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A METHOD FOR GENERATING INTEGRAL IDENTITIES

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Abstract: Based on the Parseval relation for certain integral transforms related to the Fourier transform, a method is presented for producing integral identities. When these are constructed to include two or more parameters, the selection of special values for the later can be used to simplify, and in many cases, to evaluate classes of integrals containing special functions. A number of examples is provided.

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1. Introduction

The Parseval formula, the relation of the integral of a product of two functions to the integral of the product of their transforms, is a feature of all integral transforms stemming from the Fourier transform. Besides the latter, this family includes the Laplace, and the various Hankel transforms. Fortunately, compendious tabulations of integral transforms exist, particularly the volumes edited by A. Erdélyi [1] as part of the Bateman manuscript series. As a strat-

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egy for evaluating definite integrals, the Parseval method is widely known [2], and in one form or another frequently lies behind many of Ramanujan's beautiful identities. Unfortunately its use is not as frequent as it could be, since it simplifies many derivations. In this note we will examine a variant which, while it therefore cannot be called exactly new, does not seem to appear in the literature. From the few examples provided it can be seen to lead to striking and potentially useful results. The principle is very simple. Let $f(a, b, \ldots; x)$ be a function possessing one of the transforms listed above; let $g(a, b, \ldots; x)$ be its transform, where $a, b, \ldots$ are parameters. Then, from the Parseval relation

$$\int_0^\infty f(a, b, \ldots; x) g(\alpha, \beta, \ldots; x) dx = \int_0^\infty f(\alpha, \beta, \ldots; x) g(a, b, \ldots; x) dx. \tag{1}$$

We shall see that by fixing some of the parameters appropriately, the integral of, say, the right hand side of (1) simplifies substantially or can even be expressed in closed form, while the integral on the left hand side remains hitherto unknown. The one-parameter case will rarely lead to anything interesting as (1) will generally result from a change of integration variable. As an elementary example, note that the Laplace transform of the characteristic function of the interval $[a, b]$ is $(e^{-at} - e^{-bt})/t$. From (1) we have the pretty identity

$$\int_a^b \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx = \int_\alpha^\beta \frac{e^{-ax} - e^{-bx}}{x} dx. \tag{2}$$

In all cases, such as (2), there may be restrictions on the values of the parameters, but these conditions may be determined from convergence requirements and they will not be given explicitly.

2. Examples

By reading down the pages of [1], one can produce by inspection several hundred integral identities such as (2). A few examples should suffice. Since the Laplace transform of $t^{\nu-1}e^{-at}$ is $\Gamma(\nu)(t+a)^{-\nu}$, by the definition of the Gamma function we have

$$\Gamma(\mu) \int_0^\infty \frac{t^{\nu-1} e^{-at}}{(t+b)^\mu} dt = \Gamma(\nu) \int_0^\infty \frac{t^{\mu-1} e^{-bt}}{(t+a)^\nu} dt. \tag{3}$$

Next we note that since the Fourier cosine transform of $x^{\nu-1}e^{-ax}$ is $\Gamma(\nu)(x^2 + a^2)^{-\nu/2} \cos[\nu \tan^{-1}(x/a)]$, we have the identity

$$\Gamma(\mu) \int_0^\infty \frac{x^{\nu-1}}{(b^2 + x^2)^{\mu/2}} \cos[\mu \tan^{-1}(x/b)] e^{-ax} dx$$

$$= \Gamma(\nu) \int_0^\infty \frac{x^{\mu-1}}{(a^2+x^2)^{\nu/2}} \cos[\nu \tan^{-1}(x/a)] e^{-bx} dx. \quad (4)$$

Now, by setting $\mu = 1$ in (4), after a simple change of variable we have

$$\int_0^\infty e^{-bx} \frac{\cos[\nu \tan^{-1}(x/a)]}{(x^2+a^2)^{\nu/2}} dx = \frac{b^{\nu-1}}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} e^{-abx}}{(x^2+1)} dx$$
$$= \frac{\pi b^{\nu-1}}{\Gamma(\nu)\sin(\pi\nu)} V_\nu(2ab, 0), \quad (5)$$

where $V_\nu(x,y)$ is a Lommel function of two variables [3]. By this means we have worked out a, so far, untabulated Laplace transform. A similar calculation leads to

$$\int_0^1 \frac{J_0^2(ax)}{\sqrt{1-x^2}} \cos[2n \sin^{-1} x] dx = \int_0^1 \frac{J_n^2(ax)}{\sqrt{1-x^2}} dx$$
$$= \frac{a^{2n}\Gamma^2(n+1/2)}{2(n!)^2(2n)!} \, {}_2F_3(n+1/2, n+1/2; n+1, n+1, 2n+1; -a^2). \quad (6)$$

As an illustration that this procedure can produce identities of great generality, we have [4] for $m \le n$, $k \le l$,

$$\Gamma(\nu) \int_0^\infty x^{\nu-\mu-1} \, {}_mF_n(a_1,\ldots,a_m; b_1,\ldots,b_n; -x)$$
$${}_{k+1}F_l(c_1,\ldots,c_k,\nu; d_1,\ldots,d_l; -q/x) dx = \Gamma(\mu) \int_0^\infty x^{\mu-\nu-1}$$
$${}_{m+1}F_n(a_1,\ldots,a_m,\mu; b_1,\ldots,b_n; -1/x) \, {}_kF_l(c_1,\ldots,c_k; d_1,\ldots,d_l; -qx) dx. \quad (7)$$

Even the special case $q = 0$, which we write

$$\Gamma(\nu) \int_0^\infty x^{\mu-\nu-1} \, {}_mF_n(a_1,\ldots,a_m; b_1,\ldots,b_n; -x) dx$$
$$= \Gamma(\mu) \int_0^\infty x^{\mu-\nu-1} \, {}_{m+1}F_n(a_1,\ldots,a_m,\mu; b_1,\ldots,b_n; -x) \quad (8)$$

provides something interesting. Equations (7) and (8) are valid for all parameter values in the sense that either both sides converge to the same value or both diverge.

As an example resulting from a Hankel transform, we have (see [5]) for $|\nu| < 1/2$

$$\int_0^\infty \frac{\sinh[2\nu \sinh^{-1} x]}{x\sqrt{1+x^2}} e^{-x} dx = 2^{2\nu-1}\Gamma(2\nu-1) \, {}_1F_2(\frac{1}{2}-\nu; , 1-2\nu, \frac{3}{2}$$
$$-\nu; -\frac{1}{4}) - 2^{-2\nu-1}\Gamma(-1-2\nu) \, {}_1F_2(\frac{1}{2}+\nu; \frac{3}{2}+\nu, 1+2\nu; -\frac{1}{4})$$

$$+(2\nu)^{-1}\ {}_2F_3(\frac{1}{2},1;\frac{3}{2},1-\nu,1+\nu;-\frac{1}{4})+\frac{\pi}{2}\tan(\pi\nu). \tag{9}$$

This integral could not be found in tables nor evaluated by *Mathematica*. The limit $\nu \to 1/2$ can be evaluated and reproduces a known integral representation of a Struve function. Similarly, we find

$$\int_0^1 P_n(1-2x^2)J_{2m+1}(x)dx = \int_0^1 P_m(1-2x^2)J_{2n+1}(x)dx. \tag{10}$$

This relation appears to be new, although the separate integrals are known from the work of Bose [6] in terms of the generalized hypergeometric function. The insertion of his expression into (10) suggests the existence of an undocumented transformation formula for the ${}_2F_3$ function.

3. Conclusion

It is hoped that these few examples show that the procedure, even if carried out at random, is capable of producing interesting, informative identities contributing to our knowledge of special functions. Since widely used symbolic integration packages can produce integral transforms, it might be possible to carry out this procedure automatically for a given class of functions and then filter out the interesting results.

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References

[1] *Tables of Integral Transforms I, II* (Ed. A. Erdélyi), McGraw-Hill Book Company, New York (1954).

[2] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, London (1948).

[3] Reference [1], Volume 1, p. 372.

[4] Reference [1], equation 4.23 (17), p. 219.

[5] Reference [1], Volume 2, equation 8.2 (56), p. 18.

[6] B.N. Bose, *Bull. Calcutta Math. Soc.*, **36** (1944), 125-132.