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ON THE WALRASIAN-SAMUELSON PRICE ADJUSTMENT MODEL

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Abstract: We introduce a type of Itô process that models the adjustment of the market price of a traded security to new information affecting supply and demand of an asset. It is based on supply and demand functions and the Walrasian price adjustment assumption that proportional price increase is driven by excess demand. When supply and demand curves are linearised about the equilibrium point, the process turns out to be a logistic form of Brownian motion with random element of the Wiener type. Finally we derive the modified Black-Scholes Merton partial differential equation.

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1. Introduction

Stochastic processes play an important role in the mathematical treatment of financial instruments such as equities, commodities and derivatives contracts based on these. These processes can be used to model (or price) not only traded products subject to random movements of markets, but also fixed-income products such as bonds, options and futures. The Brownian motion of market prices,

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also known in financial mathematics as the Wiener-Bachelier process (Feller 1957), can be traced back to the start of 20-th Century when a French mathematician, Louis Bachelier [2] presented his PhD thesis in which he analysed stock market fluctuations.

2. The Black-Scholes-Merton (B-S-M) Model

One of the major assumptions of Black-Scholes-Merton model is that the asset price follows a linear geometric Brownian motion. This is given by

$$dX(t) = aX(t)dt + bX(t)dZ(t), \quad a > 0,\ b > 0,\ X(0) = X_0, \tag{1}$$

where a represents the annual mean of returns $\frac{dX(t)}{X(t)}$ and b represents the annual volatility of returns. In particular, for interval $[0, t]$ equation (1) is expressible as an integral form as

$$X(t) = X_0 + a\int_0^{t_i} X(s)ds + b\int_0^{t_i} X(s)dZ(s), \tag{2}$$

where $X_0 > 0$. Equation (1) can also be expressed as a law of asset returns

$$\frac{dX(t)}{X(t)} = adt + bdZ(t), \quad X(0) = X_0.$$

2.1. Asset Price Models: European Option Models

In this section we summarize the Black-Scholes-Merton model which is used to price a European option of an asset.

The theory of option pricing has a long history, dating back to Louis Bachelier [2], Paul Samuelson [10], and several others. The later seminal contributions of Fischer Black and Myron Scholes [3] and Robert Merton [5] have brought the theory to point of widespread use in finance.

In the next section we present models that have been used to price European type options of traded assets.

2.2. The Black-Scholes Option Pricing Model (BSOPM)

In their seminal paper, Black and Scholes [3] and Merton [5] derived an explicit solution to the problem of pricing and hedging a European call or put option on a non-dividend paying asset by using the Samuelson's model. The option prices being determined only by observable variables of the asset: the current price $S(t)$, of the underlying asset, the strike price E, the expiration date, T, of the contract and the interest rate, r, plus the volatility s, of the underlying asset. One of the assumptions is that the underlying asset-price $S(t)$, follows a lognormal random walk process, also known as linear geometric Brownian motion process with drift. This process is expressed as

$$dS(t) = \mu S(t)dt + \sigma S(t)dZ(t), \tag{3}$$

where μ (drift or rate of return) and σ (volatility) are constants.

Suppose we have an option whose value is $V(S,t)$, and depends on S and t by the Itô's Lemma, the Black-Scholes-Merton's partial differential equation is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{4}$$

This differential equation does not depend on the rate of return. With its extension and variants, this equation plays a major role in the determination of the value of an option.

3. Relaxation of the Black-Scholes-Merton Model: Walrasian-Samuelson Price Adjustment Models

In this section we relax one of the assumptions used to derive the Black-Scholes-Merton model. We focus on the possibility of non-linear drifting as markets adjust to radically new perceptions of the value of asset. The basic driver of such price adjustment, taking a neo-Walrasian view (Walras [12]; Samuelson, [9], [13]; Walker, 1996) is the excess demand over supply at the trading point. We use a linearised version of this driving force to drive an Itôprocess that models price adjustment in non-steady markets. It is a logistic type process with diffusive Wiener variation – referred to as 'Verhulst-logistic Brownian motion'.

3.1. The Model

Since the 1960s, a model of Walrasian (Walras, [12]) tatonnement has been used to study stability of general price equilibrium, Samuelson [10], Arrow and Hahn (1971), Anderson (2000) and Asparouhova and others (2000), among others. In this presentation, we take the core principle of the standard Walrasian model. That is: asset price changes are directly driven by the excess demand for the asset. For simplicity we do not allow cross-asset effects that might be experienced in a multi-security market. The dynamic price-adjustment rule in such simplified markets may be expressed in continuous-time Walrasian-Samuelson form:

$$\frac{1}{p(t)}\frac{dp(t)}{dt} = \begin{cases} kED(p(t)), \\ 0, \quad \text{if } p(t) = 0 \text{ and } ED(P(T)) = 0, \end{cases} \tag{5}$$

where t represents continuous time, and $k > 0$ is a positive market adjustment coefficient and $ED(Pt)) = Q_D(P(t)) - QS(P(t))$ is excess demand taken as a continuous function of price $P(t)$. Equation (5) can be expressed as a linear function, thus, we have

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = k(Q_D(P(t)) - Q_S(P(t))) \tag{6}$$

(Takayama, 1996; Kumar and Shubik, 2002; Onyango, 2003).

3.2. Deterministic Price Adjustment Model

To make the price adjustment more computational, we begin by taking supply and demand functions to be fixed functions of instantaneous price $P(t)$. Then at equilibrium asset price point, P^*, demand $Q_D(P^*)$ is equal to supply, $Q_S(P^*)$. On the assumption of fixed supply and demand curves, P^* is constant. Away from equilibrium, excess demand for the security will raise its price, P, and an excess supply will lower its price. Thus the sign of the rate of change of price, P, with respect to time, t, will depend on the sign of the excess demand. If we linearise $Q_D(P(t))$ and $Q_S(P(t))$ about the constant equilibrium price P^*, the deterministic model of price adjustment becomes

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = k(\alpha + \beta)(P^* - P(t)), \tag{7}$$

where $Q_D(P(t)) = \alpha(P^* - P(t))$, $Q_S(P(t)) = -\beta(P^* - P(t))$, and constants α and β are demand and supply sensitivities respectively. Putting $r = k(\alpha + \beta)$ in equation (7), we get the deterministic logistic equation

$$\frac{dP(t)}{dt} = rP(t)(P^* - P(t)) \tag{8}$$

(Lungu and Øksendal, 1997; Pollett, 2002; Dwyer, 2003; Onyango, 2003, 2005). This is a deterministic logistic (first- order) ordinary differential equation in $P(t)$. Thus the fractional growth of $P(t)$ is linear in $P(t)$. This contrasts with exponential growth, where the fractional growth is constant (independent of $P(t)$).

The logistic equation was first investigated by Pierre-Francois Verhulst (Verhulst, 1838) as an improvement on the Malthusian model of population dynamics, hence it is also known as Verhulst-logistic differential equation. Since then it has been applied in several areas.

The solution set of equation (8) is given by

$$P(t) = \frac{P^* P(0)}{P(0) + (P^* - P(0))e^{-rP^*t}}, \tag{9}$$

where $P(0)$ is a parameter interpreted as the initial price an asset. From equation (9) we observe as $t \to 8$, the term $P(t) \to \frac{P^* P(0)}{P(0)} = P^*$ in the denominator will be negligibly small. The asset price thus settles into a constant level, called a steady state or equilibrium, at which no further change will occur.

4. Logistic Price Adjustment Model

In this section we model random fluctuations in supply and demand by changes $\delta\alpha$, $\delta\beta$ in the respective sensitivities. Consider that, $Q_D(P(t))$ and $Q(P(t))$ to represent averaged effects of supply and demand respectively, and suppose that both curves steepen or level off in response to random observed trades: cumulatively they execute a random walk or Wiener diffusion process. From equation (7) we have

$$\frac{dP(t)}{dt} = k(\alpha + \beta)(P^* - P(t)) + k(\delta\alpha + \delta\beta)(P^* - P(t))$$

or

$$\frac{dP(t)}{P(t)(P^* - P(t))} = k(\alpha + \beta)dt + k(\delta\alpha + \delta\beta)dt. \tag{10}$$

From equation (10), we may put $\mu = k(\alpha + \beta)$ (logistic growth parameter) and $\sigma dZ = k(\delta\alpha + \sigma\beta)dt$ (noise process) to obtain

$$\frac{dP(t)}{P(t)(P^* - P(t))} = \mu dt + \sigma dZ. \tag{11}$$

Equation (10) defines an Itô process evolving according to the stochastic differential equation

$$dP(t) = \mu P(t)(P^* - P(t))dt + \sigma P(t)(P^* - P(t))dZ. \tag{12}$$

We refer to equation (11) as logistic price adjustment model (LBM model), or Verhulst-price adjustment model (VPAM) (Onyango, 2003). In the risk-less case ($\sigma = 0$), equation (11) reduces to the logistic equation (8) with solution

(10). Using Itô's Lemma, the solution of (11) is expressed as

$$ln\left(\frac{P(t)}{|P^* - P(t)|}\right) = ln\left(\frac{P(0)}{|P^* - P(0)|} + \mu P^*(t - t_0) + \sigma P^* Z(t)\right). \tag{13}$$

Re-arranging and simplifying equation (13), we get

$$P(t) = \frac{P^* P(0)}{P(0) + (P^* - P(0))e^{-(\mu P^*(t-t_0)+\sigma P^* Z(t))}}. \tag{14}$$

This price dynamics is referred to as logistic Brownian motion of asset price, $P(t)$. When $\sigma = 0$, then we get the deterministic logistic equation (9).

5. Partial Differential Equation for Logistic Price Adjustment Model

In this section we derive the partial differential equation for logistic price adjustment model. Let $V(S,t)$ be the option value depending on asset price, S and time t, then by Itô's Lemma we have

$$dV\Big(S(t),t\Big) = \frac{\partial V(S,t)}{\partial t}dt + \frac{\partial V(S,t)}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V(S,t)}{\partial S^2}dS^2. \tag{15}$$

For logistic Brownian motion we have

$dS = \mu S(S^* - S)dt + \sigma S(S^* - S)dZ$ and $dS^2 = \mu S(S^* - S)dt + \sigma^2 S^2 (S^* - S)^2 dt$.

Substituting in (15) and simplifying we get

$$dV\Big(S(t),t\Big) = \mu S(S^* - S)\frac{\partial V(S,t)}{\partial S} + \frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 (S^* - S)^2 \frac{\partial^2 V(S,t)}{\partial S^2}dt + \sigma S(S^* - S)\frac{\partial V(S,t)}{\partial S}dZ(t). \tag{16}$$

By using the no-arbitrage argument, which implies that the percentage return of the portfolio over the time interval dt should equal the risk-free interest rate, r. That is

$$d\pi(t)_{risk-free} = r\pi(t)_{risk-free}. \tag{17}$$

Thus we get

$$\left(\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2}{2}S^2(S^* - S)^2\frac{\partial^2 V(S,t)}{\partial S^2}\right)dt = r\left(V(S,t) - \frac{\partial V(S,t)}{\partial S}S\right)dt. \tag{18}$$

Further simplification of (18) yields a partial differential equation given as

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2}{2}S^2(S^* - S)^2\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0. \tag{19}$$

This is the modified Black-Scholes-Merton partial differential equation. It is a non-linear partial differential equation whose solution might be got by using numerical methods or Monte Carlo methods.

6. Conclusion

In this paper we have relaxed one of the assumptions made by Black and Scholes [5] and Merton [3] and used the excess demand functions and Walrasian price adjustment principle to derive a logistic Brownian motion in terms of the market equilibrium, S^* and the asset price S at any time t. Finally we have used the no-arbitrage argument to derive a modified Black-Scholes-Merton partial differential equation which may be solved to get the value of an option at expiry date. Further research is needed to get the solution of this non-linear partial differential equation.

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