and n with $m < n$. This assures that there exists two sequences (m_k) and (n_k) of natural numbers with $m_k < n_k$ such that $d(y_{2m_k}, y_{2n_k+1}) > \epsilon$ for all $k \geq 1$. Let n_k be the least positive integer exceeding m_k satisfying $d(y_{2m_k}, y_{2n_k+1}) > \epsilon$. Then

$$d(y_{2m_k}, y_{2n_k-1}) \leq \epsilon \quad \text{and} \quad d(y_{2m_k}, y_{2n_k+1}) > \epsilon.$$

Proceeding in the same way as in Theorem 3.4, it can be shown that

$$\lim_{k\to\infty} d(y_{2m_k+\ell_1}, y_{2n_k+\ell_2}) = \epsilon$$

for any $\ell_1, \ell_2 \in \mathbb{N}$. Choose ℓ_1 and ℓ_2 so that $2m_k + \ell_1$ is even, $2n_k + \ell_2$ is odd and $(2n_k + \ell_2) - (2m_k + \ell_1) = 1$. Hence by (4) we have

$$d(y_{2m_k+\ell_1+1}, y_{2n_k+\ell_2+1}) \leq d(y_{2m_k+\ell_1}, y_{2n_k+\ell_2}) - \phi(d(y_{2m_k+\ell_1}, y_{2n_k+\ell_2})).$$

Now taking $k \to \infty$ we have, $\epsilon \leq \epsilon - \phi(\epsilon) \implies \phi(\epsilon) \leq 0$, which is a contradiction as $\epsilon > 0$ and $\phi(t) > 0$ for t positive. Hence the sequence $\{y_n\}$, i.e. the sequence $\{f(x_n)\}$ is a Cauchy sequence.

As $f(X)$ is a complete subspace of X, $\{f(x_n)\}$ has a limit q in $f(X)$. Consequently, we obtain p in X such that $f(p) = q$. Thus, $f(x_{2k+1}) \to q$ and

$f(x_{2k}) \to q$ as $k \to \infty$. Now

$$\begin{aligned} d(f(p), S(p)) &\leq d(f(p), f(x_{2k+1})) + d(f(x_{2k+1}), S(p)) \\ &\leq d(f(p), f(x_{2k+1})) + H(T(x_{2k}), S(p)) \\ &\leq d(f(p), f(x_{2k+1})) + d(f(x_{2k}), f(p)) - \phi(d(f(x_{2k}), f(p))). \end{aligned}$$

Taking $k \to \infty$ we have $d(f(p), S(p)) \leq 0$, i.e., $d(f(p), S(p)) = 0$ and so, $f(p) \in S(p)$. Similarly we can show, $f(p) \in T(p)$ and hence is the theorem. □

Corollary 3.7.1. (see [8, Theorem 2.1]) *Let (X, d) be a metric space and let T be a weakly contractive mapping with respect to f. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have a coincidence point in X, that is, there exists a point p in X such that $f(p) = T(p)$.*

Proof. This result follows from Theorem 3.7 when we take $T = S$ and $T : X \to X$. □

Next corollary follows from Theorem 3.7 when the mappings T and S are single-valued self-mappings of X.

Corollary 3.7.2. (see [3, Theorem 2.3]) *Let (X, d) be a metric space and T, S, f be self-mappings of X and for each x, y in X*

$$d(Tx, Sy) \leq d(f(x), f(y)) - \phi(d(fx, fy)),$$

where ϕ is as given in Theorem 3.7 above. If $T(X) \cup S(X) \subset f(X)$ and $f(X)$ is a complete subspace of X, then there exists a point p in X such that $fp = Tp = Sp$.

4. Fixed Point Theorem for Fuzzy Mappings

In this section we obtain the following common fixed point theorem for fuzzy mappings.

Theorem 4.1. *Let (X, d) be a complete metric linear space and $T, S : X \to W(X)$ be a pair of fuzzy mappings such that for all $x, y \in X$*

$$D(T(x), S(y)) \leq M^*(x, y) - \phi(M^*(x, y)), \tag{6}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$,

$$M^*(x, y) = \max\{d(x, y), p(T(x), x), p(S(y), y), \frac{1}{2}[p(y, T(x)) + p(x, S(y))]\}.$$

Then there there exists a point $u \in X$ such that $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$.

Proof. First we show that $M^*(x,y)=0$ if and only if $x=y$ is a common fixed point of T and S. In fact, if $x=y$, $\{x\}\subset T(x)$, $\{y\}\subset S(y)$, then by Remark 2.13,

$$d(x,y)=0,\ p(T(x),x)=0,\ p(S(y),y)=0,\ p(y,T(x))=0,\ p(x,S(y))=0$$

and so $M^*(x,y)=0$. Now let $M^*(x,y)=0$. Then from $d(x,y)\le M^*(x,y)$, $p(T(x),x)\le M^*(x,y)$, $p(S(y),y)\le M^*(x,y)$, we have

$$d(x,y)=p(T(x),x)=p(S(y),y)=0.$$

Hence by Remark 2.13,

$$x=y,\ \{x\}\subset T(x),\ \{y\}\subset S(y).$$

Now let x_0 be an arbitrary but fixed element of X. We construct a sequence $\{x_n\}$ of points of X as follows. By Lemma 2.16, there exists $x_1\in X$ such that $\{x_1\}\subset T(x_0)$. By Lemma 2.16 and Lemma 2.1, we can choose $x_2\in X$ such that $\{x_2\}\subset S(x_1)$ and

$$d(x_1,x_2)\le H(T(x_0)_1,S(x_1)_1).$$

Thus in view of the inequality (6) we have

$$d(x_1,x_2)\le D_1(T(x_0),S(x_1))\le D(T(x_0),S(x_1))\le M^*(x_0,x_1)-\phi(M^*(x_0,x_1)).$$

Continuing this process, having chosen $x_n\in X$, we obtain $x_{n+1}\in X$ such that $\{x_{2k+1}\}\subset T(x_{2k})$, $\{x_{2k+2}\}\subset S(x_{2k+1})$, and

$$\begin{aligned}d(x_{2k+1},x_{2k+2})&\le D(T(x_{2k}),S(x_{2k+1}))\\&\le M^*(x_{2k},x_{2k+1})-\phi(M^*(x_{2k},x_{2k+1})),\end{aligned}\tag{7}$$

$$\begin{aligned}d(x_{2k+2},x_{2k+3})&\le D(S(x_{2k+1}),T(x_{2k+2}))\\&\le M^*(x_{2k+1},x_{2k+2})-\phi(M^*(x_{2k+1},x_{2k+2})).\end{aligned}\tag{8}$$

From (7) we have

$$\begin{aligned}d(x_{2k+1},x_{2k+2})&\le M^*(x_{2k},x_{2k+1})-\phi(M^*(x_{2k},x_{2k+1}))\le M^*(x_{2k},x_{2k+1})\\&\le\max\{d(x_{2k},x_{2k+1}),p(T(x_{2k}),x_{2k}),p(S(x_{2k+1}),x_{2k+1}),\\&\qquad\frac{1}{2}[p(x_{2k+1},T(x_{2k}))+p(x_{2k},S(x_{2k+1}))]\}\\&\le\max\{d(x_{2k},x_{2k+1}),d(x_{2k+1},x_{2k})+p(T(x_{2k}),x_{2k+1}),d(x_{2k},\\&\quad x_{2k+1})+p(S(x_{2k+1}),x_{2k})\}[\text{because } p(x_{2k+1},T(x_{2k}))=0=p(x_{2k},S(x_{2k+1}))]\\&=\max\{d(x_{2k},x_{2k+1}),d(x_{2k+1},x_{2k}),d(x_{2k},x_{2k+1})\}=d(x_{2k},x_{2k+1}).\end{aligned}$$

Similarly from (8), it follows $d(x_{2k+2},x_{2k+3})\le M^*(x_{2k+1},x_{2k+2})\le d(x_{2k+1},x_{2k+2})$. Thus for $n=0,1,2,\cdots$ we have $d(x_{n+1},x_{n+2})\le d(x_n,x_{n+1})$, which shows that $\{d(x_n,x_{n+1})\}$ is a non-increasing bounded sequence of positive real numbers and therefore tends to a limit $\ell \geq 0$. If possible, let $\ell > 0$. For any $n \geq 1$ we have, $d(x_{n+1}, x_{n+2}) \leq M^*(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$. Hence taking $n \to \infty$ we have $\lim_{n\to\infty} M^*(x_n, x_{n+1}) = \ell$.

Again $d(x_{n+1}, x_{n+2}) \leq M^*(x_n, x_{n+1}) - \phi(M^*(x_n, x_{n+1}))$. Hence taking $n \to \infty$ we have, $\ell \leq \ell - \phi(\ell) \implies \phi(\ell) \leq 0$, which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now proceeding in the same way as in Theorem 3.4, it can be shown that $\{x_n\}$ is Cauchy sequence in X. It follows from the completeness for X, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Moreover, $x_{2n} \to u$ and $x_{2n+1} \to u$ as $n \to \infty$.

Now we prove that $\{u\} \subset T(u)$ and $\{u\} \subset S(u)$. We know $\{x_{2k+1}\} \subset T(x_{2k})$ and $\{x_{2k}\} \subset S(x_{2k+1})$. Note that,

$$\begin{aligned}
p(u, S(u)) &\leq M^*(x_{2k}, u)\\
&= \max\{d(x_{2k}, u), p(T(x_{2k}), x_{2k}), p(S(u), u),\\
&\qquad \frac{1}{2}[p(u, T(x_{2k})) + p(x_{2k}, S(u))]\}\\
&\leq \max\{d(x_{2k}, u), d(x_{2k}, x_{2k+1}) + p(T(x_{2k}), x_{2k+1}), p(S(u), u),\\
&\qquad \frac{1}{2}[d(u, x_{2k+1}) + p(x_{2k+1}, T(x_{2k})) + d(x_{2k}, u) + p(u, S(u))]\}\\
&\leq \max\{d(x_{2k}, u), d(x_{2k}, x_{2k+1}), p(S(u), u),\\
&\qquad \frac{1}{2}[d(u, x_{2k+1}) + d(x_{2k}, u) + p(u, S(u))]\}.
\end{aligned}$$

Taking $k \to \infty$ we have

$$\begin{aligned}
p(u, S(u)) &\leq M^*(x_{2k}, u)\\
&\leq \max\{0, 0, p(S(u), u), \frac{1}{2}[0 + 0 + p(u, S(u))]\} = p(u, S(u)),
\end{aligned}$$

and so $\lim_{k\to\infty} M^*(x_{2k}, u) = p(u, S(u))$. Now

$$\begin{aligned}
p(u, S(u)) &\leq d(u, x_{2k+1}) + p(x_{2k+1}, S(u))\\
&\leq d(u, x_{2k+1}) + D(T(x_{2k}), S(u))\\
&\leq d(u, x_{2k+1}) + M^*(x_{2k}, u) - \phi(M^*(x_{2k}, u)).
\end{aligned}$$

Hence taking $k \to \infty$ we have

$$\begin{aligned}
&p(u, S(u)) \leq 0 + p(u, S(u)) - \phi(p(u, S(u)))\\
&\implies \phi(p(u, S(u))) \leq 0 \implies \phi(p(u, S(u))) = 0 \implies p(u, S(u)) = 0.
\end{aligned}$$

Hence by Remark 2.13 we have $\{u\} \subset S(u)$. Similarly we can show, $\{u\} \subset T(u)$. □

Corollary 4.1.1. *Take $S = T$ and $M^*(x, y) = d(x, y)$, then we get*

Theorem 4.2 of [2].

Next theorem proves the existence of a coincidence point of a single-valued mapping f and a fuzzy mapping T, where T is ϕ-weakly contractive with respect to f.

Theorem 4.2. *Let (X, d) be a complete metric linear space. Let $f : X \to X$ be a self-mapping, and $T : X \to W(X)$ be a ϕ-weakly contractive fuzzy mapping with respect to f (see Definition 2.18). Suppose $\cup\{T(X)\}_\alpha \subseteq f(X)$ for $\alpha \in [0,1]$, and $f(X)$ is complete. Then there exists $u \in X$ such that u is a coincidence point of f and T, that is $\{f(u)\} \subset T(u)$. Here $T(x)_\alpha = \{y \in X : T(x)(y) \geq \alpha\}$.*

Proof. Let $x_0 \in X$ and $y_0 = f(x_0)$. Since $\cup\{T(X)\}_\alpha \subset f(X)$ for each $\alpha \in [0,1]$, by Remark 2.17 for $x_0 \in X$ there exist points $x_1, y_1 \in X$ such that $y_1 = f(x_1)$ and $\{y_1\} \subset T(x_0)$. Again by Remark 2.17 and Lemma 2.1, for $x_1 \in X$ there exist points $x_2, y_2 \in X$ such that $y_2 = f(x_2)$ and $\{y_2\} \subset T(x_1)$, and

$$\begin{aligned} d(y_1, y_2) \leq H(T(x_0)_1, T(x_1)_1) &\leq D(T(x_0), T(x_1)) \\ &\leq d(f(x_0), f(x_1)) - \phi(d(f(x_0), f(x_1))) \\ &\leq d(y_0, y_1). \end{aligned}$$

By repeating this process for any $k \geq 1$, we can select points $x_k, y_k \in X$ such that $y_k = f(x_k)$ and $\{y_k\} \subset T(x_{k-1})$, and

$$\begin{aligned} d(y_k, y_{k+1}) \leq H(T(x_{k-1})_1, T(x_k)_1) &\leq D(T(x_{k-1}), T(x_k)) \\ &\leq d(f(x_{k-1}), f(x_k)) - \phi(d(f(x_{k-1}), f(x_k))) \\ &\leq d(y_{k-1}, y_k). \end{aligned} \tag{9}$$

Thus for $n = 0, 1, 2, \cdots$ we have $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$, which shows that $\{d(y_n, y_{n+1})\}$ is a non-increasing bounded sequence of positive real numbers and therefore tends to a limit $\ell \geq 0$. If possible, let $\ell > 0$. For any $n \geq 0$ by (9) we have $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}) - \phi(d(y_n, y_{n+1}))$. Hence taking $n \to \infty$, we have $\ell \leq \ell - \phi(\ell)$, which implies $\phi(\ell) \leq 0$ which is a contradiction as $\ell > 0$, and $\phi(t) > 0$ for $t > 0$. Therefore, $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$. As in the proof of Theorem 3.4, it can also be shown that $\{y_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, $\{y_n\}$ converges to some point in $f(X)$. Let $y = \lim_{n\to\infty} y_n$ and $u \in X$ be such that $y = f(u)$. Now

$$\begin{aligned} p(f(u), T(u)) = p(y, T(u)) &\leq d(y, y_{k+1}) + p(y_{k+1}, T(u)) \\ &\leq d(y, y_{k+1}) + D(T(x_k), T(u)) \quad \text{[by Lemma 2.15]} \\ &\leq d(y, y_{k+1}) + d(f(x_k), f(u)) - \phi(d(f(x_k), f(u))) \end{aligned}$$

$$\leq d(y, y_{k+1}) + d(y_k, y).$$

Now taking $k \to \infty$, we have $p(f(u), T(u)) = 0$, and so Lemma 2.12 gives $\{f(u)\} \subset T(u)$, i.e., u is a coincidence point of f and T. □

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