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SUCCESSIVE APPROXIMATIONS FOR OPTIMAL CONTROL IN SOME STOCHASTIC SYSTEMS WITH SMALL PARAMETER

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Abstract: Optimal control synthesis is constructed for systems with the help of perturbations in the form of small nonlinear terms of order ϵ, and also of the form of standard Gaussian white noise. The minimized functional differs from the quadratic one by some nonlinear term. When $\epsilon = 0$, the optimal control synthesis is found in an exact analytic form. Successive approximations to the optimal control are constructed with the help of the perturbation method.

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1. Introduction and Problem Statement

An important class of nonlinear control systems is bilinear systems. Such systems are linear on phase coordinates when the control is fixed, and linear on the control when the coordinates are fixed. The first point for the study of bilinear systems is to investigate the dynamic processes of nuclear reactors, kinetics of neutrons, and heat transfer, which started at the beginning of the 60s' of the past century [6]. Further investigations show that many processes in engineering, biology, ecology and other areas can be described by the bilinear systems [8]. As an example, in [7] it is shown that bilinear systems may be applied to describe some chemical reactions and many physical processes in the growth of the human population. Then, in [2] we can see the theoretical and applied aspects of bilinear systems, and their structural properties. In the excellent

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survey [9] various results for deterministic and stochastic bilinear systems are systematized and the basis for the further development of the mathematical interest to such systems is created.

We consider control nonlinear stochastic systems that can be described in the form

$$\dot{x}(t) = \epsilon f_1(t, x) + B(t)x(t)u(t) + \sigma\dot{w}(t),$$
$$x(0) = x_0, \quad 0 \le t \le T. \tag{1}$$

Here the vector $x(t)$ is from the Euclidean space E_n, the control $u(t) \in E_m$, the matrices σ and B have continuous and bounded elements, $\epsilon \ge 0$ is a small parameter, and the initial vector $x_0 \in E_n$ and the constant $T \ge 0$ are given. The function $f_1(t, x) \in E_n$ is continuous in the totality of its arguments, and for all x and y satisfies

$$|f_1(t, x_1) - f_1(t, x_2)| \le a_1|x_1 - x_2|,$$
$$|f_1(t, x)|^2 \le a_2(1 + |x|^2), \tag{2}$$

where a_1 and a_2 some positive constants, and $|\cdot|$ is the Euclidean norm of x. Standard Wiener process $w(t)$ satisfies the conditions

$$w(0) = 0, \quad Mw(t) = 0, \quad Mw(t)w'(t) = It, \tag{3}$$

where M is mathematical expectation, I is identical matrix and a prime indicates the transpose. The matrix $\sigma(t)$ in (1) is such that $\sigma(t)\sigma'(t)$ is positive definite. We understand the equation (1) in the sense of Ito [3]. Note that if $\epsilon = 0$, initial system (1) is bilinear, that is, it contains a nonlinearity of form $x(t)u(t)$. It is known, that if $u = 0$, there exists only one solution of the Cauchy problem (1), see [3]. The problem is to find a control u minimizing the functional $J(0, u)$, where

$$J(t, u) = M\left[x'(t)H_1x(t) + \int_t^T (\, x'(s)H_2(s)x(s) + u'(s)H_3(s)u(s) + f(s, x(s)))\, ds\right]. \tag{4}$$

Here H_i, $i = 1, 2, 3$ are given matrices, so that H_1, $H_2(t)$ are non-negative defined, $H_3(t)$ is positive defined in the interval $[0, T]$, and the matrices $H_2(t)$ and $H_3(t)$ are measurable and bounded. The vector $f(t, x)$ is determined below. The functional (4) differs from the quadratic cost criterion and is called *nonclassical*, see [5].

Note, that the control problem of the system

$$\dot{y}(t) = A(t)y(t) + \epsilon f_1(t, y) + B(t)y(t)u(t) + \sigma\dot{w}(t) \tag{5}$$

with the cost functional (refcost) may be easily reduced to the problem (1), (4) with the help of the substitution

$$y(t) = \left(\exp \int_0^t A(s)ds\right) x(t). \tag{6}$$

2. Algorithm of Successive Approximations

We denote by $u_0(t,x)$ and $V_0(t,x)$ the optimal control and Bellman function, respectively, in problem (1), (4) with $\epsilon = 0$. Suppose that $V_0(t,x)$ satisfies the Bellman equation (see [1])

$$inf_u \left[\frac{\partial V_0}{\partial t} + \left(\frac{\partial V_0}{\partial x}\right)' Bxu + x'H_2x + u'H_3u \right.$$
$$\left. + u + f + \frac{1}{2}Tr\left(\sigma\sigma' \frac{\partial^2 V_0}{\partial x^2}\right)\right] = 0. \; V_0(T,x) = x'H_1x. \tag{7}$$

Here $\frac{\partial V_0}{\partial t}$ is the partial derivative with respect to time, and $\frac{\partial V_0}{\partial x}$ – the vector of partial derivatives with respect to coordinates of the vector x, $\partial^2 V_0/\partial x^2$ is the matrix of the second partial derivatives with respect to x, and Tr means the matrix trace.

From (7) follows that $u_0(t,x)$ is given by

$$u_0(t,x) = -\frac{1}{2}H_3^{-1}(t)B'(t)\frac{\partial V_0}{\partial x}\,x. \tag{8}$$

Now we require the Bellman function $V_0(t,x)$ in the following form

$$V_0(t,x) = x'P(t)x + g(t), \tag{9}$$

where the symmetric matrix P and the scalar function g will be defined.

If we substitute (8) and (9) in the Bellman equation (7), we obtain

$$x'\dot{P}x + \dot{g} - (x')^2 PBH_3^{-1}B'Px^2 + x'H_2x + f + Tr(P\sigma\sigma') = 0. \tag{10}$$

Equating to zero the quadratic term coefficients gives the linear matrix differential equation for the matrix P, that is,

$$\dot{P} + H_2 = 0, \quad P(T) = H_1. \tag{11}$$

Moreover, the matrix P is bounded and non-negative defined in the interval $0 \le t \le T$. In the similar way we equal to zero the terms non depending on x, and after that the remaining terms to obtain the equation for the function g and also the formula for the function $f(t,x)$ from (4)

$$\dot{g} + Tr(P\sigma\sigma') = 0, \quad g(T) = 0, \tag{12}$$

and

$$f(t,x) = (x')^2 P(t)B(t)H_3^{-1}(t)B'(t)P(t)x^2. \tag{13}$$

Thus, the synthesis of the control for (1), (4) with $\epsilon = 0$ is reduced to solution of ordinary differential equations (11)-(13). After finding the matrix P and the function g we have for the Bellman function V_0

$$V_0(t,x) = x'P(t)x + Tr\int_t^T P(s)\sigma(s)\sigma'(s)ds. \tag{14}$$

Now, let $v(t,x)$ and $V(t,x)$ be the optimal control and the optimal value of the cost functional $J(t,u)$, respectively in problem (1), (4), under the condition that system (1) begins to move from the state x at time moment t. We introduce the following notations

$$\frac{\partial V}{\partial t} = V_t,\ \frac{\partial V}{\partial x} = V_x,\ L = \frac{\partial}{\partial t} + \frac{1}{2}Tr\left(\sigma\sigma'\frac{\partial^2}{\partial x^2}\right) \tag{15}$$

and suppose that $V(t,x)$ satisfies the Bellman equation

$$sup_u\left[LV + (V_x)'(Bxu + \epsilon f_1) + x'H_2x + u'H_3u + f\right] = 0,$$
$$V(T,x) = x'H_1x. \tag{16}$$

From this equation follows, that the optimal control $v(t,x)$ is given by

$$v(t,x) = -\frac{1}{2}H_3^{-1}(t)B'(t)V_x(t,x)x. \tag{17}$$

We introduce the notation

$$B_1(t) = B(t)H_3^{-1}(t)B'(t) \tag{18}$$

and substitute (17) in (16) to obtain for the Bellman function $V(t,x)$

$$LV - \frac{1}{4}x'\left(V_x'B_1V_x\right)x + \epsilon V_x'f_1 + x'H_2x + f = 0,$$
$$V(T,x) = x'H_1x. \tag{19}$$

After finding the solution of the Cauchy problem (19) the optimal control will be immediately defined by (17).

Now we describe the algorithm of the successive approximations to the optimal control $v(t,x)$. The Bellman function $V(t,x)$ is represented as a series of the parameter ϵ (see [4])

$$V(t,x) = v_0(t,x) + \epsilon V_1(t,x) + \epsilon^2 V_2(t,x) + \dots . \tag{20}$$

Here the function $V_0(t,x)$ is the Bellman function for problem (1), (4) with $\epsilon = 0$, and defined according to (9) and (11).

Substituting (20) into (19) and equating the coefficients of the same powers of ϵ gives the linear equations for determining the rest of the functions V_j, $j \geq 1$,

$$LV_j + f'V_{x,j-1} - \frac{1}{4}\sum_{k=0}^{j} x'\left((V_{x,k})'B_1V_{x,j-k}\right)x = 0,$$
$$V_j(T,x) = 0,\ j \geq 1. \quad (21)$$

According to (17) and (20), the optimal control will be given by the formula

$$v(t,x) = -\frac{1}{2}H_3^{-1}(t)B'(t)\left(V_{x,0}(t,x) + \epsilon V_{x,1}(t,x) + \ldots\right)x, \quad (22)$$

and i-approximation, $u_i(t,x)$ to the optimal control – by (17) in which the partial sum of (20) instead of V is substituted, that is, by

$$u_i(t,x) = -\frac{1}{2}H_3^{-1}(t)B'(t)\left(V_{x,0}(t,x) + \epsilon V_{x,1}(t,x) + \ldots + \epsilon^i V_{x,i}(t,x)\right)x. \quad (23)$$

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