International Journal of Pure and Applied Mathematics
Volume 61 No. 3 2010, 301-313

INERTIA SUBGROUPS FOR OCTIC 2-ADIC FIELDS

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Abstract: We present an method for computing the inertia subgroup for the normal closure of an octic extension of a 2-adic number field. The principal application is to octic extensions of $\mathbf{Q}_2$.

AMS Subject Classification: 11E10, 20K27, 20Fxx
Key Words: inertia subgroup, number field, normal closure, octic extensions

1. Introduction

Let F/K be a finite extension of number fields, $F \cong K[x]/\langle g(x)\rangle$. There is a considerable literature on computing Galois groups of polynomials which one can apply to the computation of $G = \mathrm{Gal}(g(x)) = \mathrm{Gal}(\hat{F}/K)$ where $\hat{F}$ is the normal closure for F/K. From a number theoretic point of view, more refined information is desirable.

If P is a prime ideal of the ring of integers $\mathcal{O}_K$, then for each prime ideal $\mathfrak{P}_j$ of $\mathcal{O}_F$ above P, we have subgroups of G, $D_j \geq I_j$, the decomposition and inertia subgroups of $\mathfrak{P}_j$ respectively. Computing D_j is equivalent to computing the Galois group of the completion $\widehat{F_{\mathfrak{P}_j}}/K_P$, i.e., $\mathrm{Gal}(g_j(x))$ where $g_j(x)$ is the corresponding irreducible factor of $g(x)$ over K_P. Thus, existing techniques for computing Galois groups can be applied, provided they can be carried out effectively over K_P.

This leaves the computation of inertia subgroups I_j, for which there is very little literature. We address the case where $\deg(g_j) = 8$ and $K_P \supseteq \mathbf{Q}_2$. Lower

Received: March 12, 2010

degree cases are much simpler, as are tamely ramified extensions and degree p extensions of a p-adic field (see Jones et al [8]). The fact that the case of inertia groups for 2-adic octics is so much more complicated than these other cases makes makes it more compelling. When considering general conjectures for local fields, one should be able to test against moderately complicated examples of wild ramification, e.g., where higher ramification groups are not automatically known from the field discriminant. Moreover, it is useful to have a substantial number of fields to test against. The octic extensions of $\mathbf{Q}_2$ are the smallest degree examples to provide both.

The principal theorem here is that one can compute the inertia subgroup of the normal closure an octic extension of a 2-adic field K_P by the method discussed below. The case of greatest interest is where $K_P = \mathbf{Q}_2$, and we have implemented the method detailed below with results posted at the web site [7]. The results here complements the results of Jones et al [5] where the 2-adic octic fields of [7] are discussed. However, the method presented here is more than a one-off computation since it applies to arbitrary 2-adic base fields.

As mentioned above, there is very little literature on computing inertia groups. *Magma* (see [1]) has a function for computing the intertia subgroup of a number field. However, there are two problems with using it for the situation considered here. First, it works with number fields rather than with extensions of $\mathbf{Q}_2$, although this is probably not a serious problem. Secondly, it works directly with Galois extensions. Here we compute the inertia group for the splitting field of an octic working with the degree 8 extension rather than the Galois extension, which can have degree as large as 384. The latter is fairly impractical.

Section 2 gives notation and background. Section 3 deals with cases which can be handled fairly easily, and Section 4 treats the remaining more complicated cases.

2. Preliminaries

2.1. Notation

We now adjust our notation to be suitable for the case at hand. The base field K will be a finite extension of the 2-adic numbers, $\mathbf{Q}_2$, and F is an extension of K with $[F : K] = 8$. We assume $g(x) \in K[x]$ is irreducible, with $F \cong K[x]/\langle g(x)\rangle$, so $\deg(g(x)) = 8$. As above, $\hat{F}$ denotes the normal closure of F/K, i.e., is a splitting field of $g(x)$, and $G = \mathrm{Gal}(\hat{F}/K) = \mathrm{Gal}(g(x))$.

Let $\hat{F}^{\mathrm{un}}$ be the maximum unramified subextension for $\hat{F}/K$. In particular $\hat{F}^{\mathrm{un}}$ is the fixed field of I, the inertia subgroup of G. The residue degree f for F/K is equal to 1 if and only if $g(x)$ remains irreducible over $\hat{F}^{\mathrm{un}}$. So, when $f = 1$, we use the standard classification of transitive subgroups of S_n by T-numbers (see Butler et al [2]) for describing I. However, if $f > 1$, then I is an intransitive subgroup of S_n, and there is no standard classification of these. In this case, we will only seek to classify I up to isomorphism, i.e., as an abstract group. We then give I by its group number in the program `gap`. For example, $[8,3]$ specifies a group of order 8, and it has been numbered 3 in `gap`. For convenience, we also give a more descriptive name for the group when possible, so for example, $[8,3] = D_4$.

Note, S_8 has transitive subgroups which are isomorphic, but not conjugate. For example, $\mathrm{T}_{19} \cong \mathrm{T}_{20}$. So, computing T-numbers of inertia subgroups in the case $f = 1$ provides more refined information than specifying I as an abstract group.

2.2. Assumptions

We assume that we have the following available information:

1. The Galois group $G = \mathrm{Gal}(\hat{F}/K)$.
2. The unramified degree f for F/K.
3. The size of the inertia subgroup $|I|$ (given condition (1) this is equivalent to the unramified degree for $\hat{F}/K$).
4. subfield information for extensions of degree $\leq [F : K] = 8$.

These conditions are all met in the context of Jones et al [8], [5], [6]. In each of these cases, for a given field, one first computes subfields, the Galois group, and the slope filtration for higher ramification groups (in that order) before computing inertia subgroups. As we will see, this data is not sufficient to compute inertia groups, so in some cases we also compute various resolvent algebras of the octic extension.

2.3. Resolvents

The most common resolvents we use are as follows. The polynomial discriminant of $g(x)$ is in essence a resolvent, $x^2 - \mathrm{disc}(g(x))$, which allows us to construct the discriminant root field for F/K, $\mathrm{disc}(F) := K(\sqrt{\mathrm{disc}(g)})$. This field is independent of the choice of defining polynomial $g(x)$, and corresponds via Galois theory to the fixed field of $\mathrm{Gal}(\hat{F}/K) \cap A_8$.

We also use the *discriminant polynomial*. If the roots of $g(x)$ are $\alpha_1, \ldots, \alpha_8 \in \overline{\mathbf{Q}}_2$, then

$$g_{\mathrm{disc}}(x) = \prod_{i<j} x - (\alpha_i - \alpha_j)^2 .$$

This is an absolute resolvent corresponding to the intransitive subgroup $S_2 \times S_6 < S_8$. It can be computed easily as a resultant by the formula $g_{\mathrm{disc}}(x^2) = \mathrm{Resultant}_y(g(y), g(x+y))/x^8$. Here, $g_{\mathrm{disc}}(x)$ has degree 28.

The polynomial discriminant and the discriminant polynomial can be applied to polynomials of any degree. In a few cases, we use two other resolvents which are specific to octics. One is the absolute resolvent of degree 35 corresponding to $\mathrm{T}_{47} < S_8$, which we denote by $f_{35}(x)$. The other is an octic resolvent $f_8(x)$ which is defined for octic fields containing a quartic subfield. Details on both are given in Jones et al [5].

In all cases where we need to compute resolvents, if the resulting polynomial is not separable we use the standard technique of applying a Tschirinhaus transformation to $g(x)$ and trying again until the result is separable (see, e.g., Cohen [3, §6.3]).

One use of resolvents below is to compute sibling fields. A subfield $F^{sib} \subset \hat{F}$ is a *sibling* of F if F and F^{sib} are not isomorphic, but $\hat{F}$ is also the normal closure of F^{sib}. A *sibling set* is a maximal set of non-isomorphic siblings. Note, a sibling set of octic fields do not necessarily have the same T-number. For example, since $\mathrm{T}_{19} \cong \mathrm{T}_{20}$ as groups, a T_{19} octic field will have siblings with Galois group T_{20}, and conversely. See Jones et al [5] for more information on sibling sets for octic fields.

2.4. Overview

The approach to computing inertia subgroups is similar in some ways to methods for computing Galois groups of polynomials. Group theory cuts down the number of possibilities to a finite list, and then one looks for computable invariants which distinguish these possibilities from each other.

For each Galois group G, we start with candidates for the ramification filtration (see, e.g. Serre [9, Chapter 4]). The group G has normal subgroups I and W, its inertia subgroup and wild ramification group, with $W \leq I \leq G$.

Moreover, G/I is cyclic corresponding to the Galois group of the maximum unramified extension of $\hat{F}/K$, I/W is cyclic of order dividing $|k|^{[G:I]}-1$ corresponding to the totally ramified tame part of the extension, and W is a 2-group. Here, k denotes the residue field of K.

For the Galois theory of octics, there are 50 conjugacy classes of subgroups of S_8 to consider. Our first step in determining the inertia group of an extension with Galois group G is to compute all possible pairs (I, W). Then, we look for simple ways of distinguishing the candidates for I. To a certain extent, one is forced into an extensive case by case analysis. Consequently, we only report the results of this analysis below. Note, in each case the proposed procedure can be checked for correctness by a computation with finite groups. Many computations for this paper were carried out with `gap`, see [4].

3. The 31 Easiest Cases

We now consider the octic Galois groups G, i.e., the 50 conjugacy classes of transitive subgroups of S_8. In this section, we describe the results for 31 of these groups where the information in Section 2 is sufficient to compute the inertia subgroup.

For some groups, knowledge of G alone is sufficient to determine the inertia subgroup I. For example, computing possible ramification filtrations, one can quickly show that some groups cannot be the Galois group of an octic 2-adic extension. The result is as follows.

Proposition 1. *If K is a 2-adic field and $g(x) \in K[x]$ is an irreducible octic polynomial, then Gal$(g(x))$ cannot be T_j with $j = 37$, 43, or $j \geq 45$.*

Some groups admit a unique filtration as described above. Not all of these groups appear as Galois groups over $\mathbf{Q}_2$, but with a general 2-adic base field in mind, we can still quickly gives their inertia groups as described in the following proposition.

Proposition 2. *If K is a 2-adic field and $g(x) \in K[x]$ is an irreducible octic polynomial and Gal$(g(x)) = T_j$ with $j = 14$, 23, 24, 34, 36, 39, 40, 41, 42, or 44, then the inertial subgroup for $g(x)$ depends only on j, and is as shown in Table 1.*

The next simplest cases are those where knowledge of G, f, and $|I|$ suffices.

Proposition 3. *If K is a 2-adic field and $g(x) \in K[x]$ is an irreducible*

G	I
T_{14}	$[12,3]$
T_{23}	T_{12}
T_{24}	T_{13}

G	I
T_{34}	$[48,50]$
T_{36}	T_{25}
T_{39}	T_{32}

G	I
T_{40}	T_{32}
T_{41}	T_{33}

G	I
T_{42}	$[48,50]$
T_{44}	T_{38}

Table 1: Inertia groups completely determined by the Galois group. Note, $[12,3] = A_4$ and $[48,50] = C_2^4 : C_3$. The group T_{39} does not appear as a Galois group over $\mathbf{Q}_2$.

octic polynomial and $Gal(g(x)) = T_j$ *with* $j = 1$, 3, 5, 7, 10, 12, 13, 16, 20, 25, 32, 33, *or* 38, *then the inertial subgroup for* $g(x)$ *depends only on* j *and* f, *the residue field degree. The results are shown in Table 2.*

4. Remaining 19 Cases

In all of the remaining cases, G is a 2-group. Certainly, if $|I| = |G|$, then $I = G$. We henceforth assume $I \neq G$, in which case $[G : I] = 2^j$ with $j \geq 1$. In these cases, it is often helpful to be able to pinpoint quadratic subfields of $\hat{F}$. We start by noting how to distinguish quadratic subfields coming from D_4 fields.

4.1. D_4-Quartics

Quartic fields K_4 whose normal closure, $\hat{K}_4$, has Galois group D_4, play an important role in determining the remaining inertia subgroups. The octic field $\hat{K}_4$ has three quadratic fields which can be readily distinguished. Figure 1 shows the full subfield diagram for $\hat{K}_4$. Here, K_4^{sib} is a sibling of K_4. Primes, as in K_4', denote that the field is isomorphic to another subfield, and the isomorphisms are explicitly noted. The subfield $\hat{K}^{center}$ is the fixed field of the center of D_4.

Since we assume that we have access to subfields of an extension, we can determine $\mathrm{sub}(K_4)$, the quadratic subfield of K_4, which we express for the moment as $K(\sqrt{a})$. Second, we have the discriminant root field $\mathrm{disc}(K_4) = K(\sqrt{\mathrm{disc}(g_4)})$ where $g_4(x)$ is a quartic polynomial defining K_4. Finally, the third quadratic subfield is $\mathrm{rot}(K_4) := K(\sqrt{a \cdot \mathrm{disc}(g_4)})$. This last field is the fixed field of the rotation in D_4 in the Galois correspondence.

Note, if we start with the octic field $\hat{K}_4$, there is nothing to distinguish K_4 from K_4^{sib}. Group theoretically, this is because the corresponding subgroups of

G	$(\lvert I\rvert, f)$	**Inertia group**
T_1	$(8,1)$	T_1
	$(4,2)$	$[4,1] = C_4$
	$(2,4)$	$[2,1] = C_2$
	$(1,8)$	$[1,1] = C_1$
T_3	$(8,1)$	T_3
	$(4,2)$	$[4,2] = V_4$
T_5	$(8,1)$	T_5
	$(4,2)$	$[4,1] = C_4$
T_7	$(16,1)$	T_7
	$(8,1)$	T_1
	$(8,2)$	$[8,2] = C_4C_2$
	$(4,2)$	$[4,1] = C_4$
	$(4,4)$	$[4,2] = V_4$
T_{10}	$(16,1)$	T_{10}
	$(8,1)$	T_2
	$(8,2)$	$[8,5] = C_2^3$
	$(4,2\|4)$	$[4,2] = V_4$
T_{12}	$(24,1)^*$	T_{12}
	$(8,1)$	T_5
T_{13}	$(24,1)^*$	T_{13}
	$(12,2)$	$[12,3] = A_4$
	$(8,1)$	T_3
	$(4,4)$	V_4

G	$(\lvert I\rvert, f)$	**Inertia group**
T_{16}	$(32,1)$	T_{16}
	$(16,1)$	T_7
	$(16,2)$	$[16,11] = D_4C_2$
	$(8,2\|4)$	$[8,5] = C_2^3$
T_{20}	$(32,1)$	T_{20}
	$(16,1)$	T_{10}
	$(16,2)$	$[16,2] = D_4C_2$
	$(8,2)$	$[8,2] = C_4C_2$
	$(8,4)$	$[8,5] = C_2^3$
T_{25}	$(56,1)^*$	T_{25}
	$(8,1)$	T_3
T_{32}	$(96,1)^*$	T_{32}
	$(32,1)$	T_{22}
T_{33}	$(96,1)^*$	T_{33}
	$(48,2)$	$[48,50]$
	$(32,1)$	T_{18}
	$(16,2)$	$[16,14] = C_2^4$
T_{38}	$(192,1)^*$	T_{38}
	$(96,1)$	T_{32}
	$(64,1)$	T_{31}
	$(32,1)$	T_{22}

Table 2: Groups where f and $|I|$ suffice. Here, we write $(|I|, f)^*$ if the combination cannot occur for base field $K = \mathbf{Q}_2$.

D_4 can be interchanged by an (outer) automorphism. For quadratic subfields, one similarly cannot distinguish $\mathrm{disc}(K_4)$ from $\mathrm{sub}(K_4)$ based on the octic field $\hat{K}_4$. However, the subfield $\mathrm{rot}(K_4)$ can be distinguished from these other two fields since $\mathrm{rot}(K_4) = \mathrm{rot}(K_4^{sib})$. In one case below, we make use of this fact, and set $\mathrm{rot}(\hat{K}) = \mathrm{rot}(K_4)$, i.e., pick a quartic subfield which is not Galois, and then compute its rotation field.

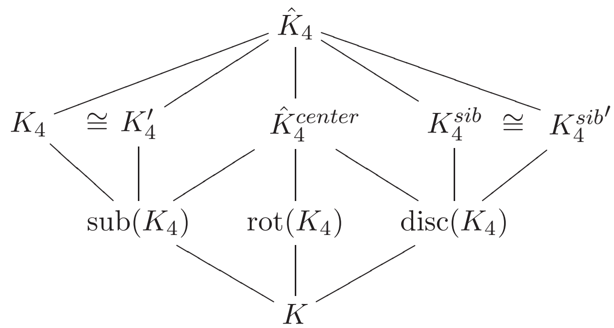

Figure 1: A D_4 field and its subfields.

G	$(\|I\|, f)$	Sub	Disc.	Rot.	None
T_4		$[4,2] = V_4$	$[4,2] = V_4$	$[4,1] = C_4$	$-$
T_6		$[8,3] = D_4$	T_4	T_1	$-$
T_8		$[8,3] = D_4$	T_5	T_1	$-$
T_9	$(8,1)$	$-$	T_3	T_2	T_4
	$(8,2)$	$[8,5] = C_2^3$	$-$	$-$	$[8,3] = D_4$

Table 3: Inertia subgroups determined using a quartic subfield with normal closure D_4. For each octic group, if the specified quadratic subfield for the D_4 is unramified, then the inertia group is given. Information on $(|I|, f)$ are only given when needed. An entry of $-$ means that combination is not possible.

4.2. Applying D_4-Quartic Subfields

In four cases, T_4, T_6, T_8, and T_9, the octic extension F/K has a D_4-quartic intermediate field K_4, from which we can determine the inertia group, using also information as described above.

Proposition 4. *If K is a 2-adic field and $g(x) \in K[x]$ is an irreducible octic polynomial and Gal$(g(x)) = T_j$ with $j = 4$, 6, 8, or 9, then the inertia subgroup for $g(x)$ depends only on j, the residue degree f, and the computation of quadratic fields associated with a D_4 quartic subfield of $K[x]/(g)$. The results are shown in Table 3.*

The groups T_4 and T_9 each have two subfields which can play the role of

K_4; one can choose either when applying Table 3.

4.3. Final Group by Group Analysis

The remaining Galois groups, for the most part, need to be treated separately. Rather than stating these cases as formal propositions, we describe the results in a case by case manner. Recall that in all cases, we are only considering where I is a proper subgroup of G.

Group $\mathrm{T}_2 \cong C_4C_2$. Here $I \neq G$ implies that I is an intransitive subgroup of G. If $|I| = 2$, then clearly $I = [2,1] = C_2$. Otherwise $|I| = 4$. The field F has two C_4 subfields, and one V_4 subfield. The two C_4 quartics have the same quadratic subfield. If this is unramified, $I = [4,2] = V_4$. If one of the other two quadratic subfields are unramified, $I = [4,1] = C_4$.

Group $\mathrm{T}_{11} \cong Q_8 : C_2$. These fields are part of sibling sets of size 3. Starting from one field, we can compute the other two by factoring $g_{\text{disc}}(x)$ (see Jones et al [5]). We compute the sibling set and see how many of the three fields have residue degree $f = 2$. This could be none, one, or two of the three siblings.

If none of them have $f = 2$, then $I = \mathrm{T}_5$; if only one does and it is not the current field, then $I = \mathrm{T}_2$; if the other two have $f = 2$ but the current field is totally ramified, then $I = \mathrm{T}_4$.

Now if the current field has $f = 2$ and it is the only one, $I = [8,2] \cong C_4C_2$. If $f = 2$ and one other sibling has $f = 2$, then $I = [8,3] \cong D_4$.

Group T_{15}. We first look at the quartic D_4-subfield of F, K_4. If $\mathrm{disc}(K_4)$ is unramified, $I = \mathrm{T}_{11}$; if $\mathrm{rot}(K_4)$ is unramified $I = \mathrm{T}_7$. Finally, if the quadratic subfield of K_4 is unramified, $f = 2$ and $I = [16,11] \cong D_4C_2$.

If none of the subfields of the D_4 are unramified, then $f = 1$. We factor $g_{\text{disc}}(x)$ over K, where it will have factors of degrees 4, 8, and 16. We test the degree 16 factor to see if it is totally ramified or not. If so, $I = \mathrm{T}_8$, otherwise $I = \mathrm{T}_6$.

Group T_{17}. First, if $(|I|, f) = (16,1)$, we consider $\mathrm{disc}(F)$. If it is unramified, $I = \mathrm{T}_{11}$; otherwise $I = \mathrm{T}_7$.

If $(|I|, f) = (8,1)$, we factor $g_{\text{disc}}(x)$ and take the degree 16 factor. It has one octic and 3 quartic subfields. If one of the quartic subfields is unramified, then $I = \mathrm{T}_4$; otherwise $I = \mathrm{T}_5$.

Finally, if $f > 1$ we have $I = [16,2] \cong C_4^2$.

Group $\mathrm{T}_{18} \cong C_2^3 : C_2^2$. Here if $f > 1$, then $I = [16, 14] \cong C_2^4$. If $f = 1$, we consider the three D_4-quartic subfields of F. If the discriminant root field of any of these is unramified (all three have to be checked), then $I = \mathrm{T}_9$, otherwise $I = \mathrm{T}_{10}$.

Group T_{19}, T_{20}, and T_{21}. These three Galois groups correspond to sibling fields. One can repeatedly compute and factor the resolvent $g_{\mathrm{disc}}(x)$ to start with one field and compute the full sibling set, which consists of two T_{19} fields, and one each of a T_{20} and a T_{21} (see Jones et al [5]). Note that T_{20} appears above in Table 2.

For T_{19}, if $(|I|, f) = (16, 1)$ we consider a D_4-quartic subfield K_4. If $\mathrm{disc}(K_4)$ is unramified, $I = \mathrm{T}_9$; otherwise $\mathrm{rot}(K_4)$ is unramified and $I = \mathrm{T}_{10}$. On the other hand, if $(|I|, f) = (16, 2)$, then $I = [16, 3] \cong (C_4C_2) : C_2$. The only remaining cases have $(|I|, f) = (8, 1)$, in which case we consider the T_{20} sibling. Either the octic T_{20} field is totally ramified, in which case $I = \mathrm{T}_2$, or it has residue field degree of 4 in which case $I = \mathrm{T}_3$.

For T_{21}, all possibilities have $f > 1$ so we are only identifying I as an abstract group. So, we compute the inertia subgroup of a any sibling T_{19} or T_{20} field, and use that abstract group for I.

Group T_{22}. If $(|I|, f) = (16, 2)$, then $I = [16, 11] \cong D_4C_2$. Otherwise, $(|I|, f) = (16, 1)$. The resolvent $g_{\mathrm{disc}}(x)$ will have three octic factors. If any of them residue degree greater than 1, $I = \mathrm{T}_9$; if all three are totally ramified, $I = \mathrm{T}_{11}$. Note, these octic factors of $g_{\mathrm{disc}}(x)$ define sibling T_{22} fields. They are part of a sibling set of size 6. We also note that T_{22} does not occur over $\mathbf{Q}_2$ because the Galois closure has 15 quadratic subfields, and $\mathbf{Q}_2$ has only 7 quadratic extensions.

Group T_{26}. If $(|I|, f) = (32, 2)$, then $I = [32, 34] = C_4^2 : C_2$. Otherwise $f = 1$ and we consider K_4, a D_4-quartic subfield. If $\mathrm{disc}(K_4)$ is unramified, $I = \mathrm{T}_{22}$, and if $\mathrm{rot}(K_4)$ is unramified, $I = \mathrm{T}_{16}$.

If neither is unramified, then we compute the octic resolvent $f_8(x)$, which will be a T_{18} field. Let I_{18} denoted its inertia subgroup. If $I_{18} = \mathrm{T}_{10}$, then $I = \mathrm{T}_{17}$, and otherwise $I = \mathrm{T}_{15}$.

Group T_{27}. Here, we have a simple chart based on $|I|$, f, and $\mathrm{disc}(F)$.

$$\begin{aligned}
(|I|, f) = (32, 2) &\implies I = [32, 27] \cong C_2^4 : C_2, \\
(|I|, f) = (16, 2) &\implies I = [16, 3] \cong (C_4C_2) : C_2, \\
(|I|, f) = (16, 4) &\implies I = [16, 14] \cong C_2^4, \\
(|I|, f) = (32, 1) \text{ and } \mathrm{disc}(F) \text{ is unram.} &\implies I = \mathrm{T}_{20},
\end{aligned}$$

$$(|I|, f) = (32, 1) \text{ and disc}(F) \text{ is ram.} \implies I = \mathrm{T}_{16}.$$

Group $\mathbf{T_{28}}$. Some cases are easy to distinguish.

$$(|I|, f) = (32, 1) \text{ and rot}(K_4) \text{ is unram.} \implies I = \mathrm{T}_{16}$$
$$(|I|, f) = (32, 1) \text{ and disc}(K_4) \text{ is unram.} \implies I = \mathrm{T}_{21}$$
$$(|I|, f) = (32, 2) \implies I = [32, 27] \cong C_2^4 : C_2$$

The remaining cases have $(|I|, f) = (16, 2)$. Octic fields with Galois group T_{28} have T_{27} siblings which can be computed by factoring a degree 35 resolvent (see Jones et al [5]). In that case we compute a T_{27} sibling and read off the answer for that field as an abstract group.

Group $\mathbf{T_{29}}$. The group T_{29} has six normal subgroups N such that $\mathrm{T}_{29}/N \cong D_4$. The six corresponding octic D_4-fields come in three pairs. Each pair shares the same three quadratic subfields. From the point of view of an octic D_4 field E_8, the only quadratic field which is distinguished is $\mathrm{rot}(E_8)$. For each of the three pairs of octic D_4 fields, $E_{8,a}$ and $E_{8,b}$, it turns out that $\mathrm{rot}(E_{8,a}) = \mathrm{rot}(E_{8,b})$.

With a T_{29} octic field, its quartic subfield K_4 gives us access to one D_4 field in the Galois closure. We compute a representative from each of the other two pairs by factoring the octic resolvent $f_8(x)$ of Jones et al [5]. Denote these quartic fields by K_4' and K_4''.

Now, if $(|I|, f) = (32, 2)$, then $I = [32, 27] \cong C_2^4 : C_2$. Otherwise, $(|I|, f) = (32, 1)$. We distinguish these as follows

$$\text{disc}(K_4) \text{ is unram.} \implies I = \mathrm{T}_{22},$$
$$\text{rot}(K_4) \text{ is unram.} \implies I = \mathrm{T}_{20},$$
$$\text{neither rot}(K_4') \text{ nor rot}(K_4'') \text{ is unram.} \implies I = \mathrm{T}_{18},$$
$$\text{rot}(K_4') \text{ is unram. or rot}(K_4'') \text{ is unram.} \implies I = \mathrm{T}_{19}.$$

Group $\mathbf{T_{30}}$. These fields have an D_4-quartic subfield K_4. Note, for a T_{30} octic field F, it is always the case that $\mathrm{disc}(F) = \mathrm{rot}(K_4)$. If $|I| = 32$ and $f = 2$, then $I = [32, 34] \cong C_4^2 : C_2$. If $|I| = 32$ and $f = 1$, then $I = \mathrm{T}_{20}$ when $\mathrm{disc}(F)$ is unramified, and $I = \mathrm{T}_{21}$ when $\mathrm{disc}(K_4)$ is unramified.

The remaining cases have $(|I|, f) = (16, 2)$. We factor $f_8(x)$, the octic resolvent from Jones et al [5], which necessarily has Galois group T_{19}. We compute the inertia subgroup I_{19} for this T_{19} field. If $I_{19} \cong [8, 2] \cong C_4C_2$, then $I = [16, 2]$; if $I_{19} \cong [8, 5] \cong C_2^3$, then $I = [16, 11]$. Ultimately, determining I_{19} will lead us to its T_{20} sibling, and the distinction is made by the residue field degree of that octic field.

Group $\mathbf{T_{31}}$. This case is simple, where the only extra piece of data to compute is disc(F).

$$(|I|, f) = (32, 2) \implies I = [32, 27] \cong C_2^4 : C_2$$
$$(|I|, f) = (32, 1) \text{ and } \mathrm{disc}(F) \text{ is unram.} \implies I = \mathrm{T}_{22}$$
$$(|I|, f) = (32, 1) \text{ and } \mathrm{disc}(F) \text{ is ram.} \implies I = \mathrm{T}_{21}$$

Group $\mathbf{T_{35}}$. The splitting field of a T_{35} polynomial contains 7 quadratic subfields. It turns out that when $I \neq \mathrm{T}_{35}$, there are 7 possibilities for the inertia group of the extension, and they correspond to which of these 7 quadratic subfields is unramified. We present the correspondence in a table based on F, a D_4 subfield K_4, and utilizing the following notation for quadratic extensions: $K(\sqrt{a}) * K(\sqrt{b}) := K(\sqrt{ab})$.

$(\lvert I\rvert, f)$	Unramified quadratic field	I
$(64, 1)$	$\mathrm{disc}(F) * \mathrm{disc}(K_4)$	T_{26}
$(64, 1)$	$\mathrm{rot}(K_4)$	T_{27}
$(64, 1)$	$\mathrm{disc}(F) * \mathrm{sub}(K_4)$	T_{28}
$(64, 1)$	$\mathrm{disc}(F)$	T_{29}
$(64, 1)$	$\mathrm{disc}(F) * \mathrm{rot}(K_4)$	T_{30}
$(64, 1)$	$\mathrm{disc}(K_4)$	T_{31}
$(64, 2)$	$\mathrm{sub}(K_4)$	$[64, 226] \cong D_4^2$

That completes the results for all 50 octic Galois groups.

4.4. Conclusion

Since [7] contains a representative of each isomorphism class of octic extensions of $\mathbf{Q}_2$, we effectively now have a second efficient approach to computing octic inertia groups over $\mathbf{Q}_2$, namely to use [7] to match an octic extension with an entry in its database and simply read off the corresponding inertia subgroup. However, the ability to find inertia subgroups in this manner is predicated on the results given here.

The method given here for computing inertia groups may be inelegant due to its ad-hoc nature. As mentioned above, there are general approaches, but they would require working with much larger degree fields, and consequently be much more expensive computationally. The question remains open as to whether there is a general method for computing inertia groups which will be as efficient as the method discussed here.

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