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PROPERTIES OF WEAK SOLUTIONS OF AN ELLIPTIC PROBLEM IN WEIGHTED SPACES

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Abstract: This paper is concerned with the Dirichlet problem for second order elliptic operators in divergence form with singular coefficients in weighted Sobolev spaces. The leading coefficients are VMO functions. The hypotheses on the other coefficients involve the weight function.

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1. Introduction

In some recent papers (see [5] and [6]) we studied a Dirichlet problem associated to a uniformly elliptic operator of second order. In these papers we extended the well known results of F. Chiarenza, M. Frasca and P. Longo (see [7], [8], [12], [13]) to the case of open sets Ω with singular boundary and coefficients singular near a subset of the boundary $\partial\Omega$. In the study of the bounds for the solutions of this problem, we built a bilinear form related, in an appropriate way, to the coefficients of the operator. In this paper we want to investigate some properties of solutions of the Dirichlet problem associated to such bilinear form.

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More precisely, given $t \in \mathbb{R}$, we consider the bilinear form

$$a_t(u,v) = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}\, u_{x_i}\, v_{x_j} + b\, u\, v \Big)\, \sigma^{2t}\, dx\,, \tag{1}$$

where the elliptic and bounded coefficients a_{ij} admit extensions in the class VMO, the coefficient b verifies a suitable sommability condition, and the weight function σ is related to the distance from a nonempty subset of $\partial\Omega$. The bilinear form (1) is defined on a weighted Sobolev space $\overset{\circ}{W}{}^{1,2}_t(\Omega)$, where the weight is an appropriate power of the function σ. Fixed a function g such that $\sigma^{-t+1}\, g \in L^2(\Omega)$, we determine some conditions on Ω and on the coefficients in order to prove that the problem

$$\begin{cases} u \in \overset{\circ}{W}{}^{1,2}_t(\Omega)\,, \\ a_t(u,v) = \displaystyle\int_\Omega g\, v\, dx \qquad \forall\, v \in \overset{\circ}{W}{}^{1,2}_t(\Omega)\,, \end{cases} \tag{2}$$

is uniquely solvable. Moreover, if $\sigma^{-t+1}\, g \in L^r(\Omega)$, for $r > 1$, then the solution u of problem (2) is such that $\sigma^{t-1}\, u \in L^r(\Omega)$ and verifies the bound

$$\|\sigma^{t-1}\, u\|_{L^r(\Omega)} \le c\, \|\sigma^{-t+1}\, g\|_{L^r(\Omega)}\,,$$

where $c \in \mathbb{R}_+$ depends on Ω, n, r, t, σ and on the regularity of the coefficients.

2. Notation and Function Spaces

Let G be any Lebesgue measurable subset of $\mathbb{R}^n$ and let $\Sigma(G)$ be the collection of all Lebesgue measurable subsets of G. If $F \in \Sigma(G)$, denote by $|F|$ the Lebesgue measure of F and by $\mathfrak{D}(F)$ (resp. $\mathfrak{D}^0(F)$) the class of restrictions to F of functions $\zeta \in C^\infty_o(\mathbb{R}^n)$ (resp. $\zeta \in C^0_o(\mathbb{R}^n)$) with $\bar{F} \cap \operatorname{supp} \zeta \subseteq F$. Moreover, for $p \in [1, +\infty]$, let $L^p_{\mathrm{loc}}(F)$ be the class of functions g such that $\zeta\, g \in L^p(F)$ for all $\zeta \in \mathfrak{D}(F)$.

Let Ω be an open subset of $\mathbb{R}^n$. We set

$$\Omega(x,r) = \Omega \cap B(x,r) \qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, r \in \mathbb{R}_+\,,$$

where $B(x,r)$ is the open ball with center x and radius r.

We denote by $\mathcal{A}(\Omega)$ the class of all measurable functions $\rho : \Omega \to\ R_+$ such that

$$\gamma^{-1}\, \rho(y) \le \rho(x) \le \gamma\, \rho(y) \qquad \forall\, y \in \Omega\,, \quad \forall\, x \in \Omega(y, \rho(y))\,, \tag{3}$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y. For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \{z \in \partial\Omega : \lim_{x \to z} \rho(x) = 0\}\,. \tag{4}$$

It is known that S_ρ is a closed subset of $\partial\Omega$ and that

$$\rho \in L^\infty_{\rm loc}(\bar\Omega)\,, \qquad \rho^{-1} \in L^\infty_{\rm loc}(\bar\Omega \setminus S_\rho) \tag{5}$$

(see [10], [4]).

If $r \in \mathbb{N}$, $1 \le p \le +\infty$, $s \in \mathbb{R}$ and $\rho \in \mathcal{A}(\Omega)$, consider the space $W^{r,p}_s(\Omega)$ of distributions u on Ω such that $\rho^{s+|\alpha|-r}\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le r$, equipped with the norm

$$\|u\|_{W^{r,p}_s(\Omega)} = \sum_{|\alpha|\le r} \|\rho^{s+|\alpha|-r}\partial^\alpha u\|_{L^p(\Omega)}\,.$$

Moreover, denote by $\overset{\circ}{W}{}^{r,p}_s(\Omega)$ the closure of $C^\infty_\circ(\Omega)$ in $W^{r,p}_s(\Omega)$ and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$. Clearly the following imbeddings

$$\overset{\circ}{W}{}^{r,p}_s(\Omega) \hookrightarrow W^{r,p}_s(\Omega) \hookrightarrow L^p_{s-r}(\Omega)$$

hold.

A more detailed account of properties of the above defined spaces can be found, for instance, in [9], [2] and [11].

We now recall the definitions of some function spaces in which the coefficients of the operator will be chosen. For $p \in [1, +\infty[$ and $s \in \mathbb{R}$, let $K^p_s(\Omega)$ be the set of all functions $g \in L^p_{\rm loc}(\bar\Omega \setminus S_\rho)$ such that

$$\|g\|_{K^p_s(\Omega)} = \sup_{x\in\Omega}\Big(\rho^{s-n/p}(x)\,\|g\|_{L^p(\Omega(x,\rho(x)))}\Big) < +\infty\,, \tag{6}$$

endowed with the norm defined by (6). It is easy to prove (see [3]) that the spaces $L^\infty_s(\Omega)$ and $C^\infty_o(\Omega)$ are subset of $K^p_s(\Omega)$. Therefore we can define two new spaces of functions: we denote by $\widetilde{K}{}^p_s(\Omega)$ and $\overset{\circ}{K}{}^p_s(\Omega)$ the closures of $L^\infty_s(\Omega)$ and $C^\infty_o(\Omega)$, respectively, in $K^p_s(\Omega)$.

Now, it is useful to recall that for a function $g \in \widetilde{K}{}^p_s(\Omega)$ the following characterization holds:

$$g \in \widetilde{K}{}^p_s(\Omega) \Longleftrightarrow \lim_{h\to+\infty} \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}\frac{|\Omega(x,\rho(x))\cap E|}{\rho^n(x)}\le 1/h}} \|g\,\chi_E\|_{K^p_s(\Omega)} = 0\,, \tag{7}$$

where χ_E is the characteristic function of E.

If Ω has the property

$$|\Omega(x,r)| \ge A\,r^n \qquad \forall\, x \in \Omega\,, \quad \forall\, r \in]0,1]\,,$$

where A is a positive constant independent of x and r, it is possible to consider the space $BMO(\Omega,t)$ ($t\in\ R_+$) of functions $g\in L^1_{\rm loc}(\bar\Omega)$ such that

$$[g]_{BMO(\Omega,t)}=\sup_{\substack{x\in\Omega\\ r\in]0,t]}}\ \fint_{\Omega(x,r)}\Big|g-\fint_{\Omega(x,r)}g\Big|<+\infty\,,$$

where

$$\fint_{\Omega(x,r)}g=|\Omega(x,r)|^{-1}\int_{\Omega(x,r)}g.$$

If $g\in BMO(\Omega)=BMO(\Omega,t_A)$, where

$$t_A=\sup_{t\in\mathbb{R}_+}\Big(\sup_{\substack{x\in\Omega\\ r\in]0,t]}}\ \frac{r^n}{|\Omega(x,r)|}\le\frac{1}{A}\Big)\,,$$

we shall say that $g\in VMO(\Omega)$ if $[g]_{BMO(\Omega,t)}\to 0$ for $t\to 0^+$. A function $\eta[g]:\mathbb{R}_+\to\mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega,t)}\le\eta[g](t)\quad\forall\,t\in\mathbb{R}_+\,,\qquad\lim_{t\to 0^+}\eta[g](t)=0\,.$$

3. Preliminary Results

Let Ω be an open subset of $\mathbb{R}^n$ ($n>2$) with the segment property[1] and fix $\rho\in\mathcal{A}(\Omega)\cap L^\infty(\Omega)$ such that $S_\rho\neq\emptyset$.

We suppose that the following condition on Ω holds:

h_1) there exists an open subset Ω_o of $\mathbb{R}^n$ with the uniform $C^{1,1}$-regularity property such that

$$\Omega\subset\Omega_o\,,\qquad\qquad\partial\Omega\setminus S_\rho\subset\partial\Omega_o\,.$$

Remark 1. We observe that if condition h_1) holds, then there exists $\theta\in]0,\frac{\pi}{2}[$ such that

$$\forall x\in\Omega\qquad\exists C_\theta(x)\ :\qquad\overline{C_\theta(x,\rho(x))}\subset\Omega\,,$$

where $C_\theta(x)$ is an open indefinite cone with vertex in x and opening θ and $C_\theta(x,\rho(x))=C_\theta(x)\cap B(x,\rho(x))$ (see Remark 3.1 of [4]). As a consequence, there exists a function $\sigma\in\mathcal{A}(\Omega)\cap C^\infty(\Omega)\cap C^{0,1}(\bar\Omega)$ which is equivalent to ρ and such that

$$|\partial^\alpha\sigma(x)|\le c_\alpha\,\sigma^{1-|\alpha|}(x)\qquad\forall\,x\in\Omega\,,\quad\forall\,\alpha\in\mathbb{N}^n_o\,,\tag{8}$$

where $c_\alpha\in\mathbb{R}_+$ is independent of x (see [10]).

[1] For all definitions of regularity properties of open subsets of $\mathbb{R}^n$ we will refer to [1].

The assumption h_1) assure only that the functions $\big(\sigma(x)\big)_{x_i}$ $(i = 1, \cdots, n)$ are bounded near S_ρ. Thus we introduce a sequence of functions $(\eta_k)_{k\in\mathbb{N}}$, related to the function σ, such that the sequence of gradients goes to 0 near S_ρ.

Choose a cutoff function $g \in C_\circ^\infty(\bar{\mathbb{R}}_+)$ such that

$$0 \le g \le 1\,, \qquad g(t) = 1 \quad \text{if} \quad t \ge 1\,, \qquad g(t) = 0 \quad \text{if} \quad t \le \frac{1}{2}\,.$$

For any $k \in \mathbb{N}$ we set

$$\eta_k(x) = \frac{1}{k}\,\zeta_k(x) + (1 - \zeta_k(x))\,\sigma(x)\,, \qquad x \in \Omega\,,$$

where $\zeta_k(x) = g(k\,\sigma(x))$, $x \in \Omega$. Clearly, $\eta_k \in C^\infty(\Omega)$ for each $k \in \mathbb{N}$ and

$$\eta_k(x) = \begin{cases} \dfrac{1}{k} & \text{if } x \in \overline{\Omega}_k, \\ \sigma(x) & \text{if } x \in \Omega \setminus \Omega_{2k}\,, \end{cases}$$

where

$$\Omega_k = \Big\{ x \in \Omega \ : \ \sigma(x) > \frac{1}{k} \Big\}\,.$$

It is easy to show that for each $k \in \mathbb{N}$,

$$\sigma(x) \le \eta_k(x) \le 2\,\sigma(x) \qquad x \in \Omega \setminus \overline{\Omega}_k\,, \tag{9}$$

$$c'_k\,\sigma(x) \le \eta_k(x) \le 2\,\sigma(x) \qquad x \in \Omega_k\,, \tag{10}$$

$$\big(\eta_k(x)\big)_x \le c_1\big(\sigma(x)\big)_x \qquad x \in \Omega\,, \tag{11}$$

where $c'_k \in \mathbb{R}_+$ depends on k and σ, and $c_1 \in \mathbb{R}_+$ depends only on n.

For $p \ge 2$ and $t \in \mathbb{R}$, we put

$$V_t^p(\Omega) = \big\{ u \in W_t^{1,2}(\Omega) \ : \ |\eta_k^{t-1}\, u|^{p-2}\, u \in W_t^{1,2}(\Omega) \quad \forall\, k \in \mathbb{N} \big\},$$

$$\overset{\circ}{V}{}_t^p(\Omega) = \big\{ u \in \overset{\circ}{W}{}_t^{1,2}(\Omega) \ : \ |\eta_k^{t-1}\, u|^{p-2}\, u \in \overset{\circ}{W}{}_t^{1,2}(\Omega) \quad \forall\, k \in \mathbb{N} \big\}\,.$$

We can prove the following result:

Lemma 2. *Suppose that condition h_1) holds. Then for any $p \ge 2$ and $t \in \mathbb{R}$ we have*

$$W_t^{1,2}(\Omega) \cap W_t^{2(p-1)}(\Omega) \subset V_t^p(\Omega), \tag{12}$$

$$\overset{\circ}{W}{}_t^{1,2}(\Omega) \cap \overset{\circ}{W}{}_t^{1,2(p-1)}(\Omega) \subset \overset{\circ}{V}{}_t^p(\Omega)\,. \tag{13}$$

Proof. The inclusion (12) is a consequence of the relations (9), (10) and the equivalence of ρ and σ. We prove now (13). It follows from condition h_1) and Theorem 4.1 of [10], that there exists a sequence $(\delta_h)_{h\in\mathbb{N}}$ in $\mathfrak{D}(\bar{\Omega} \setminus S_\rho)$ such that

$$\lim_{h\to+\infty} \partial^\alpha(1 - \delta_h(x)) = 0 \qquad \forall\, x \in \Omega\,, \quad \forall\, \alpha \in \mathbb{N}_o^n\,, \tag{14}$$

$$\sup_{h\in\mathbb{N}} |\partial^\alpha \delta_h(x)| \le c_\alpha\, \rho^{-|\alpha|}(x) \qquad \forall\, x\in\Omega\,, \quad \forall\,\alpha\in\mathbb{N}_o^n\,, \tag{15}$$

where $c_\alpha \in \mathbb{R}_+$ is independent of x. Fix $u \in \overset{\circ}{W}{}_t^{1,2}(\Omega)\cap \overset{\circ}{W}{}_t^{1,2(p-1)}(\Omega)$. Let $(\varphi_m)_{m\in\mathbb{N}}$ be a sequence in $C_\circ^\infty(\Omega)$ such that

$$\|\varphi_m - u\|_{W_t^{1,2(p-1)}(\Omega)} \to 0 \qquad \text{for} \quad m\to+\infty. \tag{16}$$

Relations (5) and (15), after some calculations, allow us to obtain the estimate

$$\|\delta_h\,\varphi_m - \delta_h\, u\|_{W^{1,2(p-1)}(\Omega)} \le c_1\,\|\varphi_m - u\|_{W_t^{1,2(p-1)}(\Omega)}\,, \tag{17}$$

where $c_1 \in \mathbb{R}_+$ is independent of m. Since for any fixed $h\in\mathbb{N}$ the functions $\tilde\varphi_m = \delta_h\,\varphi_m$ $(m\in\mathbb{N})$ belong to $C_\circ^\infty(\Omega)$, from (16) and (17) we deduce that $\delta_h\, u \in \overset{\circ}{W}{}^{1,2(p-1)}(\Omega)$.

For $h\in\mathbb{N}$, let $\zeta_h \in \mathfrak{D}(\bar\Omega\setminus S_\rho)$ be such that $\zeta_h = 1$ on $\operatorname{supp}\delta_h$. Recalling that $\eta_k\in C^\infty(\Omega)$, we have

$$\|\eta_k^{t-1}\,\zeta_h\,\tilde\varphi_m - \eta_k^{t-1}\,\delta_h\, u\|_{W^{1,2(p-1)}(\Omega)} \le c_2\,\|\tilde\varphi_m - \delta_h\, u\|_{W^{1,2(p-1)}(\Omega)}\,, \tag{18}$$

where $c_2\in\mathbb{R}_+$ is independent of m. Since for any fixed $k, h\in\mathbb{N}$ the functions $\eta_k^{t-1}\,\zeta_h\,\tilde\varphi_m$ $(m\in\mathbb{N})$ belong to $C_\circ^\infty(\Omega)$, from (16)-(18) we deduce that $\eta_k^{t-1}\,\delta_h\, u\in \overset{\circ}{W}{}^{1,2(p-1)}(\Omega)$.

Fix $k\in\mathbb{N}$. For any $h\in\mathbb{N}$, put $v_h = (\eta_k^{t-1}\,\zeta_h\,\tilde\varphi_m)_o$ where $(-)_o$ denotes the extension of $(-)$ to $\mathbb{R}^n$ with zero values out of Ω. Obviously $v_h\in\overset{\circ}{W}{}^{1,2(p-1)}(\Omega)$ for each $h\in\mathbb{N}$ and therefore, from well known results, we deduce

$$|v_h|^{p-2}\, v_h \in \overset{\circ}{W}{}^{1,2}(\Omega_o)\,.$$

For any fixed $h\in\mathbb{N}$, let $(\psi_m^h)_{m\in\mathbb{N}}$ be a sequence in $C_\circ^\infty(\Omega_o)$ such that

$$\|\psi_m^h - |v_h|^{p-2}\, v_h\|_{W^{1,2}(\Omega_o)} \to 0 \qquad \text{for} \quad m\to+\infty. \tag{19}$$

With easy computations we obtain

$$\begin{aligned} &\left\|\eta_k^{1-t}\,\zeta_h\,\psi_m^h - |\eta_k^{t-1}\, u|^{p-2}\, u\right\|_{W_t^{1,2}(\Omega)} \\ &\le c_3\left\|\eta_k^{1-t}\,\zeta_h\left(\psi_m^h - |v_h|^{p-2}\, v_h\right)\right\|_{W^{1,2}(\Omega_o)} \\ &+\left\||\eta_k^{t-1}\, u|^{p-2}\, u\left(|\delta_h|^{p-2}\,\delta_h - 1\right)\right\|_{W_t^{1,2}(\Omega)} \end{aligned} \tag{20}$$

$$\le c_4\left\|\psi_m^h - |v_h|^{p-2} v_h\right\|_{W^{1,2}(\Omega_o)} + \left\||\eta_k^{t-1} u|^{p-2}\, u\left(|\delta_h|^{p-2}\delta_h - 1\right)\right\|_{W_t^{1,2}(\Omega)}\,,$$

where c_3, $c_4\in\mathbb{R}_+$ depend on n, ρ, t and h. From (19) we deduce that for any fixed $h\in\mathbb{N}$ there exists $m_h\in\mathbb{N}$ such that

$$\begin{aligned} &\left\|\eta_k^{1-t}\,\zeta_h\,\psi_{m_h}^h - |\eta_k^{t-1}\, u|^{p-2}\, u\right\|_{W_t^{1,2}(\Omega)} \\ &\le \frac{1}{h} + \left\||\eta_k^{t-1}\, u|^{p-2}\, u\left(|\delta_h|^{p-2}\,\delta_h - 1\right)\right\|_{W_t^{1,2}(\Omega)}\,. \end{aligned} \tag{21}$$

Now we prove that the last norm in (21) also goes to zero for $h\to+\infty$. In fact, for any fixed $k \in \mathbb{N}$, set $w_k = |\eta_k^{t-1}\, u|^{p-2}\, u$ and observe that from (14) and (15) we have

$$\left\|\rho^{t-1}\, w_k\left(|\delta_h|^{p-2}\,\delta_h - 1\right)\right\|_{L^2(\Omega)} \to 0 \qquad \text{for} \quad h \to +\infty. \tag{22}$$

Moreover, since $\rho \in L^\infty(\Omega)$, we also have for $h \to +\infty$

$$\left\|\rho^t\Big(w_k\big(|\delta_h|^{p-2}\,\delta_h - 1\big)\Big)_x\right\|_{L^2(\Omega)} \tag{23}$$

$$\leq c_5\left(\left\|\rho^t (w_k)_x\left(|\delta_h|^{p-2}\delta_h - 1\right)\right\|_{L^2(\Omega)} + \left\|\rho^t w_k\left(|\delta_h|^{p-2}\delta_h\right)_x\right\|_{L^2(\Omega)}\right) \to 0\,.$$

where $c_5 \in \mathbb{R}_+$ is independent on h.

Since $\eta_k^{1-t}\,\zeta_h\,\psi_{m_h}^h \in C_\circ^\infty(\Omega)$ for any $k, h \in \mathbb{N}$, from (21)-(23) we deduce that $|\eta_k^{t-1}\, u|^{p-2}\, u \in \overset{\circ}{W}{}_t^{1,2}(\Omega)$. □

Suppose that the following condition on ρ holds:

$h_2)$ $$\lim_{x \to x_o} \sigma_x(x) = \lim_{|x| \to +\infty} \big(\sigma(x) + \sigma_x(x)\big) = 0 \qquad \forall\, x_o \in S_\rho\,.$$

We will assume that $p \in [2, +\infty[$, $t \in \mathbb{R}$. Consider in Ω the differential operator

$$L_t = -\sum_{i,j=1}^n a_{ij}\,\frac{\partial^2}{\partial_{x_i}\,\partial_{x_j}} - \sum_{i=1}^n\Big(\sum_{j=1}^n\big((a_{ij})_{x_j} + 2\,t\,a_{ij}\,\sigma^{-1}\,\sigma_{x_j}\big)\Big)\frac{\partial}{\partial_{x_i}} + b,$$

with the following conditions on the coefficients:

$h_3)$ there exist extensions a_{ij}^o of a_{ij} to Ω_o such that

$$\begin{cases} a_{ij}^o = a_{ji}^o \in L^\infty(\Omega_o) \cap VMO(\Omega_o) & i, j = 1, \cdots, n, \\ \exists\, \nu \in \mathbb{R}_+ \;:\; \sum\limits_{i,j=1}^n a_{ij}^o\,\xi_i\,\xi_j \geq \nu\,|\xi|^2 & \text{a.e. in } \Omega_o, \quad \forall\, \xi \in \mathbb{R}^n, \end{cases}$$

$h_4)$ $$\operatorname*{ess\,inf}_\Omega\,(\sigma^2\, b) = b_o > 0\,.$$

In the following preliminary result we also assume that

$i_1)$ $$(a_{ij})_{x_k} \in L^\infty(\Omega)\,, \quad i, j, k = 1, \cdots, n, \qquad b \in L_2^\infty(\Omega)\,.$$

Note that by $h_2)$, $i_1)$ and Lemma 2.1 of [4], the functions $\sigma^{-1}\,\sigma_{x_i}$, $(a_{ij})_{x_k}$ $(i, j, k = 1, \cdots, n)$ belong to $\overset{\circ}{K}{}_1^q(\Omega)$ for any $q \geq 1$.

Observe that under assumptions $h_1) - h_4)$ and $i_1)$, according to Theorem 3.1 of [4], for any $p \in [2, +\infty[$ and $t \in \mathbb{R}$ the operator $L_t : W_{t+1}^{2,p}(\Omega) \to L_{t+1}^p(\Omega)$ is bounded.

Now we prove the solvability of a Dirichlet problem and derive, under additional assumption, higher regularity for our weak solution.

Lemma 3. *Suppose that conditions $h_1) - h_4)$ and $i_1)$ hold. Then for any $p \in [2, +\infty[$ and $t \in \mathbb{R}$, the problem*

$$\begin{cases} u \in W^{2,p}_{t+1}(\Omega) \cap \overset{\circ}{W}{}^{1,p}_t(\Omega)\,, \\ L_t\, u = f, \qquad f \in L^p_{t+1}(\Omega)\,, \end{cases} \tag{24}$$

is uniquely solvable. If moreover $f \in \mathfrak{D}^0(\bar{\Omega} \setminus S_\rho)$, then the solution of problem (24) has the property

$$u \in W^{2,q}_{\tau+1}(\Omega) \cap \overset{\circ}{W}{}^{1,q}_\tau(\Omega) \qquad \forall\, q \geq 2, \quad \forall\, \tau \in \mathbb{R}\,. \tag{25}$$

Proof. The existence and uniqueness result for problem (24) follows from Theorem 4.2 of [6]. Suppose now that $f \in \mathfrak{D}^0(\bar{\Omega} \setminus S_\rho)$ and observe that $f \in L^q_{\tau+1}(\Omega)$ for any $q \geq 2$ and $\tau \in \mathbb{R}$. Let u_{pt} be the solution of problem (24) and $u_{q\tau}$ be the solution of the problem

$$\begin{cases} u \in W^{2,q}_{\tau+1}(\Omega) \cap \overset{\circ}{W}{}^{1,q}_\tau(\Omega)\,, \\ L_t\, u = f, \qquad f \in L^q_{\tau+1}(\Omega)\,. \end{cases}$$

Assume first that $q \leq p$. By Theorem 3.4 and Lemma 3.2 of [6], it follows that $u_{q\tau} \in W^{2,p}_{t+1}(\Omega) \cap \overset{\circ}{W}{}^{1,p}_t(\Omega)$. From the uniqueness of solution, we have $u_{q\tau} = u_{pt}$. Let now $q > p$. Following the same argument, we obtain also $u_{pt} = u_{q\tau}$. Fix now $\tau \geq t$. Since $\rho \in L^\infty(\Omega)$, we deduce that $u_{p\tau} \in W^{2,p}_{t+1}(\Omega) \cap \overset{\circ}{W}{}^{1,p}_t(\Omega)$ and therefore, from the uniqueness,

$$u_{pt} = u_{p\tau} \in W^{2,p}_{\tau+1}(\Omega) \cap \overset{\circ}{W}{}^{1,p}_\tau(\Omega). \tag{26}$$

If $\tau < t$, using the same argument, we have again the (26). This completes the proof. □

4. Main Results

Throughout this section we assume that the following condition holds:

$h_5)$ $$b \in \widetilde{K}^{n/2}_2(\Omega)\,.$$

Fix $t \in \mathbb{R}$ and consider the bilinear form

$$a_t(u, v) = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}\, u_{x_i}\, v_{x_j} + b\, u\, v \Big)\, \sigma^{2t}\, dx\,, \tag{27}$$

with $u, v \in \overset{\circ}{W}{}^{1,2}_t(\Omega)$. For $g \in L^2_{-t+1}(\Omega)$, we consider the problem

$$\begin{cases} u \in \overset{\circ}{W}{}^{1,2}_t(\Omega)\,, \\ a_t(u,v) = \displaystyle\int_\Omega g\, v\, dx \qquad \forall\, v \in \overset{\circ}{W}{}^{1,2}_t(\Omega)\,. \end{cases} \tag{28}$$

Let us set, for any $i, j = 1, \cdots, n$ and $h \in \mathbb{N}$,

$$\tilde{a}_{ij} = \begin{cases} a^o_{ij} & \text{on } \Omega_o\,, \\ \delta_{ij}\, \nu & \text{on } \mathbb{R}^n \setminus \Omega_o\,, \end{cases} \qquad\qquad a^h_{ij} = J_h * \tilde{a}_{ij}\,,$$

where $(J_h)_{h\in\mathbb{N}}$ is a sequence of mollifiers. Using condition h_3), it is easy to prove that for each $h \in \mathbb{N}$ we have

$$\begin{cases} a^h_{ij} = a^h_{ji} \in L^\infty(\Omega_o) \cap VMO(\Omega_o) \qquad i,j = 1, \cdots, n, \\ \displaystyle\sum_{i,j=1}^n a^h_{ij}\, \xi_i\, \xi_j \geq \nu\, |\xi|^2 \qquad \text{a.e. in } \Omega_o, \quad \forall\, \xi \in \mathbb{R}^n, \end{cases} \tag{29}$$

$$\left(a^h_{ij}\right)_{x_k} \in L^\infty(\Omega) \qquad \forall\, i,j,k = 1, \cdots, n\,. \tag{30}$$

For any $h \in \mathbb{N}$ we define

$$E_h = \left\{ x \in \Omega \,:\, \rho^2(x)\, b(x) > \gamma^2\, \|b\|_{K^{n/2}_2(\Omega)}\, h^{2/n} \right\},$$

with γ given in (3), and put

$$\alpha_h = \gamma^2\, \|b\|_{K^{n/2}_2(\Omega)}\, h^{2/n}\,.$$

From (3), it is easy to verify that for any fixed $x \in \Omega$ we obtain the estimate

$$\frac{|\Omega(x,\rho(x)) \cap E_h|}{\rho^n(x)} \leq \frac{1}{\rho^n(x)} \int_{\Omega(x,\rho(x))\cap E_h} \left| \frac{\rho^2(y)\, b(y)}{\alpha_h} \right|^{n/2} dy$$
$$\leq \frac{\gamma^n}{\alpha_h^{n/2}} \int_{\Omega(x,\rho(x))} \left| b(y)\, \chi_{E_h} \right|^{n/2} dy,$$

where χ_{E_h} is the characteristic function of E_h. Thus we deduce

$$\sup_\Omega \frac{|\Omega(x,\rho(x)) \cap E_h|}{\rho^n(x)} \leq \frac{\gamma^n}{\alpha_h^{n/2}}\, \|b\|^{n/2}_{K^{n/2}_2(\Omega)} \leq \frac{1}{h}\,.$$

Then, from condition h_5) and (7), it follows that

$$\lim_{h\to+\infty} \|b\, \chi_{E_h}\|_{K^{n/2}_2(\Omega)} = 0\,. \tag{31}$$

Now for any $h \in \mathbb{N}$ we introduce the functions

$$b^h = b - b\, \chi_{E_h}\,,$$

and observe that

$$b^h \in L^\infty_2(\Omega) \qquad \forall\, h \in \mathbb{N}\,, \tag{32}$$

$$\lim_{h\to+\infty} \|b - b^h\|_{K_2^{n/2}(\Omega)} = 0\,. \tag{33}$$

First we prove a bound for the solution of the problem (28) assuming that the function g is more regular.

Lemma 4. *Suppose that conditions $h_1) - h_5)$ hold. Then the problem (28) is uniquely solvable. Moreover, if $g \in \mathfrak{D}^0(\bar\Omega \setminus S_\rho)$ then the solution u belongs to $L^p_{t-1}(\Omega)$ for all $p \in [2, +\infty[$ and verifies the bound*

$$\|u\|_{L^p_{t-1}(\Omega)} \leq c\, \|g\|_{L^p_{-t+1}(\Omega)}\,, \tag{34}$$

where $c \in \mathbb{R}_+$ depends on Ω, n, p, t, ρ, ν, $\|a^o_{ij}\|_{L^\infty(\Omega)}$ and b_o.

Proof. Fix $u,\, v \in \overset{\circ}{W}{}^{1,2}_t(\Omega)$. We observe that if $b \in K_2^{n/2}(\Omega)$, then $b^{1/2} \in K_1^n(\Omega)$ and $\|b^{1/2}\|^2_{K_1^n(\Omega)} = \|b\|_{K_2^{n/2}(\Omega)}$. Using Theorem 3.1 of [4], we can prove the bounds

$$a_t(u,u) \geq \nu \int_\Omega \sigma^{2t}\, u_x^2\, dx + b_o \int_\Omega \sigma^{2(t-1)}\, u^2\, dx \geq c_1\, \|u\|^2_{W^{1,2}_t(\Omega)}\,,$$

$$a_t(u,v) \leq c_2 \Big(\|u\|_{W^{1,2}_t(\Omega)}\, \|v\|_{W^{1,2}_t(\Omega)} + \|b^{1/2}\|^2_{K_1^n(\Omega)}\, \|u\|_{W^{1,2}_t(\Omega)}\, \|v\|_{W^{1,2}_t(\Omega)} \Big)\,,$$

where c_1, $c_2 \in \mathbb{R}_+$ are independent on u and v. Then from Lax-Milgram Theorem we deduce that the problem (28) is uniquely solvable.

Now we prove the bound (34). To this aim, for any fixed $h \in \mathbb{N}$, we put

$$a^h_t(u,v) = \int_\Omega \Big(\sum_{i,j=1}^n a^h_{ij}\, u_{x_i}\, v_{x_j} + b^h\, u\, v \Big)\, \sigma^{2t}\, dx\,, \tag{35}$$

with $u, v \in \overset{\circ}{W}{}^{1,2}_t(\Omega)$.

Fixed $u \in \overset{\circ}{W}{}^{1,2}_t(\Omega)$, from conditions $h_3)$ and $h_4)$ it follows that

$$a^h_t(u,u) \geq \nu \int_\Omega \sigma^{2t} u_x^2 dx + b_o \int_\Omega \sigma^{2(t-1)} u^2 dx - \int_\Omega b \chi_{E_h} \sigma^{2t} u dx\,. \tag{36}$$

Using Theorem 3.1 of [4], we obtain

$$\begin{aligned}\int_\Omega b\, \chi_{E_h}\, \sigma^{2t}\, u\, dx &\leq c_3\, \|b^{1/2}\, \chi_{E_h}\|^2_{K_1^n(\Omega)}\, \|u\|^2_{W^{1,2}_t(\Omega)} \\ &\leq c_3\, \|b\, \chi_{E_h}\|_{K_2^{n/2}(\Omega)}\, \|u\|^2_{W^{1,2}_t(\Omega)}\,,\end{aligned} \tag{37}$$

where $c_3 \in \mathbb{R}_+$ depends only on Ω, n, t and ρ. From (31) we deduce that there exists $h_o \in \mathbb{N}$ such that

$$\|b\, \chi_{E_h}\|_{K_2^{n/2}(\Omega)} \leq \frac{\min\{\nu,\, b_o\}}{2\, c_3} \qquad \forall\, h \geq h_o\,. \tag{38}$$

Thus from (36)-(38) we obtain

$$a_t^h(u,u) \geq c_4 \, \|u\|^2_{W_t^{1,2}(\Omega)} \qquad \forall\, h \geq h_o \,, \tag{39}$$

where $c_4 \in \mathbb{R}_+$ depends on Ω, n, t, ρ, ν, b_o.

Let $v \in V_t^p(\Omega)$. Arguing as in Lemma 4.1 of [5] we can find $k_o \in \mathbb{N}$, which depends on n, p, t, ν, $\|a_{ij}^h\|_{L^\infty(\Omega)}$, b_o, such that

$$\begin{aligned} & a_t^h(v, |\eta_{k_o}^{t-1} v|^{p-2} v) \geq (p-1)\frac{\nu}{2} \int_\Omega |\eta_{k_o}^{t-1} v|^{p-2} \, |\sigma^t \, v_x|^2 \, dx \\ & + \frac{b_o}{2} \int_\Omega |\eta_{k_o}^{t-1} v|^{p-2} \, |\sigma^{t-1} v|^2 \, dx - \int_\Omega b \, \chi_{E_h} \, v^2 \, \sigma^{2t} \, |\eta_{k_o}^{t-1} v|^{p-2} \, dx \,. \end{aligned} \tag{40}$$

Again from Theorem 3.1 of [4], we have

$$\begin{aligned} \Big| \int_\Omega b \, \chi_{E_h} \, v^2 \, \sigma^{2t} \, |\eta_{k_o}^{t-1} \, v|^{p-2} \, dx \Big| & \leq c_4 \, \big\| \sigma \, b^{1/2} \, \chi_{E_h} \, |\eta_{k_o}^{t-1} \, v|^{p/2} \big\|^2_{L^2(\Omega)} \\ & \leq c_5 \, \|b \, \chi_{E_h}\|_{K_2^{n/2}(\Omega)} \, \big\| |\eta_{k_o}^{t-1} \, v|^{p/2} \big\|^2_{W_1^{1,2}(\Omega)} \,, \end{aligned} \tag{41}$$

where $c_4 \in \mathbb{R}_+$ depends on k_o and $c_5 \in \mathbb{R}_+$ depends on k_o, Ω, ρ. Since $\big(\eta_{k_o}\big)_x = 0$ on Ω_{k_o} and $\frac{\sigma}{\eta_{k_o}} \leq 1$ on $\Omega \setminus \overline{\Omega}_{k_o}$, we obtain

$$\begin{aligned} & \big\| |\eta_{k_o}^{t-1} \, v|^{p/2} \big\|^2_{W_1^{1,2}(\Omega)} \leq c_6 \Big(\int_\Omega |\eta_{k_o}^{t-1} \, v|^p \, dx \\ & + \int_\Omega \big(|\eta_{k_o}^{t-1} \, v|^p \, (\eta_{k_o})_x^2 + |\eta_{k_o}^{t-1} \, v|^{p-2} \, |\eta_{k_o}^t \, v_x|^2 \big) \, dx \Big) \,, \end{aligned} \tag{42}$$

where $c_6 \in \mathbb{R}_+$ depends on n, p, t and ρ.

Relations (11), (41) and (42) imply

$$\begin{aligned} & \Big| \int_\Omega b \, \chi_{E_h} \, v^2 \, \sigma^{2t} \, |\eta_{k_o}^{t-1} \, v|^{p-2} \, dx \Big| \leq c_7 \, \|b \, \chi_{E_h}\|_{K_2^{n/2}(\Omega)} \\ & \times \Big(\int_\Omega |\eta_{k_o}^{t-1} \, v|^{p-2} \, |\sigma^{t-1} v|^2 \, dx + \int_\Omega |\eta_{k_o}^{t-1} \, v|^{p-2} \, |\sigma^t \, v_x|^2 \, dx \Big) \,, \end{aligned} \tag{43}$$

where $c_7 \in \mathbb{R}_+$ depends on the same parameters as c_5. On the other hand, from (31), there exists $h_1 \geq h_o$ such that

$$\|b \, \chi_{E_h}\|_{K_2^{n/2}(\Omega)} \leq \frac{\min\{(p-1)\nu, \, b_o\}}{4 \, c_7} \qquad \forall\, h \geq h_1 \,. \tag{44}$$

Finally from (40), (43) and (44), we obtain, for any $h \geq h_1$,

$$a_t^h(v, |\eta_{k_o}^{t-1} \, v|^{p-2} v) \geq c_8 \, \|v\|^p_{L^p_{t-1}(\Omega)} \,, \tag{45}$$

where $c_8 \in \mathbb{R}_+$ depends on Ω, n, p, t, ρ, ν, $\|a_{ij}^h\|_{L^\infty(\Omega)}$, b_o.

For $h \in \mathbb{N}$ and $t \in \mathbb{R}$, we consider now the differential operator

$$L_t^h = -\sum_{i,j=1}^n a_{ij}^h \frac{\partial^2}{\partial_{x_i}\,\partial_{x_j}} - \sum_{i=1}^n \Big(\sum_{j=1}^n \, (a_{ij}^h)_{x_j} + 2\,t\,a_{ij}\,\sigma^{-1}\,\sigma_{x_j} \Big)\Big) \frac{\partial}{\partial_{x_i}} + b^h .$$

We observe that $a_t^h(\cdot\,,\,\cdot)$ is the bilinear form associated to the operator $\sigma^{2t}\,L_t^h$. In fact, for $u, v \in \overset{\circ}{W}{}_t^{1,2}(\Omega)$, we have

$$a_t^h(u,v) = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}^h\, u_{x_i}\, v_{x_j} + b^h\, u\, v \Big)\, \sigma^{2t}\, dx\,, = \int_\Omega \sigma^{2t}\, L_t^h u\; v\, dx\,.$$

Now, for each $h \in \mathbb{N}$, consider the problem

$$\begin{cases} u \in W_{t+1}^{2,p}(\Omega) \cap \overset{\circ}{W}{}_t^{1,p}(\Omega)\,, \\ L_t^h\, u = f, \qquad f \in L_{t+1}^p(\Omega)\,. \end{cases} \tag{46}$$

From Lemma 3 we deduce that the problem (46) is uniquely solvable for any $h \in \mathbb{N}$. Moreover, if $f \in \mathfrak{D}^0(\bar\Omega \setminus S_\rho)$, the solution u_h of (46) has the property

$$u_h \in W_{\tau+1}^{2,q}(\Omega) \cap \overset{\circ}{W}{}_\tau^{1,q}(\Omega) \qquad \forall\, q \ge 2, \quad \forall\, \tau \in \mathbb{R}\,.$$

On the other hand, if we put $f = \sigma^{-2t}\, g$, the function u_h is also solution of the problem

$$a_t^h(u,v) = \int_\Omega g\, v\, dx \qquad\quad \forall\, v \in \overset{\circ}{W}{}_t^{1,2}(\Omega)\,. \tag{47}$$

So, if $g \in \mathfrak{D}^0(\bar\Omega \setminus S_\rho)$, from Lemma 2, the solution u_h of problem (47) belongs to $\overset{\circ}{V}{}_t^p(\Omega)$.

Using the same tools of Theorem 4.2 of [5], from (45) we deduce that for any $h \ge h_1$

$$\begin{aligned} &\|u_h\|^p_{L^p_{t-1}(\Omega)} \le c_9\; a_t^h(u_h, |\eta_{k_o}^{t-1}\, u_h|^{p-2} u_h) \\ &= c_9 \int_\Omega \sigma^{2t} L_t^h u_h\, |\eta_{k_o}^{t-1}\, u_h|^{p-2} u_h\, dx \le c_{10} \|\rho^{t+1}\, L_t^h u_h\|_{L^p(\Omega)}\, \|\rho^{t-1}\, u_h\|^{\frac{p-1}{p}}_{L^p(\Omega)}\,, \end{aligned}$$

and so

$$\|u_h\|_{L^p_{t-1}(\Omega)} \le c_{11}\, \|L_t^h\, u_h\|_{L^p_{t+1}(\Omega)} = c_{11}\, \|g\|_{L^p_{-t+1}(\Omega)}\,, \tag{48}$$

where $c_9 - c_{11} \in \mathbb{R}_+$ depend on the same parameters as c_8.

Moreover from (39) we have

$$\begin{aligned} &\|u_h\|^2_{W_t^{1,2}(\Omega)} \le c_{12}\, a_t^h(u_h, u_h) \\ &= c_{12} \int_\Omega g\, u_h\, dx \le c_{12}\, \|g\|_{L^2_{-t+1}(\Omega)}\, \|u_h\|_{W_t^{1,2}(\Omega)}\,, \end{aligned}$$

from which it follows that

$$\|u_h\|_{W_t^{1,2}(\Omega)} \leq c_{12}\,\|g\|_{L^2_{-t+1}(\Omega)}\,, \tag{49}$$

where $c_{12} \in \mathbb{R}_+$ depends on Ω, n, t, ρ, ν, b_o.

By (49), we deduce that there exists a subsequence $(u'_h)_{h\geq h_1}$ of $(u_h)_{h\geq h_1}$ converging weakly in $L^2_{t-1}(\Omega)$ to a function u'. On the other hand, from (48), we also deduce that there exists a subsequence $(u''_h)_{h\geq h_1}$ of $(u'_h)_{h\geq h_1}$ converging weakly in $L^p_{t-1}(\Omega)$ to a function u''. Since $C^\infty_o(\Omega)$ is dense in $L^p_{t-1}(\Omega)$ $\forall\, p \geq 2$, from the uniqueness of weak limit we deduce that $u' = u''$. Moreover from (48) we have

$$\|u'\|_{L^p_{t-1}(\Omega)} \leq \liminf_h \|u''_h\|_{L^p_{t+1}(\Omega)} \leq c_{11}\,\|g\|_{L^p_{-t+1}(\Omega)}\,. \tag{50}$$

Now we prove that the function u' is equal to the solution u of the problem (28). To this aim fix $w \in \overset{\circ}{W}{}^{1,2}_t(\Omega)$ and $v \in C^\infty_o(\Omega)$ and observe that from Theorem 3.1 of [4] we have

$$\begin{aligned}
&|a^h_t(w,v) - a_t(w,v)| \\
&= \Big| \int_\Omega \Big(\sum_{i,j=1}^n (a^h_{ij} - a_{ij})\,\sigma^{2t}\,w_{x_i}\,v_{x_j} + (b^h - b)\,\sigma^{2t}\,w\,v\Big)\,dx\Big| \\
&\leq \sum_{i,j=1}^n \int_\Omega |a^h_{ij} - a_{ij}|^2\,|\sigma^t\,v_{x_j}|^2\,dx \cdot \int_\Omega |\sigma^t\,w_{x_i}|^2\,dx \\
&+ c_{13}\,\|b^h - b\|^2_{K_2^{\frac{n}{2}}(\Omega)} \cdot \|w\|_{W_t^{1,2}(\Omega)} \cdot \|v\|_{W_t^{1,2}(\Omega)}\,,
\end{aligned} \tag{51}$$

where $c_{13} \in \mathbb{R}_+$ depends on Ω, n, t and ρ. Since a^h_{ij} converge to a_{ij} (for $h \to +\infty$) in $L^p_{\mathrm{loc}}(\Omega)$ for any $p \geq 2$, we deduce that

$$\sum_{i,j=1}^n \int_\Omega |a^h_{ij} - a_{ij}|^2\,|\sigma^t\,v_{x_j}|^2\,dx \to 0 \qquad \text{for} \quad h \to +\infty\,. \tag{52}$$

Therefore from (51), (52) and (33) we obtain $\forall\, w \in \overset{\circ}{W}{}^{1,2}_t(\Omega)$ and $\forall\, v \in C^\infty_o(\Omega)$

$$a^h_t(w,v) \to a_t(w,v) \qquad \text{for} \quad h \to +\infty\,. \tag{53}$$

Furthermore, from (53) we deduce that for any $v \in C^\infty_o(\Omega)$

$$\begin{aligned}
&|a^h_t(u''_h, v) - a_t(u', v)| \\
&\leq |a^h_t(u''_h, v) - a^h_t(u', v)| + |a^h_t(u', v) - a_t(u', v)| \to 0 \quad \text{for } h \to +\infty\,.
\end{aligned} \tag{54}$$

On the other hand

$$a^h_t(u''_h, v) = \int_\Omega g\,v\,dx \qquad \forall\, v \in \overset{\circ}{W}{}^{1,2}_t(\Omega)\,.$$

Then, by (54), we have

$$a_t(u', v) = \int_\Omega g\, v\, dx \qquad \forall\, v \in C_o^\infty(\Omega)\,. \tag{55}$$

Relation (55) and the uniqueness of solution of (28) complete the proof. □

Now we can prove our main result.

Theorem 5. *Suppose that conditions $h_1) - h_5)$ hold. If $g \in L^r_{-t+1}(\Omega)$ with $r \in]1, +\infty[$, then the solution u of problem (28) belongs to $L^r_{t-1}(\Omega)$ and verifies the bound*

$$\|u\|_{L^r_{t-1}(\Omega)} \le c\, \|g\|_{L^r_{-t+1}(\Omega)}\,, \tag{56}$$

where $c \in \mathbb{R}_+$ depends on Ω, n, r, t, ρ, ν, $\|a^o_{ij}\|_{L^\infty(\Omega)}$ and b_o.

Proof. Suppose first that $r \le 2$. Then $r' \ge 2$ (r' is the exponent conjugate of r). Fix $\varphi \in \mathfrak{D}^0(\bar\Omega \setminus S_\rho)$. From Lemma 4 we deduce that the problem

$$\begin{cases} w \in \overset{\circ}{W}{}^{1,2}_t(\Omega)\,, \\ a_t(w, v) = \displaystyle\int_\Omega \varphi\, v\, dx \qquad \forall\, v \in \overset{\circ}{W}{}^{1,2}_t(\Omega)\,, \end{cases} \tag{57}$$

is uniquely solvable and the solution w verifies the bound

$$\|w\|_{L^{r'}_{t-1}(\Omega)} \le c_1\, \|\varphi\|_{L^{r'}_{-t+1}(\Omega)}\,, \tag{58}$$

where $c_1 \in \mathbb{R}$ depends on Ω, n, r, t, ρ, ν, $\|a^o_{ij}\|_{L^\infty(\Omega)}$ and b_o. Since the solution u of problem (28) belongs to $\overset{\circ}{W}{}^{1,2}_t(\Omega)$, we obtain

$$\int_\Omega u\, \varphi\, dx = a_t(w, u) = a_t(u, w) = \int_\Omega g\, w\, dx\,,$$

thus, by (58),

$$\begin{aligned} &\Big|\int_\Omega u\, \varphi\, dx\Big| = \Big|\int_\Omega g\, w\, dx\Big| \le \|g\|_{L^r_{-t+1}(\Omega)} \cdot \|w\|_{L^{r'}_{t-1}(\Omega)} \\ &\le c_1\, \|g\|_{L^r_{-t+1}(\Omega)} \cdot \|\varphi\|_{L^{r'}_{-t+1}(\Omega)}\,. \end{aligned} \tag{59}$$

Since the bound (59) holds for any function $\varphi \in \mathfrak{D}^0(\bar\Omega \setminus S_\rho)$, we deduce, from Hahn-Banach Theorem, that there exists a linear and continuous functional

$$F : v \in L^{r'}_{-t+1}(\Omega) \to F(v) = \int_\Omega u\, v\, dx\,.$$

Finally, by Riesz Representation Theorem, we have

$$u \in L^r_{t-1}(\Omega) \qquad \|u\|_{L^r_{t-1}(\Omega)} \le c_1\, \|g\|_{L^r_{-t+1}(\Omega)}\,. \tag{60}$$

Suppose now that $r > 2$. The previous case can be used to prove that the solution w of problem (57) belongs to $L^{r'}_{t-1}(\Omega)$ and verifies the bound (58). Following again previous functional argument, we deduce the result also in this case. □

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