International Journal of Pure and Applied Mathematics
Volume 61 No. 2 2010, 157-160

KAEHLER MANIFOLDS WITH THE RICCI TENSOR SATISFYING CERTAIN CONDITIONS

Yoshio Matsuyama
Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku, Tokyo, 112-8551, JAPAN
e-mail: matuyama@math.chuo-u.ac.jp

Abstract: The purpose of the present paper is to prove that the Ricci tensor of a Kaehler manifold with the cyclic parallel Ricci tensor is parallel.

AMS Subject Classification: 53C40, 53B25
Key Words: Kaehler maniolds, nearly parallel Ricci tensor, cyclic parallel Ricci tensor

1. Introduction

Let M be an n-dimensional Kaehler manifold. Then the Kaehler metric g of M satisfies $g(JX, JY) = g(X, Y)$ and $\nabla J = 0$, where J and ∇ denote the complex structure and the covariant differentiation of M, respectively. Let S be the Ricci tensor of M. It is well known that a Kaehler manifold with the parallel Ricci tensor is of Einstein if M is irreducible. The Ricci tensor S of type (1, 1) is called the *harmonic curvature* if M satisfies $(\nabla_X S)Y = (\nabla_Y S)X$ for any X and Y tangent to M. Matsushima [1] proved that the Ricci tensor of a Kaehler manifold with the harmonic curvature is parallel.

The Ricci tensor S of type (1, 1) is calleld the *nearly parallel Ricci tensor* if M satisfies $(\nabla_X S)X = 0$ for any X tangent to M. And the Ricci tensor S of type (0, 2) is called the *cyclic parallel Ricci tensor* if M satisfies $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$ for any X, Y and Z tangent to M.

Now, we consider the case of M being a real n-dimensional hypersurface of

Received: March 8, 2010

a real space form. Then Ono, Sekiguchi and the author showed (see [4], [5]) : Let M^n be a hypersurface of dimension $n \geq 2$ with constant mean curvature. Then the following are equaivalent: (1) M has the cyclic parallel Ricci tensor and (2) has the parallel Ricci tensor.

Hence, we can also consider the case of M being a Kaehler hypersurface of a complex space form (see [3], [6], [7]).

The purpose of this paper is to prove that the Ricci tensor of a Kaehler manifold with the cyclic parallel Ricci tensor is parallel, i.e., $\nabla S = 0$. We note that this condition is weaker than $\nabla S = 0$. We prove the following theorem:

Theorem 1. *Let M be a Kaehler manifold of complexdimension n with the cyclic parallel Ricci tensor. Then the Ricci tensor of M is parallel.*

2. Preliminaries

Let M be a Kaehler manifold of complex dimension n with the Kaehler metric g. Denoting the complex structure and the covariant differentiation on M by J and ∇, respectively, we have $\nabla J = 0$ and $g(JX, JY) = g(X, Y)$.

Then we have the following lemma.

Lemma 1. *In a Kaehler manifolds J and S commute.*

Proof. Let $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ denote an orthonormal basis for the tangent space T_xM at any point x of M. From the definition of S we have

$$\begin{aligned} g(SX, JY) &= \sum g(R(X, e_i)e_i, JY) + \sum g(R(X, Je_i)Je_i, JY) \\ &= \sum g(R(X, e_i)e_i, JY) + \sum g(R(X, Je_i)e_i, Y), \end{aligned}$$

where R denotes the curvature tensor of M. On the other hand, we have

$$\begin{aligned} g(JSY, X) &= \sum g(JR(Y, e_i)e_i, X) + \sum g(JR(Y, Je_i)Je_i, X) \\ &= -\sum g(R(Y, e_i)e_i, JX) - \sum g(R(Y, Je_i)e_i, X) \\ &= -\sum g(R(JX, e_i)e_i, Y) - \sum g(R(X, e_i)Je_i, Y) \\ &= \sum g(R(X, Je_i)e_i, Y) + \sum g(R(X, e_i)e_i, JY). \end{aligned}$$

Thus it holds $g(SX, JY) = g(JSY, X)$, i.e., $SJ = JS$. □

Next, we prove the following lemma.

Lemma 2. *Assume $(\nabla_X S)X = 0$. Then we have $(\nabla_X S)Y = 0$.*

Proof. From the assumption we have

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) = 0,$$
$$(\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0,$$
$$(\nabla_Z S)(X,Y) + (\nabla_X S)(Y,Z) = 0$$

for any X, Y and Z. Hence, we obtain

$$2(\nabla_X S)(Y,Z) + 2(\nabla_Y S)(Z,X) + 2(\nabla_Z S)(X,Y) = 0.$$

Thus we have $(\nabla_X S)(Y,Z) = 0$, i.e., $(\nabla_X S)Y = 0$. □

3. Proof of Theorem

Now, we prove the following:

Theorem 2. *If* $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$ *in a Kaehler manifold*, then $(\nabla_X S)Y = 0$.

Proof. We first remark the following:

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \nabla_X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)\\ &= \nabla_X(g(SY,Z)) - g(S\nabla_X Y, Z) - g(SY, \nabla_X Z)\\ &= g((\nabla_X S)Y, Z) + g(S\nabla_X Y, Z) + g(SY, \nabla_X Z)\\ &\quad - g(S\nabla_X Y, Z) - g(SY, \nabla_X Z)\\ &= g((\nabla_X S)Y, Z).\end{aligned}$$

From the assumption of theorem we have

$$\begin{aligned}g((\nabla_{JX} S)JY, Z) &= -g((\nabla_{JY} S)Z, JX) - g((\nabla_Z S)JX, JY)\\ &= -g((\nabla_{JY} S)JX, Z) - g((\nabla_Z S)X, Y).\end{aligned}$$

Similarly, we have

$$g((\nabla_X S)Y, Z) = -g((\nabla_Y S)Z, X) - g((\nabla_Z S)X, Y).$$

Hence we obtain

$$(\nabla_X S)Y + (\nabla_Y S)X = (\nabla_{JX} S)JY + (\nabla_{JY} S)JX.$$

Taking a place JX for Y, we have

$$(\nabla_X S)JX + (\nabla_{JX} S)X = 0.$$

Hence we get

$$-(\nabla_X S)X + (\nabla_{JX} S)JX = 0.$$

On the other hand, we know that JS is also cyclic. Then we have

$$g((\nabla_Y JS)X, JX) + g((\nabla_X JS)JX, Y) + g((\nabla_{JX} JS)Y, X) = 0.$$

Hence we get

$$g((\nabla_Y S)X, X) - g((\nabla_X S)X, Y) - g((\nabla_{JX} S)JX, Y) = 0.$$

Since we have

$$g((\nabla_Y S)X, X) + g((\nabla_X S)X, Y) + g((\nabla_{JX} S)JX, Y) = 0,$$

Thus we obtain

$$-3(\nabla_X S)X - (\nabla_{JX} S)JX = 0$$

Therefore $(\nabla_X S)X = 0$. From Lemma 2 we obtain the conclusion. □

References

[1] A.L. Besse, *Einstein Manifolds Reprint of 1987 Editon, Classics in Mathematics*, Springer Verlag, Berlin (2008).

[2] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Volume II, Interscience Tracts, John Wiley and Sons, New York (1963).

[3] K. Nomizu, B. Smyth, Differential geometry of complex hypersurfaces, II, *J. Math. Soc. Japan*, **20** (1968), 498-521.

[4] T. Ono, S. Sekiguchi, Y. Matsuyama, On a hypersurface with the Ricci tensor satisfying certain conditons, *International Journal of Pure and Applied Mathematics*, **51**, No. 2 (2009), 293-301.

[5] P.J. Ryan, Hypersurfaces with parallel Ricci tensor, *Osaka J. Math.*, **8** (1971), 251-259.

[6] P.J. Ryan, A class of complex hypersurfaces, *Colloquium Mathematicum*, **26** (1972), 177-182.

[7] B. Smyth, *Differential Geometry of Complex Hypersurfaces.*