International Journal of Pure and Applied Mathematics
Volume 61 No. 3 2010, 253-267

THE INITIAL VALUE SEMILINEAR DIFFUSION EQUATIONS ON ORLICZ SPACES

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Abstract: This paper is an extension of the L^{p} solutions of an initial value semilinear diffusion equation to include Orlicz spaces. It is shown that if Φ is a $\triangle_2$N-function such that the initial value belongs to $\mathrm{L}^{\Phi}(\mathrm{R}^{\mathrm{n}})$, then the maximal solution u^* of the initial value problem belongs to $\mathrm{L}^{\Phi}(\mathrm{R}^{\mathrm{n}})$ and is unique. Some regularity and almost everywhere point-wise convergence results in $\mathrm{L}^{\Phi}(\mathrm{R}^{\mathrm{n}})$ are also given.

AMS Subject Classification: 31B10, 35G10, 35C15, 35D10, 46E35
Key Words: Orlicz spaces, semiliear diffusion equation, regularity, maximal solution

1. Introduction

We consider here an initial value semilinear diffusion equation of the form

$$u_{\mathrm{t}} - \Delta u = f(u(x,t)), \quad (x,t) \in R^{\mathrm{n}} \times [0,\infty), \tag{1}$$

$$u(x,0) = h(g(x)), \quad x \in R^{\mathrm{n}}. \tag{2}$$

The function f is such that there exist a $\triangle_2$N-function ϕ and a convex function

Received: March 5, 2010

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χ satisfying $\left\|f(u^*)\right\|_\phi \leq \chi(\|u^*\|_\phi)$. The case where f is a power function of the form $x^{\alpha+1}$; $\alpha > 0$ is a real number (or an integer) and h is the identity function $h(x) = x$ problem (1)-(2) has been extensively studied in [6], [7], [8], [9], [10], [13], [16], [22], [24], [25] for various restrictions on α in different Sobolev spaces and the like, such as the L^p and Hilbert spaces.

In an earlier paper [13], the application of potential operator theory was employed to study the weak solutions of problem (1)-(2) when the initial data is in some L^p spaces. Using the techniques employed in [2], [3], [5], [12], [14] the existence and uniqueness of maximal solutions was established when the restriction $\frac{n}{2} > \frac{\alpha+1}{\alpha}$; $n > 2$ is imposed on the spatial dimension and the initial data is very small in norm. In the same paper, the regularity results of maximal solutions and pointwise a.e. convergence of solutions in some L^p spaces where $p = \frac{\alpha n}{2}$ were also given.

Since the development of Orlicz spaces arose naturally from the need for a more inclusive class of spaces than the L^p-families [11], [17], [18] and subsequent efforts have been made in the direction of extending the L^p result to include Orlicz spaces, the work in [13] could not be complete without the extension of the results found there to Orlicz spaces. Motivated by this need, in this paper we will extend the results of [13] to the setting of Orlicz spaces. Thus, in essence the paper is an application of Orlicz spaces. As in [13], we will begin by introducing some standard notations and the derivation of the weak solution of problem (1)-(2). Throughout the paper, $x = (x_1, ..., x_n)$ will denote a point in R^n and $|x| = \left(x_1^2 + ... + x_n^2\right)^{\frac{1}{2}}$ will take the usual meaning of the distance of the point x from the origin.

If we let W(x,t) be the fundamental solution of the diffusion equation $u_t = \triangle u$, then it is well known that a formal integral solution of (1)-(2) is given by

$$u(x,t) = \int_0^t \int_{R^n} W(x-y,t-\tau)u^{\alpha+1}(y,\tau)dyd\tau + \int_{R^n} W(x-y,t)g(y)dy. \quad (3)$$

If we further let ϕ be the perturbed nonlinear operator, then we apply Orlicz spaces in consideration of the nonlinear integral equation

$$(\phi u)(x,t) = \int_0^t \int_{R^n} W(x-y,t-\tau)u^{\alpha+1}(y,\tau)dyd\tau + \int_{R^n} W(x-y,t)g(y)dy. \quad (4)$$

Throughout this paper, we consider a solution u of (3) to be a weak solution of (1)-(2) for all $t > 0$ whenever the integrals involved exist in the Lebesgue sense over R^n for all values $t > 0$. We will demonstrate that the existence and uniqueness result in [13] (see Lemma 5) is much simpler and neatly obtained in Orlicz spaces than with the L^p spaces.

In [13] we made use of the standard properties of the fundamental solution W, the Hardy-Littlewood-Sobolev result, and the interpolation results of L^p spaces. Following the same tradition, we begin this paper by extending to the setting of Orlicz spaces results about the Hardy-Littlewood-Sobolev Theorem or, as it is sometimes known, the Riesz-Thorin Interpolation Theorem. In Section 2 we review some basic properties of the theory of Orlicz spaces and restate some of the L^p results and their generalization in Orlicz spaces. Proofs of new results are given and we have indicated where the proofs of the known results can be found. In Section 3 the main results are given, and in Section 4 a regularity result and proof is given.

In what follows, we shall consider the following standard Solonnikov estimates of the fundamental solution W which are given in [3], [5], [12], [13], [14], [19], [20]

$$|W(x,t;\ y,\tau)| \leq c\left(|x - y| + |t - \tau|^{\frac{1}{2}}\right)^{-n}, \tag{5}$$

where $c > 0$ is a constant and n is the space dimension. The estimate for the spatial derivatives of W is given by

$$\left|D_x^\beta W(x,t;\ y,\tau)\right| \leq C\left(|x - y| + |t - \tau|^{\frac{1}{2}}\right)^{-n-|\beta|}, \tag{6}$$

where the derivatives exist in the distribution sense, and $\beta = (\beta_1, ..., \beta_n)$, $|\beta| = \beta_1 + ... + \beta_n$, $D_x = \left(\frac{\partial}{\partial x_1}\cdot\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n}\right)$, $D_x^\beta = \left(\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}\cdot\frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}}\cdots\frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}\right)$. We will also consider the Hardy-Littlewood Maximal function f^* of a function f defined as $f^*(x) = \sup \frac{1}{|Q|}\int_Q |f(y)|dy$, where the supremum is taken over all cubes Q with center at x and edges parallel to the coordinate axes, in Orlicz spaces. As in [13], we shall define $u^*(x) = \sup_{t>0} |u(x, t)|$ to be the maximal function of u(x, t) over $t > 0$ defined on R^n. We note the following well known properties of W:

(i) $\int_{R^n} W(x,t)dx = 1$.

(ii) $\sup_{t>0} \int_{R^n} W(x-y,t)f(y)dy \leq cf^*(x)$, where c is independent of f.

(iii) $\sup_{t>0} \left|\int_{R^n} \frac{t^{|\beta|}f(x-y)dy}{|y|^{n+|\beta|} + t^{n+|\beta|}}\right| \leq Cf^*(x)$.

(iv) $W \in C_0^\infty$.

2. Orlicz Spaces

In this section, we shall provide only definitions and theorems that are relevant to obtaining the desired results in Sections 3 and 4. We refer the reader to details about the theory of Orlicz spaces, its properties, characterization as a Banach space and applications to [1], [11], [15], [17], [18].

Definition 1. A convex function $\Phi : R \to R_+$ satisfying the conditions: $\Phi(\text{-x}) = \Phi(x)$; $\Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$ is called a Young's function. Associated with Φ is another convex function Ψ: $R \to R_+$ having similar properties and defined as $\Psi(y) = \sup\{x|y| - \Phi(x): x \geq 0\}$; $y \in R$. It is known that the pair (Φ, Ψ) of complementary functions satisfy the Young's inequality $xy \leq \Phi(x) + \Psi(y)$.

Definition 2. If in addition to the properties in Definition 1, Φ satisfies: $\Phi(x) = 0$ iff $x = 0$, $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$ and $\Phi(R) \subset R_+$, then Φ is called a nice Young's function, or an N-Function.

Definition 3. Let Φ: $R \to R_+$ be a Young function. Then it can be represented as $\Phi(x) = \int_0^x \varphi(t)dt$; $x \in R_+$, where $\varphi(0) = 0$, φ: $R_+ \to R_+$ is non-decreasing left continuous and if $\varphi(x) = \infty$ for $x \geq a$ then $\varphi(x) = \infty$; $x \geq a \geq 0$. If we let ψ be the generalized inverse of the monotone function φ, then we define $\Psi(y) = \int_0^y \psi(t)dt$; $y \geq 0$.

In this setting it is clear that Ψ is convex with $\Psi(0) = 0$. Thus, for the remainder of the text, we shall adopt Definition 3 as the means by which complementary functions can be obtained.

Definition 4. A Young's function Φ: $R \to R_+$ is said to satisfy the $\triangle_2$-condition, denoted $\Phi \in \triangle_2$, if $\Phi(2x) \leq K\Phi(x)$; $x \geq x_0 \geq 0$ for some absolute constant $K > 0$.

Let Φ be an N-function and let Ω be an open set in R^n. We denote by $L^\Phi(\Omega)$ the set, called an Orlicz class, of measurable functions f on Ω such that

$$\rho(f, \Phi, \Omega) := \int_\Omega \Phi(f(x))dx \ < \ \infty.$$

Let (Φ, Ψ) be a pair of complementary N-functions and let f be a measurable function defined almost everywhere in Ω. The Orlicz norm of f denoted by either

$\|f\|_{\Phi,\Omega}$ or $\|f\|_{\Phi}$ is defined by

$$\|f\|_{\Phi} = \sup\{\int |f(x)g(x)dx| : g \in L^{\Psi}(\Omega) \text{ and } \rho(g, \Psi, \Omega) \leq 1\}.$$

The set $L^{\Phi}(\Omega)$ of measurable functions f on Ω such that $\|f\|_{\Phi} < \infty$ is called an Orlicz space. The normalized version of this definition is called the Luxemburg norm denoted by $\|f\|_{\Phi}$ and is defined in $L^{\Phi}(\Omega)$ as $\|f\|_{\Phi} = \inf\{k > 0$: $\int_{\Omega} \Phi(\frac{f(x)}{k})dx \leq 1\}$. Throughout the paper, when $\Omega = R^n$, we let $L^{\Phi}(\Omega)$ be $L^{\Phi}(R^n)$.

Definition 5. If Φ is a Young's function then Φ^{-1} is defined for $0 \leq y \leq \infty$ by $\Phi^{-1}(y) = \inf\{x\colon \Phi(x) > y\}$, $\inf \emptyset = \infty$. By this definition ψ is the inverse of φ. We now make the following important observations:

1. For $0 \leq x < \infty$, $\Phi(x) = \sup\{y\colon \Phi^{-1}(y) < x\}$, where we take the sup of the empty set to be zero.

2. The domain of Φ is $[0, \infty)$; the domain of Φ^{-1} is $[0, \infty]$.

3. In all cases $\Phi^{-1}(\infty) = \infty$.

4. Φ is continuous to the left while Φ^{-1} is continuous to the right. By allowing Φ to jump to ∞ at x_1 say, we may include L^{∞} as an Orlicz space.

In order to use the interpolation of singular integrals and the convolution of two functions in Orlicz spaces in the study of the integral equation in (4), we shall need the following generalized Young's inequalities, whose proof can be found in [15].

Lemma 1. *If Φ_i; $i = 1, 2, 3$ are N-functions such that for $x \geq 0$, $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq x\Phi_3^{-1}(x)$, then for $x, y \geq 0$, we have $xy \leq \Phi_1(x)\Phi_3^{-1}(\Phi_2^{-1}(y)) + \Phi_2(y)\Phi_3^{-1}(\Phi_1(x))$.*

Theorem 2. *Suppose Φ_i, $i = 1, 2, 3$ are N-functions which satisfy for $x \geq 0$ $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq x\Phi_3^{-1}(x)$ and that $f \in L^{\Phi_1}(\Omega)$, $g \in L^{\Phi_2}(\Omega)$ on a locally compact unimodular topological group (Ω, μ), where μ is the Haar measure. Then their convolution $h(x) := (f * g)(x) = \int_{\Omega} f(t)g(t^{-1}x)dt$ is in $L^{\Phi_3}(\Omega)$, and $\|h\|_{\Phi_3} \leq 2 \|f\|_{\Phi_1} \|g\|_{\Phi_2}$.*

Lemma 3. (Generalized Hölder Inequality) *If Φ_i, $i = 1, 2, 3$ are N-functions such that $\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \Phi_3^{-1}(x)$, and if $f \in L^{\Phi_1}(\Omega)$, $g \in L^{\Phi_2}(\Omega)$ on a measurable space (Ω, μ), then the product $h(x) := f(x)g(x) \in L^{\Phi_3}(\Omega)$ and $\|h\|_{\Phi_3} \leq 2 \|f\|_{\Phi_1} \|g\|_{\Phi_2}$.*

The convolution result in Theorem 2 is in general true for singular integrals, however the generalized convolution of two functions $h(x) := \int_{R^n} f(x-y)g(y)dy$ have well known L^p Young's inequality which also has a generalization in Orlicz spaces. That is, it is well known that if $f \in L^p$, $g \in L^q$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ - 1 and $\frac{1}{p} + \frac{1}{q} \geq 1$; $1 \leq p, q < \infty$, and r >1, then $h \in L^r$ and $\|h\|_r \leq \|f\|_p \|g\|_q$. From which it is readily seen that if q = 1, then r = p and $\|h\|_p \leq \|g\|_1 \|f\|_p$. We will only state here the equivalent results in the case of Orlicz spaces and indicate where the proof may be found.

Theorem 4. *Let Φ be a Young function and $L^\Phi(R^n)$ be the corresponding Orlicz space on the Lebesgue measure space (R^n, μ). If $L^\Phi(R^n)$ is closed under convolution in the sense that f, $g \in L^\Phi(R^n)$ implies $f * g \in L^\Phi(R^n)$, then there exists a constant $0 < K < \infty$ such that $\|f * g\|_\Phi \leq K\|f\|_\Phi \|g\|_\Phi$.*

For the proof, see [17], [18].

Next we will extend L^Φ to include distributions of functions on the measurable space (Ω, μ) and denote the new space by M^Φ as follows: If f is a complex (or real-) valued Lebesgue measurable function on a measure space (Ω, μ), then for $\lambda \geq 0$ we define $m(f, \lambda) = \mu\ (E_\lambda)$, where E_λ is the distribution $E_\lambda = \{x$: $|f(x)| > \lambda$; $- \infty < \lambda < \infty$. $m(f,\lambda)$ is a monotone non-increasing function which takes the non-negative reals into $[0, \infty]$. We define its inverse f^{-1} as $f^{-1}(x) = \inf\{\lambda$: $m(f,\lambda) \leq x$; $x \geq 0\}$. f^{-1} is called the non-increasing rearrangement of |f| onto the positive reals. For more properties of E_y we refer the reader to [23]. As stated in the introduction, we shall consider the Hardy-Littlewood maximal function denoted by f^* of a Lebesgue measurable function f on subsets of R^n. It is well known that f^* is not integrable over R^n (unless f = 0 a.e.), but does satisfy the weak-type condition $\mu\{x \in R^n : f^*(x) > \lambda\} \leq \frac{c}{\lambda} \|f\|_1$; $\lambda > 0$, where c depends only on n. The behavior of f^*on the other L^p spaces, $1 < p \leq \infty$ is given in

Theorem 5. *Let $1 < p \leq \infty$ and $f \in L^p(R^n)$. Then $f^* \in L^p(R^n)$ and $\|f^*\|_p \leq C\|f\|_p$ where C depends only on n and p.*

The equivalent of Theorem 5 in Orlicz spaces is contained in

Theorem 6. *Let Φ and Ψ be $\triangle_2$ complimentary N - functions and let f^* be the Hardy-Littlewood maximal function. If $f \in L^\Phi(R^n)$, then we have $\left\|f^*\right\|_\Phi \leq c\|f\|_\Phi$, where c depends on λ, n and Ψ.*

See [21] for the proof.

Definition 6. $f \in M^\Phi$ on (Ω, μ) iff there is a positive number K so large that $(f^{-1})^{-1} \leq K\, A^{-1}(\frac{1}{x})$. $\|f\|_{M^\Phi}$ is defined as the infimum of all such K. The fact that M^Φ is a Banach space under the norm $\|.\|_{M^\Phi}$ can be found in [15].

Lemma 7. *If Φ_1, Φ_2 are Young's functions such that $\int_0^1 \frac{\Phi_1^{-1}(t)\Phi_2^{-1}(t)dt}{t^2} < \infty$ and $\Phi_3^{-1}(x) = \int_0^x \frac{\Phi_1^{-1}(t)\Phi_2^{-1}(t)dt}{t^2}$, then Φ_3 is a Young's function.*

See [15] for the proof.

Let Φ be a Young's function satisfying the $\triangle_2$-condition, then there is a constant $p \geq 1$ such that

$$x\varphi(x) \leq p\Phi(x), \text{ where } \Phi(x) = \int_0^x \varphi(t)dt. \tag{2.1}$$

(2.1) with $p < \infty$ is equivalent to a condition on the complementary Young's function Ψ, namely

$$x\psi(x) \geq q\Psi(x), \quad q > 1, \tag{2.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\psi(x)$ is inverse to $\varphi(x)$ and $\Psi(x) = \int_0^x \psi(t)dt$. A Young's function Ψ which satisfies (2.2) gives rise to an Orlicz space L^Ψ. We shall need one fact about Young's function satisfying (2.2).

Lemma 8. *If Ψ is a Young's function such that $p\Psi(x) \leq x\psi(x)$, $p > 1$, then $\int_0^x \frac{\psi(t)}{t}dt \leq q\frac{\Psi(x)}{x} \leq \frac{q}{p}\psi(x)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Lemma 9. *If Ψ is a Young's function such that $p\Psi(x) \leq x\psi(x)$, $p > 1$, then $\int_0^{\Psi(y)} \frac{dt}{\Psi^{-1}(t)} \leq q\psi(y)$.*

With these properties, we are now ready to generalize the L^p Hardy-Littlewood-Sobolev Theorem in Orlicz spaces. We restate the well known L^p result

Theorem 10. (Hardy-Littlewood-Sobolev) *Let $0 < \mu < n$, $1 \leq p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\mu}{n}$.*

(a) If $f \in L^p(R^n)$, then $(I_\mu f)(x) = [\gamma(\mu)]^{-1} \int_{R^n} |x - y|^{\mu - n} f(y)dy$; $\gamma(\mu) = \frac{\left[\pi^{\frac{3}{2}} 2^\mu \Gamma(\frac{\mu}{2})\right]}{\Gamma(\frac{n}{2} - \frac{\mu}{2})}$ converges absolutely for almost every x.

(b) If, in addition, $1 < p$, then $\left\|(I_\mu f)\right\|_q \leq A_{p,q} \|f\|_p$.

For easy notation, we rewrite $f_\alpha(x) = \int_{R^n} G_\alpha(y)f(x - y)dy$, where $G_\alpha(y) = [\gamma(\mu)]^{-1} |x - y|^{\mu - n}$. By integrating $G_\alpha(y)$ over cubes centered at the origin it is easily seen from [1], [15], [21], [23] that $G_\alpha \in M^{\Phi_1}$, where $\Phi_1(x) = x^{\frac{n}{(n - \mu)}}$. On taking $\Phi_2(x) = x^p$, we have from $\Phi_3^{-1}(x) = \int_0^x \frac{\Phi_1^{-1}(t)\Phi_2^{-1}(t)dt}{t^2}$ that $\Phi_3(x) = \frac{x^{\frac{1}{q}}}{1/q}$. Hence $L^{\Phi_3}(R^n) = L^q(R^n)$. We, thus have

Theorem 11. *If Φ_1, Φ_2 are $\triangle_2$N-functions such that $\int_0^1 \frac{\Phi_1^{-1}(t)\Phi_2^{-1}(t)dt}{t^2} < \infty$, $p\Phi_2(x) \leq x\varphi(x)$, $p > 1$, and the $\triangle_2$N-function Φ_3 is defined by $\Phi_3^{-1}(x) = \int_0^x \frac{\Phi_1^{-1}(t)\Phi_2^{-1}(t)dt}{t^2}$, and if furthermore, $G_\alpha \in M^{\Phi_1}$,and $f \in L^{\Phi_2}$, then $f_\alpha(x) = \int_{R^n} G_\alpha(y)f(x - y)dy$ belongs to L^{Φ_3} and $\|f_\alpha\|_{\Phi_3} \leq K(n, \mu)\|f\|_{\Phi_2}$.*

Theorem 12. *Let $H_{|\beta|}(f)(x) = \sup_{t>0} \int_{R^n} \frac{t^{|\beta|}f(y)dy}{|x-y|^{n + |\beta|} + t^{n + |\beta|}}$. Let Φ and Ψ be a pair of $\triangle_2$ complementary N-functions. If $f \in L^\Phi(R^n)$, then there exist a constant c depending only on Ψ such that $H_{|\beta|}(f) \in L^\Phi(R^n)$ and $\left\|H_{|\beta|}(f)\right\|_\Phi \leq C\|f\|_\Phi$.*

Proof. The proof follows from property (iii) of Introduction and Theorem 6. □

3. Main Results

Lemma 13. *Let Φ be a $\triangle_2$ N-function and let $(T_g)(x,t) = \int_{R^n} W(x-y, t)g(y)dy$. If $g \in L^\Phi(R^n)$ then T_g: $L^\Phi(R^n) \to L^\Phi(R^n)$ and $\|T_g\|_\Phi \leq K\|g\|_\Phi$.*

Proof. For some constant $\kappa > 0$, we have $\frac{(T_g)}{\kappa} = \int_{R^n} W(x\text{-}y, t)\frac{g(y)}{\kappa}dy$. Since $\int_{R^n} W(x, t)dx = 1$ and for a Φ satisfying the conditions of the theorem, we have on applying the Jensen's integral inequality that $\Phi(\frac{T_g}{\kappa}) = \Phi\left(\frac{\int_{R^n} W(x\text{-}y,t)\frac{g(y)}{\kappa}dy}{\int_{R^n} W(x\text{-}y,t)dy}\right) \leq \int_{R^n} W(x\text{-}y, t)\Phi(\frac{g(y)}{\kappa})dy$. On integrating both sides with respect to x and applying Tonelli's Theorem, we see that $T_g \in L^\Phi(R^n)$. A similar argument gives that $L^\Phi(R^n) \to L^\Phi(R^n)$ and $\|T_g\|_\Phi \leq K\|g\|_\Phi$. □

Lemma 14. *Let Φ be a $\triangle_2$N-function. Let $(T_u)(x,t) = \int_0^t \int_{R^n} W(x\text{-}y,\ t\text{-}\tau)u^{\alpha + 1}(y,\ \tau)dyd\tau$. Let $(T_u^*)(x) = \sup_{t>0} |(T_u)(x,t)|$ be the maximal operator. If $u^*(x) \in L^\Phi(R^n)$, then T_u^*: $L^\Phi(R^n) \to L^\Phi(R^n)$ and $\left\|T_u^*\right\|_\Phi \leq C(n,\ c_0)\left\|u^*\right\|_\Phi$, where c_0 is such that $\int_0^\infty \frac{ds}{(1 + \sqrt{s})^n} \leq c_0$ and $\Phi(x) = |x|^{\frac{n\alpha}{2}}$.*

Proof. Let Φ_i; i = 1, 2, 3 be $\triangle_2$N-functions such that $\Phi_3^{-1}(x) = \int_0^{|x|} \frac{\Phi_1^{-1}(t)\Phi_2^{-1}(t)dt}{t^2}$. From $(T_u^*)(x) \leq \sup_{t>0} \int_0^t \int_{R^n} |W(x\text{-}y,\ t\text{-}\tau)|\left|u^{(\alpha + 1)}(y,\tau)\right|dyd\tau \leq cc_0 \int_{R^n} \frac{u^{*\ (\alpha + 1)}(y)dy}{|x - y|^{(n - 2)}}$, where $\int_0^\infty \frac{ds}{(1 + \sqrt{s})^n} \leq c_0$, if we let $\Phi_1(x) = |x|^{\frac{n}{n-2}}$; n > 2, $\Phi_2(x) = |x|^{\frac{p}{\alpha + 1}}$; $p > \alpha + 1$ then $\Phi_1^{-1}(x) = x^{\frac{n-2}{n}}$, $\Phi_2^{-1}(x) = x^{\frac{\alpha + 1}{p}}$, and $\Phi_3^{-1}(x) = \int_0^{|x|} \frac{t^{\frac{n-2}{n}} t^{\frac{\alpha + 1}{p}} dt}{t^2} = \frac{|x|^{(\frac{\alpha + 1}{p} - \frac{2}{n})}}{(\frac{\alpha + 1}{p} - \frac{2}{n})}$. Hence $\Phi_3(x) = |x|^r$, where $\frac{1}{r} = \frac{\alpha + 1}{p} - \frac{2}{n}$; r > 1. If we let r = p, then $p = \frac{n\alpha}{2}$ and $\Phi_2(x) = |x|^{\frac{n\alpha}{2(\alpha + 1)}} = (\Phi_3(x))^{\frac{1}{\alpha + 1}}$. If we now let $\Phi_3 = \Phi$ in Theorem 11, then since Φ is increasing, we have as required that $\left\|T_u^*\right\|_\Phi \leq C(n,\ c_0)\left\|u^*\right\|_\Phi$, where $C = C(n,\ c_0)$ is some positive constant. □

Theorem 15. *Let $u(x,t) = \int_0^t \int_{R^n} W(x\text{-}y,\ t\text{-}\tau)u^{\alpha + 1}(y,\tau)dy\ d\tau + \int_{R^n} W(x\text{-}y,t)g(y)dy$ be a weak solution of problem (1)-(2). Let Φ be a $\triangle_2$ condition N-function. If $g \in L^\Phi(R^n)$; $n > 2$, then there exist a constant K satisfying $0 < K < 1$ such that $u^*(x) \in L^\Phi(R^n)$ and $\left\|u^*\right\|_\Phi \leq \frac{c}{1 - K}\|g\|_\Phi$, where c is some positive constant and $\Phi(x) = |x|^{\frac{n\alpha}{2}}$.*

Proof. Lemmas 13 and 14 give that $\left\|u^*\right\|_\Phi \leq K\left\|u^*\right\|_\Phi + c\|g\|_\Phi$. From which it is easily seen that $\left\|u^*\right\|_\Phi \leq \frac{c}{1 - K}\|g\|_\Phi$; $0 < K < 1$. □

Theorem 16. (Contraction Result) *Let Φ and Ψ be $\triangle_2$ complementary N - functions. Let G be a convex function. Then, the nonlinear operator $(\phi u)(x,t)$ maps the ball $\{ \left\|u^*\right\|_\Phi \leq s_0 \}$ into itself if s_0 is the smallest positive root of the equation $s = G(s)$, provided that $\|G(0)\|_\Phi$ is made sufficiently small so that G remains convex on the interval $0 \leq s \leq s_0$. If $\|G'(s_1)\|_\Psi < 1$, where $0 < s_1 < s_0$, then ϕ is a contraction mapping in the ball of radius s_0.*

Proof. The fact that ϕ maps $\left\|u^*\right\|_\Phi \leq s_0$ into itself follows directly from

Lemmas 13 and 14. Combining Lemmas 13 and 14, we see from the integral equation (4) that $\left\|(\phi u^*)\right\|_\Phi \leq K\left\|u^*\right\|_\Phi + c\|g\|_\Phi$. On letting $s = G(s) = Ks + c\|g\|_\Phi$; $s = \left\|u^*\right\|_\Phi$, the result follows from the convexity of G. □

As a direct consequence of Theorem 16, we have

Theorem 17. *Let* $(\phi u)(x,t) = \int_0^t \int_{R^n} W(x\text{-}y,\ t\text{-}\tau)u^{\alpha+1}(y,\tau)dyd\tau + \int_{R^n} W(x\text{-}y,\ t)g(y)dy$ *be such that u(x,t) is a weak solution of problem (1)-(2). Let* Φ *and* Ψ *be* $\triangle_2$ *complementary N-functions and for a constant* r_0*, we let* $B_{r_0}(x)$ *be a ball of radius* r_0 *having center at x defined as* $B_{r_0}(x) = \{u^*(x) = \sup_{t>0}|u(x,t)|;\ \left\|u^*\right\|_\Phi \leq r_0$*, where u(x,t) is a weak solution of problem (1)-(2)* $\}$*. If* $g \in L^\Phi(R^n)$*, then* ϕ*:* $B_{r_0} \to B_{r_0}$ *and is a contraction map.*

Theorem 18. *The maximal solution of problem (1)-(2) is unique in Orlicz spaces.*

Proof. The proof is a consequence of Lemmas 13, 14 and Theorem 17. □

Next, we present an almost everwhere pointwise convergence results.

Lemma 19. *Let* Φ *be a* $\triangle_2$*N-function and let* $(T_g)(x,t) = \int_{R^n} W(x\text{-}y,t)g(y)dy$*. If* $g \in L^\Phi(R^n)$*, then* $(T_g)(x,\ t)$ *turns to g(x) a.e. as* $t \to 0$*.*

Proof. Let Ψ be the complementary $\triangle_2$ N-function to Φ and let $f \in L^\Psi(R^n)$ with $\|f\|_\Psi = 1$. Then on noting that $\int_{R^n} W(x,\ t)dx = 1$, we have

$$\int_{R^n} \left|(T_g)(x,t) - g(x)\right| |f(x)|\, dx \leq \int_{R^n} \{\int_{R^n} |g(x-y) - g(x)|\, |f(x)|\, dx\} W(y,t)dy$$

$$\leq 2\int_{R^n} \left\|g_y - g\right\|_\Phi W(y,t)dy,$$

where $g_y(x) = g(x - y)$. Hence $\left\|(T_g)(.,\ t) - g\right\|_\Phi \leq 2\int_{R^n} \left\|g_y - g\right\|_\Phi W(y,t)dy$. Since W has compact support and $g \in L^\Phi(R^n)$ is continuous, for every $\varepsilon > 0$ there exist a y sufficiently small such that $\left\|g_y - g\right\|_\Phi \leq \frac{\varepsilon}{2}$ and consequently $2\int_{R^n} \left\|g_y - g\right\|_\Phi W(y,t)dy \leq 2.\frac{\varepsilon}{2}.\int_{R^n} W(y,t)dy = \varepsilon$. On this account we see that $\left\|(T_g)(.,\ t) - g\right\|_\Phi \to 0$ as $t \to 0$ a.e. and the lemma follows. □

Lemma 20. *Let* Φ *be a* $\triangle_2$ *condition N-function and let* $(T_u)(x,t) =$

$\int_0^t \int_{R^n} W(x\text{-}y,\ t\text{-}\tau)u^{\alpha+1}(y,\ \tau)dyd\tau$. *If* $u^{\alpha+1}(x,\ .) \in L^{\Phi}(R^n)$, *then* $\lim_{t\to 0}(T_u(x,t) = 0$ *in* $L^{\Phi}(R^n)$.

Proof. From $(T_u)(x,t) = \int_0^t \int_{R^n} W(x\text{-}y,\ t\text{-}\tau)u^{\alpha+1}(y,\ \tau)dyd\tau$, we have by Jensen's inequality and Lemma 13 that

$$\Phi[(T_u)(x,t)] = \Phi\left[\int_0^t \int_{R^n} W(x\text{-}y,t\text{-}\tau)u^{\alpha+1}(y,\tau)dyd\tau\right]$$
$$\le \int_0^t \int_{R^n} W(x-y,t-\tau)\Phi(u^{\alpha+1}(x,\tau)dyd\tau.$$

On integrating both sides over R^n and invoking Tonelli's Theorem, we have $\|(T_u)(.,\ t)\|_{\Phi} \le K\int_0^t \|u^{\alpha+1}(.,\ t)\|_{\Phi}d\tau$. From which we see that $\lim_{t\to 0}\|(T_u)(.,\ t)\|_{\Phi} = 0$. Hence $T_u \to 0$ as $t \to 0$. □

Lemmas 19 and 20 give

Theorem 21. (Pointwise a.e. Convergence) *Let* Φ *be a* $\triangle_2$ *N-function. Let* $u(x,t) = \int_0^t \int_{R^n} W(x\text{-}y,\ t\text{-}\tau)u^{\alpha+1}(y,\tau)dyd\tau + \int_{R^n} W(x\text{-}y,t)g(y)dy$ *be a weak solution of problem (1)-(2). If* $g \in L^{\Phi}(R^n)$ *and* $u^{\alpha+1}(x,\ .) \in L^{\Phi}(R^n)$, *then* $u(x,\ t)$ *converges to* $g(x)$ *as* $t \to 0$ *a.e. in* $L^{\Phi}(R^n)$.

4. Regularity Results

We shall take for granted here the well known fact [23] that if $W \in C_0^{\infty}$, then $f * W \in C^{\infty}$ and $D^{\beta}(f * W)(x) = (f * D^{\beta}W)(x)$ to establish the regularity results for the maximal solution of problem (1)-(2) in Orlicz spaces. The main result of this section is:

Theorem 22. *Let* $|\beta| \le M$ *be the order of the spacial distributional derivative of the maximal solution. Let u(x,t) be the weak solution of problem (1)-(2). Let* Φ *be a* $\triangle_2$ *condition N-function. If* $g \in L^{\Phi}(R^n)$; $n + |\beta| > 2$, *then* $D^{\beta}u^*(x) = \sup_{t>0}|D^{\beta}u(x,\ t)| \in L^{\Phi}(R^n)$ *and* $\|D^{\beta}u^*\|_{\Phi} \le C(c_0,n)\|u^*\|_{\Phi} + c\|g\|_{\Phi}$.

Proof. Let

$$u(x,t) = \int_0^t \int_{\mathrm{R^n}} W(x-y,t-\tau)u^{\alpha+1}(y,\tau)dyd\tau + \int_{\mathrm{R^n}} W(x-y,t)g(y)dy.$$

Then, on taking the distributional derivatives of order M, we have

$$D^\beta u(x,t) = \int_0^t \int_{\mathrm{R^n}} (D_{\mathrm{x}}^\beta W(x-y,t-\tau))u^{\alpha+1}(y,\tau)dyd\tau$$

$$+ \int_{\mathrm{R^n}} (D_{\mathrm{x}}^\beta W(x-y,t))g(y)dy.$$

On using estimate (6), we have

$$\left|\mathrm{D}^\beta \mathrm{u(x,t)}\right| \le C\int_0^t \int_{\mathrm{R^n}} \frac{|\mathrm{u(y,}\tau)|^{\alpha+1}\,\mathrm{dyd}\tau}{\left(|\mathrm{x\text{-}y}| + |\mathrm{t}\text{-}\tau|^{\frac{1}{2}}\right)^{\mathrm{n}+|\beta|}} + C\int_{\mathrm{R^n}} \frac{|\mathrm{g(y)}|\,\mathrm{dy}}{\left(|\mathrm{x\text{-}y}| + \mathrm{t}^{\frac{1}{2}}\right)^{\mathrm{n}+|\beta|}}.$$

Since

$$\left(|\mathrm{x\text{-}y}| + \mathrm{t}^{\frac{1}{2}}\right)^{\mathrm{n}+|\beta|} \ge t^{-\frac{|\beta|}{2}}\left(|\mathrm{x\text{-}y}|^{\mathrm{n}+|\beta|} + \mathrm{t}^{\frac{\mathrm{n}+|\beta|}{2}}\right),$$

we have that

$$\left|\mathrm{D}^\beta \mathrm{u(x,t)}\right| \le C\int_0^t \int_{\mathrm{R^n}} \frac{|\mathrm{u(y,}\tau)|^{\alpha+1}\,\mathrm{dyd}\tau}{\left(|\mathrm{x\text{-}y}| + |\mathrm{t}\text{-}\tau|^{\frac{1}{2}}\right)^{\mathrm{n}+|\beta|}} + C\int_{\mathrm{R^n}} \frac{\mathrm{t}^{\frac{|\beta|}{2}}\,|\mathrm{g(y)}|\,\mathrm{dy}}{|\mathrm{x\text{-}y}|^{\mathrm{n}+|\beta|} + \mathrm{t}^{\frac{\mathrm{n}+|\beta|}{2}}}.$$

Therefore, on taking the supremum of both sides over t > 0, we have

$$\sup_{\mathrm{t} ¿ 0}\left|\mathrm{D}^\beta \mathrm{u(x,t)}\right| \le C_0 C\int_{\mathrm{R^n}} \frac{[\,\mathrm{u}^*(\mathrm{y})\,]^{\alpha+1}\mathrm{dy}}{|\mathrm{x\text{-}y}|^{\mathrm{n}+|\beta|-2}} + C\sup_{\mathrm{t} ¿ 0}\int_{\mathrm{R^n}} \frac{\mathrm{t}^{\frac{|\beta|}{2}}\,|\mathrm{g(y)}|\,\mathrm{dy}}{|\mathrm{x\text{-}y}|^{\mathrm{n}+|\beta|} + \mathrm{t}^{\frac{\mathrm{n}+|\beta|}{2}}},$$

where $\mathrm{C_0}$ is such that $\int_0^\infty \frac{\mathrm{ds}}{(1+\sqrt{\mathrm{s}})^{\mathrm{n}+|\beta|}} \le \mathrm{C_0}$. Now we let Φ_{i}; i = 1, 2, 3 be $\triangle_2$ condition N-functions such that $\Phi_3^{-1}(\mathrm{x}) = \int_0^{|\mathrm{x}|} \frac{\Phi_1^{-1}(\mathrm{t})\Phi_2^{-1}(\mathrm{t})\mathrm{dt}}{\mathrm{t}^2}$. Then taking the $\mathrm{L}^{\Phi_3}(\mathrm{R^n})$ norm of the last inequality, we have

$$\left\|\mathrm{D}^\beta \mathrm{u}^*\right\|_{\Phi_3} \le \frac{\mathrm{K_1}}{2}\left\|\int_{\mathrm{R^n}} \frac{[\,\mathrm{u}^*(\mathrm{y})\,]^{\alpha+1}\mathrm{dy}}{|\mathrm{x\text{-}y}|^{\mathrm{n}+|\beta|-2}}\right\|_{\Phi_3} + \frac{\mathrm{K_2}}{2}\left\|\sup_{\mathrm{t} ¿ 0}\int_{\mathrm{R^n}} \frac{\mathrm{t}^{\frac{|\beta|}{2}}\,|\mathrm{g(y)}|\,\mathrm{dy}}{|\mathrm{x\text{-}y}|^{\mathrm{n}+|\beta|} + \mathrm{t}^{\frac{\mathrm{n}+|\beta|}{2}}}\right\|_{\Phi_3}.$$

Utilizing Theorem 12 on the last term of the right hand side of the inequality we have $\left\|D^{\beta}u^{*}\right\|_{\Phi_3} \leq \frac{K_1}{2}\left\|\int\limits_{R^n} \frac{[u^{*}(y)]^{\alpha+1}dy}{|x-y|^{n+|\beta|-2}}\right\|_{\Phi_3} + C_2\left\|g\right\|_{\Phi_3}$. For the first term, we let $\Phi_1(x) = |x|^{\frac{n+|\beta|}{n+|\beta|-2}}$ from which

$$\Phi_1^{-1}(x) = x^{\frac{n+|\beta|-2}{n+|\beta|}} \text{ and } \Phi_2(x) = |x|^{\frac{p}{\alpha+1}}$$

with $\Phi_2^{-1}(x) = x^{\frac{\alpha+1}{p}}$. Then

$$\Phi_3^{-1}(x) = \int_0^{|x|} \frac{t^{\frac{n+|\beta|-2}{n+|\beta|}}.t^{\frac{\alpha+1}{p}}}{t^2}dt = \frac{|x|^{\left(\frac{\alpha+1}{p} - \frac{2}{n+|\beta|}\right)}}{\frac{\alpha+1}{p} - \frac{2}{n+|\beta|}}, \quad n+|\beta| > 2.$$

Therefore, if we choose $r > 1$ such that $\frac{1}{r} = \frac{\alpha+1}{p} - \frac{2}{n+|\beta|}$, then we have $\Phi_3(x) = |x|^r$. If we now let $r = p$, then $p = \frac{\alpha(n+|\beta|)}{2}$ and $\Phi_2(x) = |x|^{\frac{\alpha(n+|\beta|)}{2(\alpha+1)}}$. If $[u^{*}(x)]^{\alpha+1} \in L^{\Phi_2}(R^n)$, then Theorem 11 gives

$$\left\|D^{\beta}u^{*}\right\|_{\Phi_3} \leq C_1\left\|[u^{*}]^{\alpha+1}\right\|_{\Phi_2} + C_2\left\|g\right\|_{\Phi_3}.$$

On using the fact that $\Phi_2(x) = |x|^{\frac{\alpha(n+|\beta|)}{2(\alpha+1)}} = [\Phi_3(x)]^{\frac{1}{\alpha+1}}$, we have the required result that $\left\|D^{\beta}u^{*}\right\|_{\Phi_3} \leq C_1\left\|u^{*}\right\|_{\Phi_3} + C_2\left\|g\right\|_{\Phi_3}$. □

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