International Journal of Pure and Applied Mathematics
Volume 61 No. 3 2010, 269-280

NEW SUBCLASS OF MULTIVALENT HYPERGEOMETRIC MEROMORPHIC FUNCTIONS

F. Ghanim[1], M. Darus[2 §]
[1]Faculty of Management
Multimedia University
Cyberjaya, 63100, Selangor D. Ehsan, MALAYSIA
e-mail: firas.ghanim@mmu.edu.my
[2]School of Mathematical Sciences
Faculty of Science and Technology
University Kebangsaan Malaysia
Bangi, 43600, Selangor D. Ehsan, MALAYSIA
e-mail: maslina@ukm.my

Abstract: In this paper, we introduce a new class $\Sigma_p^*(A,B,k)$ for $-1 \leq B < A \leq 1$ which consists hypergeometric meromorphic functions of the form $L_p^*(a,c)f(z) = \frac{1}{z^p} + \sum\limits_{n=0}^{\infty} \frac{(a)_{n+2}}{(c)_{n+2}} a_{n+p} z^{n+p}$ in $U^* = \{z : 0 < |z| < 1\}$. We determine sufficient conditions, distortion properties and radii of starlikeness and convexity for functions in the class $L_p^*(a,c)f(z)$.

AMS Subject Classification: 30C45, 33C05
Key Words: p-valent hypergeometric functions, meromorphic functions, starlike functions, convex functions, Hadamard product

1. Introduction

Let Σ_p denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+p} z^{n+p}, \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$.

Received: March 8, 2010

§Correspondence author

For $0 \leq \beta < p$, we denote by $S_p^*(\beta)$ and $k_p(\beta)$, the subclasses of Σ_p consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in U^*.

The classes $S_p^*(\beta)$, $k_p(\beta)$ and various other subclasses of Σ_p have been studied rather extensively by Aouf et al [1], [2], [3], Joshi and Srivastava [11], Kulkarni et al [12], Mogra [17], [18], Owa et al [19], Srivastava and Owa [20], Uralegaddi and Somantha [22], and Yang [24].

For functions $f_j(z)(j = 1; 2)$ defined by

$$f_j\left(z\right) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_{n+p,j} z^{n+p-1}, \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$\left(f_1 * f_2\right) = \frac{1}{z^p} + + \sum_{n=0}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p}. \tag{1.3}$$

Let us define the function $\phi_p(a, c; z)$ by

$$\phi_p\left(a, c; z\right) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(c)_{n+1}} z^{n+p-1} \tag{1.4}$$

$(c \neq 0, -1, -2, ...,$ and $a \in C/\{0\}, p \in N = 1, 2, 3, ...)$, where $(\lambda)_n$ is the Pochhammer symbol. We note that

$$\phi_p\left(a, c; z\right) = \frac{1}{z^p} {}_2F_1\left(1, a, c; z\right), \tag{1.5}$$

where

$${}_2F_1\left(1, a, c; z\right) = \sum_{n=0}^{\infty} \frac{(1)_{n+1}\,(a)_{n+1}}{(c)_{n+1}} \frac{z^n}{n!},$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\phi_p(a, c; z)$, using the Hadamard product for $f(z) \in \Sigma_p$, we define a new linear operator $L_p^*(a, c)$ on Σ_p

$$L_p^*\left(a, c\right) f\left(z\right) = \phi_p\left(a, c; z\right) * f\left(z\right) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \left| \frac{(a)_{n+2}}{(c)_{n+2}} \right| a_{n+p} z^{n+p}. \tag{1.6}$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [7], [8], Liu [13], Liu and Srivastava [14], [15], [16], Cho and Kim [5].

Also for a function $f(z) \in \Sigma_p$, we define the integral operator $J_{\nu,p}$ by

$$J_{\nu,p} f(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t)\, dt \quad (\nu > 0). \tag{1.7}$$

There are many papers [23], [25], [2] in which the operator $J_{\nu,p}$ was investigated.

For a function $f \in \Sigma_p$ we define

$$I^0 \left(L_p^* (a,c) f(z)\right) = L_p^* (a,c) f(z),$$

and for $k = 1, 2, 3, ...$

$$I^k \left(L_p^* (a,c) f(z)\right) = z \left(I^{k-1} L_p^* (a,c) f(z)\right)' + \frac{1+p}{z}$$

$$= \frac{1}{z^p} + \sum_{n=0}^{\infty} (n+p)^k \left|\frac{(a)_{n+2}}{(c)_{n+2}}\right| a_{n+p} z^{n+p}. \tag{1.8}$$

We note that $I^k (L_1^* (a,c) f(z))$ studied by Ghanim, Daruas and Swaminathan [10] and $I^k (L_1^* (a,a) f(z))$ studied by Frasin and Darus [9].

Also, it follows from (1.6) that (see [13])

$$z \left(L_p^* (a,c) f(z)\right)' = a L_p^* (a+1,c) f(z) - (a+p) L_p^* (a,c). \tag{1.9}$$

Now, let $-1 \leq B < A \leq 1$ and for all $z \in U^*$, a function $f \in \Sigma_p$ is said to be a member of the class $\Sigma_p^* (A,B,k)$ if it satisfies

$$\left| \frac{z \left(I^k L^* (a,c) f(z)\right)' + p\, I^k L^* (a,c) f(z)}{Bz \left(I^k L^* (a,c) f(z)\right)' + Ap \left(I^k L^* (a,c) f(z)\right)} \right| < 1. \tag{1.10}$$

The class $\Sigma_1^* (A,B,0) = \Sigma^* (A,B)$ was studied by Morga [18].

Note that, for $a = c$, p=1, $\Sigma^* (1-2\alpha, -1, k)$ with $0 \leq \alpha < 1$, is the class introduced and studied in [9].

In Section 2, our first result will concern the coefficient estimates and distortion theorem for the class $\Sigma_p^* (A,B,k)$.

2. Coefficient Estimates and Distortion Theorems

Our first result provides a sufficient condition for a function, analytic in U^*, to be in $\Sigma_p^* (A,B,k)$.

Theorem 2.1. *Let the function f be defined by (1.6). If*

$$\sum_{n=0}^{\infty}(n+p)^k\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|\left[(n+p)(1-B)+p(1-A)\right]|a_{n+p}|\le p(A-B)\,, \quad (2.1)$$

where $k\in N_0$, $p\in N$, $-1\le B<A\le 1$, then $f\in\Sigma_p^*(A,B,k)$.

Proof. Suppose that (2.1) holds true. Consider the expression

$$M\left(f,f'\right)=\left|z\left(I^kL_p^*(a,c)\,f(z)\right)'+p\ I^kL_p^*(a,c)\,f(z)\right| \\ -\left|Bz\left(I^kL_p^*(a,c)\,f(z)\right)'+Ap\ \left(I^kL_p^*(a,c)\,f(z)\right)\right|.$$

Then for $0<|z|=r<1$ we have

$$M\left(f,f'\right)=\left|\sum_{n=0}^{\infty}(n+p)^k\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|(n+2p)\,a_{n+p}z^{n+p}\right| \\ -\left|\frac{p(A-B)}{z^p}+\sum_{n=0}^{\infty}(n+p)^k\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|((n+p)\,B+AP)\,a_{n+p}z^{n+p}\right|.$$

This gives

$$r^pM\left(f,f'\right)\le\sum_{n=0}^{\infty}(n+p)^k\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|\left[(n+p)(1-B)+p(1-A)\right]|a_{n+p}|\,r^{n+2p} \\ -p(A-B)\,. \quad (2.2)$$

The inequality in (2.2) holds true for all r $(0<r<1)$. Therefore, by letting $r\to 1$ in (2.2), we have

$$M\left(f,f'\right)\le\sum_{n=0}^{\infty}(n+p)^k\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|\left[(n+p)(1-B)\right. \\ \left.+p(1-A)\right]|a_{n+p}|-p(A-B)\le 0$$

by the hypothesis (2.1). Hence it follows that

$$\left|z\left(I^kL_p^*(a,c)\,f(z)\right)'+p\ I^kL_p^*(a,c)\,f(z)\right| \\ \le\left|Bz\left(I^kL_p^*(a,c)\,f(z)\right)'+Ap\ \left(I^kL_p^*(a,c)\,f(z)\right)\right|,$$

so that $f\in\Sigma_p^*(A,B,k)$.

Hence the theorem. □

Our assertion in Theorem 2.1 is sharp for functions of the form:

$$f_{n+p}(z) = \frac{1}{z^p} + \frac{p(A-B)\left|(c)_{n+2}\right|}{(n+p)^k\left[(n+p)(1-B)+p(1-A)\right]\left|(a)_{n+2}\right|} z^n \qquad (2.3)$$

$(n \geq 0,\ k \in N_0, p \in N, -1 \leq B < A \leq 1)$.

Corollary 2.1. *Let the function f be defined by (1.6) and let $f \in \Sigma_p$. If $f \in \Sigma_p^*(A, B, k)$. Then*

$$|a_{n+p}| \leq \frac{p(A-B)\left|(c)_{n+2}\right|}{(n+p)^k\left[(n+p)(1-B)+p(1-A)\right]\left|(a)_{n+2}\right|} \qquad (2.4)$$

$(n \geq 0,\ k \in N_0, p \in N, -1 \leq B < A \leq 1)$.

The result (2.4) is sharp for functions f_{n+p} given by (2.3).

Corollary 2.2. *For $k = 0$, $A = 1$ and $B = -1$ in Theorem 2.1, we have*

$$\sum_{n=0}^{\infty}\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|(n+p)|a_n + p| \leq 1$$

and therefore the function $L_p^*(a, c)f(z)$ is starlike in U^*.

Corollary 2.3. *For $k = 1$, $A = 1$ and $B = -1$ in Theorem 2.1, we have*

$$\sum_{n=0}^{\infty}\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|(n+p)^2|a_n + p| \leq 1$$

and therefore the function $L_p^*(a, c)f(z)$ is convex in U^*.

The growth and distortion property for functions in the class $\Sigma_p^*(A, B, k)$ are given in the following result.

Theorem 2.2. *If the function f be defined by (1.6) is in the class $\Sigma_p^*(A, B, k)$, then for $0 < |z| = r < 1$, we have*

$$\frac{1}{r^p} - \frac{p(A-B)}{[2p - p(A+B)]} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{p(A-B)}{[2p - p(A+B)]} r^p \qquad (2.5)$$

and

$$\frac{p}{r^{p+1}} - \frac{p(A-B)}{[2p - p(A+B)]} r^{p-1} \leq \left|f'(z)\right| \leq \frac{p}{r^{p+1}} + \frac{p(A-B)}{[2p - p(A+B)]} r^{p-1} \qquad (2.6)$$

with equality for

$$f(z) = \frac{1}{z^p} + \frac{p(A-B)}{[2p - p(A+B)]} z^p.$$

Proof. Since $f \in \Sigma_p^*(A,B,k)$, Theorem 2.1 readily yields the inequality

$$\sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} a_{n+p} \leq \frac{p\,(A-B)}{(p)^k\,[2p - p\,(A+B)]}. \tag{2.7}$$

Thus, for $0 < |z| = r < 1$, and making use of (2.7) we have

$$|f(z)| \leq \frac{1}{|z|^p} + \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} |a_{n+p}|\, |z|^{n+p}$$

$$\leq \frac{1}{r^p} + r^p \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} |a_{n+p}| \leq \frac{1}{r^p} + \frac{p\,(A-B)}{[2p - p\,(A+B)]} r^p \tag{2.8}$$

and

$$|f(z)| \geq \frac{1}{|z|^p} - \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} |a_{n+p}|\, |z|^{n+p}$$

$$\geq \frac{1}{r^p} - r^p \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} |a_{n+p}| \geq \frac{1}{r^p} - \frac{p\,(A-B)}{[2p - p\,(A+B)]} r^p. \tag{2.9}$$

Also from Theorem 2.1, it follows that

$$\sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} (n+p)\, |a_{n+p}| \leq \frac{p\,(A-B)}{(p)^{k-1}\,[2p - (A+B)]}. \tag{2.10}$$

Hence

$$\left|f'(z)\right| \leq \frac{p}{|z|^{p+1}} + \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} (n+p)\, |a_{n+p}|\, |z|^{n+p-1}$$

$$\leq \frac{p}{r^{p+1}} + r^{p-1} \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} (n+p)\, |a_{n+p}|$$

$$\leq \frac{p}{r^{p+1}} + \frac{p\,(A-B)}{[2p - p\,(A+B)]} r^{p-1} \tag{2.11}$$

and

$$\left|f'(z)\right| \geq \frac{p}{|z|^{p+1}} + \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} (n+p)\, |a_{n+p}|\, |z|^{n+p-1}$$

$$\geq \frac{p}{r^{p+1}} + r^{p-1} \sum_{n=0}^{\infty} \frac{\left|(a)_{n+2}\right|}{\left|(c)_{n+2}\right|} (n+p)\, |a_{n+p}|$$

$$\geq \frac{p}{r^{p+1}} + \frac{p\,(A-B)}{[2p - p\,(A+B)]} r^{p-1}. \tag{2.12}$$

This completes the proof of the theorem. □

3. Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the class $\Sigma_p^*(A,B,k)$ is given by the following theorems.

Theorem 3.1. *If the function f is defined by (1.6) in the class $\Sigma_p^*(A,B,k)$, then f is meromorphically starlike of order δ $(0 \le \delta < 1)$ in $|z| < r_1$,*

$$r_1 = r_1(A,B,k) = \inf_{n\geq 0}\left\{\frac{(n+p)^k(p-\delta)\left[(n+p)(1-B)+p(1-A)\right]}{p(A-B)(n+3p-\delta)}\right\}^{\frac{1}{n+2p}}, \quad (3.1)$$

where the result is sharp for the functions f_n given by (2.3).

Proof. It suffices to prove that

$$\left|\frac{z\left(L_p^*(a,c)f(z)\right)'}{L_p^*(a,c)f(z)}+p\right| \le p-\delta, \quad (3.2)$$

for $|z| < r_1$, we have

$$\left|\frac{z\left(L_p^*(a,c)f(z)\right)'}{L_p^*(a,c)f(z)}+p\right| = \left|\frac{\sum_{n=0}^{\infty}\frac{(a)_{n+2}}{(c)_{n+2}}(n+2p)a_{n+p}z^{n+p}}{\frac{1}{z^p}+\sum_{n=0}^{\infty}\frac{(a)_{n+2}}{(c)_{n+2}}a_{n+p}z^{n+p}}\right|$$

$$\le \frac{\sum_{n=0}^{\infty}\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|(n+2p)\left|a_{n+p}\right||z|^{n+2p}}{1-\sum_{n=0}^{\infty}\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|\left|a_{n+p}\right||z|^{n+2p}}. \quad (3.3)$$

Hence (3.3) holds true if

$$\sum_{n=0}^{\infty}\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|(n+2p)\left|a_{n+p}\right||z|^{n+2p} \le (p-\delta)\left(1-\sum_{n=0}^{\infty}\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|\left|a_{n+p}\right||z|^{n+2p}\right) \quad (3.4)$$

or

$$\sum_{n=0}^{\infty}\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right|\frac{(n+3p-\delta)}{(p-\delta)}\left|a_{n+p}\right||z|^{n+2p} \le 1. \quad (3.5)$$

With the aid of (2.1) and (3.5) it is true if

$$\left|\frac{(a)_{n+2}}{(c)_{n+2}}\right| \frac{(n+3p-\delta)}{(p-\delta)} |z|^{n+2p}$$
$$\leq \frac{(n+p)^k \left[(n+p)(1-B)+p(1-A)\right] \left|(a)_{n+2}\right|}{p(A-B)\left|(c)_{n+2}\right|} \quad (n \geq 0). \tag{3.6}$$

Solving (3.6) for $|z|$, we obtain

$$|z| < \left\{\frac{(n+p)^k (p-\delta)\left[(n+p)(1-B)+p(1-A)\right]}{p(A-B)(n+3p-\delta)}\right\}^{\frac{1}{n+2p}} \quad (n \geq 0). \tag{3.7}$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *If the function f is defined by (1.6) in the class $\Sigma^*(A,B,k)$, then f is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(A,B,k)$$
$$= \inf_{n \geq 0} \left\{\frac{(n+p)^{k-1} (p-\delta)\left[(n+p)(1-B)+p(1-A)\right]}{p(A-B)(n+3p-\delta)}\right\}^{\frac{1}{n+2p}}. \tag{3.8}$$

The result is sharp for the functions f_n given by (2.3).

Proof. By using the technique employed in the proof of Theorem 3.1, with the aid of Theorem 2.1, we can show that

$$\left|\frac{z\left(L_p^*(a,c)f(z)\right)''}{\left(L_p^*+(a,c)f(z)\right)'} + p + 1\right| \leq (p-\delta), \tag{3.9}$$

for $|z| < r_2$. Thus we have the assertion of Theorem 3.2. □

4. Convex Linear Combinations

Our next results involves a linear combination of several functions of the type (2.3).

Theorem 4.1. *Let*

$$f_{p-1}(z) = \frac{1}{z^p} \tag{4.1}$$

and

$$f_{n+p}(z) = \frac{1}{z^p} + \frac{p(A-B)\left|(c)_{n+2}\right|}{(n+p)^k \left[(n+p)(1-B)+p(1-A)\right]\left|(a)_{n+2}\right|} z^{n+p} \tag{4.2}$$

$(n \geq 0,\ k \in\ N_0)$. *Then* $f \in \Sigma_p^* (A, B, k)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z), \tag{4.3}$$

where $\lambda_{p+n-1} \geq 0$ *and* $\sum\limits_{n=0}^{\infty} \lambda_{p+n-1} = 1$.

Proof. From (4.1), (4.2) and (4.3), it is easily seen that

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z)$$

$$= \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{p(A-B)\left|(c)_{n+2}\right| \lambda_{n+p}}{(n+p)^k \left[(n+p)(1-B) + p(1-A)\right] \left|(a)_{n+2}\right|} z^{n+p} \tag{4.4}$$

$$=: \frac{1}{z^p} + \sum_{n=0}^{\infty} b_{n+p} z^{n+p}.$$

Since

$$\sum_{n=0}^{\infty} \frac{(n+p)^k \left[(n+p)(1-B) + p(1-A)\right] \left|(a)_{n+2}\right|}{p(A-B)\left|(c)_{n+2}\right|} \lambda_{n+p}$$

$$\frac{p(A-B)\left|(c)_{n+2}\right|}{(n+p)^k \left[(n+p)(1-B) + p(1-A)\right] \left|(a)_{n+2}\right|} = \sum_{n=0}^{\infty} \lambda_{n+p} = 1 - \lambda_{p-1} \leq 0,$$

it follows from Theorem 2.1 that the function $f \in \Sigma_p^* (A, B, k)$.

Conversely, let us suppose that $f \in \Sigma_p^* (A, B, k)$. Since

$$|a_{n+p}| \leq \frac{p(A-B)\left|(c)_{n+2}\right|}{(n+p)^k \left[(n+p)(1-B) + p(1-A)\right] \left|(a)_{n+2}\right|} \quad (n \geq 0,\ k \in N_0).$$

Setting

$$\lambda_{n+p} = \frac{(n+p)^k \left[(n+p)(1-B) + p(1-A)\right] \left|(a)_{n+2}\right|}{p(A-B)\left|(c)_{n+2}\right|} |a_{n+p}|,$$

$(n \geq 0,\ k \in\ N_0, p \in N)$ and

$$\lambda_{p-1} = 1 - \sum_{n=0}^{\infty} \lambda_{n+p},$$

it follows that $f(z) = \sum\limits_{n=0}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z)$. This completes the proof of Theorem 4.1. □

Theorem 4.2. *The class $\Sigma_p^*(A,B,k)$ is closed under convex linear combinations.*

Proof. Suppose that the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} |a_{n+p,j}|\, z^{n+p} \quad (j=1,2;\; z\in U^*) \tag{4.5}$$

are in the class $\Sigma_p^*(A,B,k)$.

Setting

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0\le \mu \le 1), \tag{4.6}$$

we find from (4.6) that

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \frac{|(a)_{n+2}|}{|(c)_{n+2}|} \{|\mu a_{n+p,1} + (1-\mu)\, a_{n+p,2}|\} z^{n+p}, \tag{4.7}$$

$0\le\mu\le 1$, $p\in N$, $z\in U^*$.

In view of Theorem 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Big((n+p)^k [(n+p)(1-B) + p(1-A)]\Big) \\
&\qquad\qquad \times \frac{|(a)_{n+2}|}{|(c)_{n+2}|} \{|\mu a_{n+p,1} + (1-\mu)\, a_{n+p,2}|\} \\
&= \mu \sum_{n=0}^{\infty} (n+p)^k [(n+p)(1-B) + p(1-A)] \frac{|(a)_{n+2}|}{|(c)_{n+2}|} |a_{n+p,1}| \\
&+ (1-\mu) \sum_{n=0}^{\infty} (n+p)^k [(n+p)(1-B) + p(1-A)] \frac{|(a)_{n+2}|}{|(c)_{n+2}|} |a_{n+p,2}| \\
&\qquad\qquad \le \mu p(A-B) + (1-\mu)\, p(A-B) = p(A-B),
\end{aligned}$$

which shows that $f\in\Sigma_p^*(A,B,k)$. Hence the theorem is proved. □

References

[1] M.K. Aouf, New criteria for multivalent meromorphic starlike functions of order α, *Proc. Japan. Acad. Ser. A. Math. Sci.*, **69** (1993), 66-70.

[2] M.K. Aouf, H.M. Hossen, New criteria for meromorphic p-valent starlike functions, *Tsukuba J. Math.*, **17** (1993), 481-486.

[3] M.K. Aouf, H.M. Srivastava, A new criteria for meromorphic p-valent convex functions of order α, *Math. Sci. Res. Hot-Line*, **1**, No. 8 (1997), 7-12.

[4] S.K. Bajpai, A note an a class of meromorphic univalent functions, *Rev. Roumaine Math. Pures Apll.*, **22**, No. 3 (1977), 295-297.

[5] N.E. Cho, I.H. Kim, Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function, *Appl. Math. and Compu.*, **187** (2007), 115-121.

[6] N.E. Cho, S.H. Lee, S. Owa, A class of meromorphic univalent functions with positive coefficients, *Kobe J. Math.*, **4**, No. 1 (1987), 43-50.

[7] J. Dziok, H.M. Srivastava, Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function, *Adv. Stud. Contemp. Math.*, **5** (2002), 115-125.

[8] J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Trans. Spec. Funct.*, **14** (2003), 7-18.

[9] B.A. Frasin, M. Darus, On certain meromorphic functions with positive coefficients, *South East Asian Bulletin of Math.*, **28** (2004), 615-623.

[10] F. Ghanim, M. Darus, A. Swaminathan, New subclass of hypergeometric meromorphic functions, *Far East J. Math. Sci.*, **34**, No. 2 (2009), 245-256.

[11] S.B. Joshi, H.M. Srivastava, A certain family of meromorphically multivalent functions, *Computers Math. Appl.*, **38** (1999), 201-211.

[12] S.R. Kukarni, U.H. Naik, H.M. Srivastava, A certain class of meromorphically p-valent quasi-convex functions, *Pan Amer. Math. J.*, **8**, No. 1 (1998), 57-64.

[13] J.L. Liu, A linear operator and its applications on meromorphic p-valent functions, *Bull. Institute Math. Academia Sinica*, **31** (2003), 23-32.

[14] J.L. Liu, H.M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.*, **259** (2001), 566-581.

[15] J.L. Liu, H.M. Srivastava, Certain properties of the Dziok-Srivastava operator, *Appl. Math. Comput.*, **159** (2004), 485-493.

[16] J.L. Liu, H.M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, *Math. Comput. Modell.*, **39** (2004), 21-34.

[17] M.L. Mogra, Meromorphic multivalent functions with positive coefficients I, *Math. Japon.*, **35** (1990), 1-11.

[18] M.L. Mogra, Meromorphic multivalent functions with positive coefficients II, *Math. Japon.*, **35** (1990), 1089-1098.

[19] S. Owa, H.E. Darwish, M.K. Aouf, Meromorphic multivalent functions with positive and fixed second coefficients, *Math. Japon.*, **46** (1997), 231-236.

[20] H.M. Srivastava, S. Owa (Eds.), *Current Topics in Analytic Function Theory*, World Scientific, Singapore-New Jersey-London-Hong Kong (1992).

[21] B.A. Uralegaddi, M.D. Ganigi, A certain class of meromorphically starlike functions with positive coefficients, *Pure Appl. Math. Sci.*, **26** (1987), 75-80.

[22] B.A. Uralegaddi, C. Somanatha, Certain classes of meromorphic multivalent functions, *Tamkang J. Math.*, **23** (1992), 223-231.

[23] B.A. Uralegaddi, C. Somanatha, New criteria for meromorphic starlike univalent functions, *Bull. Austral. Math. Soc.*, **43** (1991), 137-140.

[24] D.G. Yang, On new subclasses of meromorphic p-valent functions, *J. Math. Res. Exposition*, **15** (1995), 7-13.

[25] Yang Dinggong, On a class of meromorphic starlike multivalent functions, *Bull. Institute Math. Academia Sinica*, **24** (1996), 151-157.