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ON THE SOLUTION OF n-DIMENSIONAL OPERATOR $\circledast^k$ AND THE FOURIER TRANSFORM OF THEIR CONVOLUTION

Wanchak Satsanit

Department of Mathematics
Faculty of Science
Maejo University
Nong Han, San Sai, Chiang Mai, 50290, THAILAND
e-mail: aunphue@live.com

Abstract: In this paper, we consider the solution of equation

$$\circledast^k u(x) = \sum_{r=0}^{m} c_r \circledast^r \delta$$

where $\circledast^k$ is the circledast operator iterated k times and is defined by

$$\circledast^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right)^k,$$

$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ – the n-dimensional Euclidian space, $p + q = n$, c_r is a constant, δ is the Dirac-delta distribution, $\circledast^0 \delta = \delta$ and $k = 0, 1, 2, 3, \cdots$. It was found that the type of the solution of this equation, such as the ordinary functions, the tempered distributions and the singular distributions, depend on the relationship between the values of k and m. After that we study the Fourier transform of the elementary solution and also the Fourier transform of their convolution.

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1. Introduction

The operator $\diamondsuit^k$ has been used first by A. Kananthai (see [2]). It is named as the Diamond operator iterated k times and is defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \quad p+q=n, \tag{1.1}$$

where n is the dimension of the Euclidean space $\mathbb{R}^n$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. The operator $\diamondsuit^k$ can be expressed in the form

$$\diamondsuit^k = \triangle^k \square^k = \square^k \triangle^k, \tag{1.2}$$

where $\triangle^k$ is the Laplacian operator iterated k-times defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \right)^k \tag{1.3}$$

and $\square^k$ is the ultra-hyperbolic operator iterated k-times defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \tag{1.4}$$

A. Kananthai (see [2]) has shown that the solution of the convolution form $u(x) = (-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is a unique elementary solution of $\diamondsuit^k$, where $R_{2k}^e(x)$ and $R_{2k}^H(x)$ are defined by (2.5) and (2.2) with $\alpha = 2k$ respectively, that is

$$\diamondsuit^k \left((-1)^k R_{2k}^e(x) * R_{2k}^H(x) \right) = \delta. \tag{1.5}$$

Next, W. Satsanit has been first introduced the $\circledast^k$ operator (see [5]) which is defined by

$$\begin{aligned} \circledast^k &= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k \\ &= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right) \cdot \right. \\ &\quad \left. \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \end{aligned}$$

$$= \triangle^k \left(\triangle^2 - \frac{1}{4}(\triangle + \Box)(\triangle - \Box)\right)^k$$
$$= \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^k, \qquad (1.6)$$

where $\Diamond$, $\triangle$ and $\Box$ are defined by (1.1), (1.3) and (1.4) respectively with $k = 1$.

Now, the purpose of this paper is to find the solution of the equation

$$\circledast^k u(x) = \sum_{r=0}^{m} c_r \circledast^r \delta. \qquad (1.7)$$

In finding the solution of (1.7), we use the method of convolutions of the generalized function.

Before going to that point, the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Let

$$\upsilon = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 \qquad (2.1)$$

be the nondegenerated quadratic form, where $n = p + q$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ is the interior of forward cone and $\overline{\Gamma}_+$ denote it closure. For any complex number α, define the function

$$R^H_\alpha(\upsilon) = \begin{cases} \frac{\upsilon^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \qquad (2.2)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}. \qquad (2.3)$$

The function $R^H_\alpha(\upsilon)$ is called the Ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki (see [4], p. 72).

It is well known that $R^H_\alpha(\upsilon)$ is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R^H_\alpha(\upsilon)$ denote the support of $R^H_\alpha(\upsilon)$ and suppose supp $R^H_\alpha(\upsilon) \subset \bar{\Gamma}_+$, that is supp $R^H_\alpha(\upsilon)$ is compact.

From S.E. Trione (see [6], p. 11), $R^H_{2k}(\upsilon)$ is an elementary solution of the operator $\Box^k$ that is

$$\Box^k R^H_{2k}(\upsilon) = \delta(x). \tag{2.4}$$

Definition 2.2. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and $|x| = \left(x_1^2 + x_2^2 + ... + x_n^2\right)^{\frac{1}{2}}$ the function $R^e_\alpha(x)$ denoted the elliptic kernel of Marcel Riesz and is defined by

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)}, \tag{2.5}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{2.6}$$

α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

It can be shown that $R^e_{-2k}(x) = (-1)^k \triangle^k \delta(x)$, where $\triangle^k$ is defined by (1.3). It follows that $R^e_0(x) = \delta(x)$ (see [1], p. 118).

Moreover, we obtain $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$ that is

$$\triangle^k((-1)^k R^e_{2k}(x) = \delta(x) \tag{2.7}$$

(see [2], Lemma 2.4, p. 31).

Definition 2.3. Let $f(x) \in L_1(\mathbb{R}^n)$ be the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi.x} f(x)dx, \tag{2.8}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \cdots dx_n$. Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi.x} \widehat{f(\xi)} d\xi. \tag{2.9}$$

If f is a distribution with compact supports by (see [7], Theorem 7.4-3, p. 187), equation (2.8) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i\xi.x} \right\rangle. \tag{2.10}$$

Lemma 2.1. *The functions $R^e_\alpha(x)$ and $R^H_\alpha(x)$, defined by (2.2) and (2.5), respectively, are homogeneous distribution of order $\alpha - n$ and also a tempered distribution.*

Proof. Since $R^H_\alpha(x)$ and $R^e_\alpha(x)$ satisfy the Euler equation, that is

$$(\alpha - n)R^H_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R^H_\alpha(x)$$

and

$$(\alpha - n)R^e_\alpha(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} R^e_\alpha(x).$$

We have $R^H_\alpha(x)$ and $R^e_\alpha(x)$ are homogeneous distributions of order $\alpha - n$ and Donoghue (see [1], pp. 154-155) has proved the every homogeneous distribution is a tempered distribution. This completes the proof. □

Lemma 2.2. (The Convolution of Tempered Distribution) *The convolution $R^e_\alpha(x) * R^H_\alpha(x)$ exists and is a tempered distribution.*

Proof. Choose supp $R^H_\alpha(x) = K \subset \Gamma_+$ where K is a compact set. The $R^H_\alpha(x)$ is a tempered distribution with compact support and by Donoghue (see [1], pp. 156-159). $R^e_\alpha(x) * R^H_\alpha(x)$ exists and is a tempered distribution. □

Lemma 2.3. (The Convolution of $R^H_\alpha(x)$ and $R^e_\alpha(x)$) *Let $R^e_\alpha(x)$ and $R^H_\alpha(x)$ defined by (2.5) and (2.2) respectively, then we obtain the following formulas:*

(1) $R^e_\alpha(x) * R^e_\alpha(x) = R^e_{\alpha+\beta}(x)$, where α and β are complex parameters;

(2) $R^H_\alpha(x) * R^H_\alpha(x) = R^H_{\alpha+\beta}(x)$, for α and β are both integers and except only the case both α and β are both integers.

Proof. For the proof of the first formula see [1], p. 158.

The proof of the second formula, for the case α and β are both even integers is given in [3].

Let us consider the case α is odd and β is even, or α is even and β is odd. From Trione (see [7]):

$$\Box^k R^H_\alpha(x) = R^H_{\alpha-2k}(x) \tag{2.11}$$

and

$$\Box^k R^H_{2k}(x) = \delta(x), \quad k = 0, 1, 2, 3, \ldots, \tag{2.12}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated k-times defined by

$$\Box^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k .$$

Now let m be an odd integer, we have

$$\square^k R_m^H(x) = R_{m-2k}^H(x)$$

and

$$R_{2k}^H(x) * \square^k R_m^H(x) = R_{2k}^H(x) * R_{m-2k}^H(x),$$

or

$$\left(\square^k R_{2k}^H(x)\right) * R_m^H(x) = R_{2k}^H(x) * R_{m-2k}^H(x),$$

$$\delta * R_m^H(x) = R_{2k}^H(x) * R_{m-2k}^H(x).$$

Thus

$$R_m^H(x) = R_{2k}^H(x) * R_{m-2k}^H(x).$$

Since m is odd, hence $m - 2k$ is odd and $2k$ is a positive even. Put $\alpha = 2k,\ \beta = m - 2k$, we obtain

$$R_\alpha^H(x) * R_\beta^H(x) = R_{\alpha+\beta}^H(x)$$

for α is a nonnegative even and β is odd.

For the case α is a negative even and β is odd, by (2.8) we have

$$\square^k R_0^H(x) = R_{-2k}^H(x),$$

or

$$\square^k \delta = R_{-2k}^H(x),$$

where $R_0^H(x) = \delta$. Now for m is odd,

$$R_{-2k}^H(x) * \square^k R_m^H(x) = R_{-2k}^H(x) * R_{m-2k}^H(x),$$

or

$$\left(\square^k \delta\right) * \square^k R_m^H(x) = R_{-2k}^H(x) * R_{m-2k}^H(x)$$

$$\delta * \square^{2k} R_m^H(x) = R_{-2k}^H(x) * R_{m-2k}^H(x).$$

Thus

$$R_{m-2(2k)}^H(x) = R_{-2k}^H(x) * R_{m-2k}^H(x).$$

Put $\alpha = -2k$ and $\beta = m - 2k$, now α is a negative even and β is odd. Then we obtain

$$R_\alpha^H(x) * R_\beta^H(x) = R_{\alpha+\beta}^H(x).$$

This completes the proof. $\square$

Lemma 2.4. *(The Fourier Transform of $\circledast^k \delta$)*

$$\mathcal{F} \circledast^k \delta = \frac{(-1)^{3k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3\right]^k,$$

where $\mathcal{F}$ is the Fourier transform defined by (2.8) and if the norm of ξ is given by $\|\xi\| = \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)^{1/2}$, then

$$|\mathcal{F} \circledast^k \delta| \leq \frac{M}{(2\pi)^{n/2}} \|\xi\|^{6k},$$

where M is a constant, that is $\mathcal{F}\circledast^k$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by equation (2.9)

$$\circledast^k \delta = \mathcal{F}^{-1} \frac{(-1)^{3k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^{3k}\right].$$

Proof. By equation (2.10)

$$\begin{aligned}
\mathcal{F} \circledast^k \delta &= \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^k \delta, e^{-i\xi,x} \right\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1} \delta, \circledast e^{-i\xi,x} \right\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1} \delta, \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right) e^{-i\xi,x} \right\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1} \delta, \frac{3}{4}\Diamond\Box e^{-i(\xi,x)} \right\rangle \\
&\quad + \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1} \delta, \frac{1}{4}\triangle^3 e^{-i\xi,x} \right\rangle. \qquad (2.13)
\end{aligned}$$

Consider

$$\begin{aligned}
&\left\langle \circledast^{k-1} \delta, \frac{3}{4}\Diamond\Box e^{-i\xi,x} \right\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1} \delta, \frac{3}{4}(-1)^3 \left[\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right]\right. \\
&\qquad \left.\left[\left(\sum_{i=1}^{p} \xi_i^2\right) - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)\right] e^{-i\xi,x} \right\rangle \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1} \delta, \frac{1}{4}\triangle^3 e^{-i\xi,x} \right\rangle \\
&= \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1} \delta, \frac{1}{4}(-1)^3 \left[\left(\sum_{i=1}^{n} \xi_i^2\right)\right]^3 e^{-i(\xi,x)} \right\rangle. \qquad (2.15)
\end{aligned}$$

By (2.14) and (2.15), the equation (2.13) becomes

$$\mathcal{F} \circledast^k \delta$$
$$= \frac{1}{(2\pi)^{n/2}} \left\langle \circledast^{k-1}\delta, (-1)^3 \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^3 \right) e^{-i(\xi,x)} \right\rangle. \quad (2.16)$$

By keeping on operator $\circledast$ with $k-1$ times, we obtain

$$\mathcal{F} \circledast^k \delta = \frac{(-1)^{3k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3 \right].$$

Since

$$\begin{aligned}\left(\xi_1^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \ldots + \xi_{p+q}^2\right)^3 &= \left(\xi_1^2 + \ldots + \xi_p^2 + \xi_{p+1}^2 + \ldots + \xi_{p+q}^2\right). \\ \times \Big(\left(\xi_1^2 + \ldots + \xi_p^2\right)^2 &- \left(\xi_1^2 + \ldots + \xi_p^2\right). \left(\xi_{p+1}^2 + \ldots + \xi_{p+q}^2\right) \\ &+ \left(\xi_{p+1}^2 + \ldots + \xi_{p+q}^2\right)^2 \Big).\end{aligned}$$

That is,

$$\begin{aligned}|\mathcal{F} \circledast^k \delta| &= \frac{1}{(2\pi)^{n/2}} \left| \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3 \right| \\ &\leq \frac{M}{(2\pi)^{n/2}} \left|\xi_1^2 + \ldots + \xi_n^2\right| \\ &\quad \times \left| \left(\xi_1^2 + \ldots + \xi_n^2\right)^2 + \left(\xi_1^2 + \ldots + \xi_n^2\right)^2 + \left(\xi_1^2 + \ldots + \xi_n^2\right)^2 \right| \\ &\leq \frac{M}{(2\pi)^{n/2}} \|\xi\|^6,\end{aligned}$$

where M is constant and $\|\xi\| = \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)^{1/2}$, $\xi_i (i = 1, 2, \ldots, n) \in \mathbb{R}$. Hence we obtain $\mathcal{F} \circledast^k \delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (2.9)

$$\circledast^k \delta = \mathcal{F}^{-1} \frac{(-1)^{3k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3 \right]^k.$$

This completes the proof. □

3. Main Results

Theorem 3.1. *Given the equation*

$$\circledast^k G(x) = \delta(x), \quad (3.1)$$

then

$$G(x) = \left(R^H_{4k}(x) * (-1)^{3k} R^e_{6k}(x)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{3.2}$$

is a Green function for the operator $\circledast^k$ *iterated k-times, where* $\circledast$ *is defined by (1.6), and*

$$S(x) = \frac{3}{4}(-1)^2 R^e_4(x) + \frac{1}{4} R^H_4(x). \tag{3.3}$$

$S^{*k}(x)$ *denotes the convolution of S itself k-times,* $\left(S^{*k}(x)\right)^{*-1}$ *denotes the inverse of* $S^{*k}(x)$ *in the convolution algebra. Moreover* $G(x)$ *is a tempered distribution.*

Proof. From (3.1), we have

$$\circledast^k G(x) = \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^k G(x) = \delta(x),$$

or we can write

$$\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right) * \left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x) = \delta(x).$$

Convolving both sides of the above equation by $R^H_4(x) * (-1)^3 R^e_6(x)$,

$$\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)\left(R^H_4(x) * (-1)^3 R^e_6(x)\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x)$$
$$= \delta(x) * R^H_4(x) * (-1)^3 R^e_6(x).$$

By properties of convolution

$$\left(\frac{3}{4}\triangle\left((-1)R^e_2(x)\right) * \Box^2\left(R^H_4(x)\right) * (-1)^2 R^e_4(x) + \frac{1}{4}\triangle^3(-1)^3 R^e_6(x) * R^H_4(x)\right)$$
$$\times\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x) = \delta(x) * R^H_4(x) * (-1)^3 R^e_6(x).$$

By (2.4) and (2.7), we obtain

$$\left(\frac{3}{4}\delta * \delta * (-1)^2 R^e_4(x) + \frac{1}{4}\delta * R^H_4(x)\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x)$$
$$= \delta(x) * R^H_4(x) * (-1)^3 R^e_6(x),$$

$$\left(\frac{3}{4}(-1)^2 R^e_4(x) + \frac{1}{4}R^H_4(x)\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x) = R^H_4(x) * (-1)^3 R^e_6(x).$$

Thus

$$\left(\frac{3}{4}(-1)^2 R_4^e(x) + \frac{1}{4} R_4^H(x)\right)\left(\frac{3}{4}\Diamond\Box + \frac{1}{4}\triangle^3\right)^{k-1} G(x)$$
$$= R_4^H(x) * (-1)^3 R_6^e(x). \quad (3.4)$$

Keeping on convolving both sides of the above equation by $R_4^H(x) * (-1)^3 R_6^e(x)$ up to $k-1$ times, we obtain

$$S^{*k}(x) * G(x) = \left((-1)^3 R_6^e(x) * R_4^H(x)\right)^{*k} \quad (3.5)$$

the symbol $*k$ denotes the convolution of itself $k-$times. By properties of $R_\alpha(u)$, we have

$$\left(R_4^H(x) * (-1)^3 R_6^e(x)\right)^{*k} = R_{4k}^H(x) * (-1)^{3k} R_{6k}^e(x).$$

Thus,

$$S^{*k}(x) * G(x) = R_{4k}^H(x) * (-1)^{3k} R_{6k}^e(x). \quad (3.6)$$

Now, consider the function $S^{*k}(x)$. Since $(-1)^3 R_6^e(x) * R_{4k}^H(x)$ is a tempered distribution. Thus $S(x)$ defined by (3.3) is a tempered distribution, we obtain $S^{*k}(x)$ is a tempered distribution. Since $(-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x) \in \mathcal{S}'$, the space of tempered distribution. Choose $\mathcal{S}' \subset \mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution. Thus $(-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x) \in \mathcal{D}'_{\mathcal{R}}$. It follows that $(-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x)$ is an element of convolution algebra. Since $\mathcal{D}'_{\mathcal{R}}$ is a convolution algebra. Hence the method of Zemanian (see [8]), the equation (3.6) has a unique solution

$$G(x) = \left((-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x)\right) * \left(S^{*k}(x)\right)^{*-1}, \quad (3.7)$$

where $\left(S^{*k}(x)\right)^{*-1}$ is an inverse of S^{*k} in the convolution algebra, $G(x)$ is called the Green function of the operator $\circledast^k$. Since $(-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x)$ and $\left(S^{*k}(x)\right)^{*-1}$ are tempered distribution, then by Donoghue (see [1], p. 152) $\left((-1)^{2k} R_{6k}^e(x) * R_{4k}^H(x) *\right) * \left(S^{*k}(x)\right)^{*-1}$ is a tempered distribution. It follows that $G(x)$ is a tempered distribution. □

Theorem 3.2. *For* $0 < r < k$

$$\circledast^r \left(((-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x)) * (S^{*k}(x))^{*-1}\right)$$
$$= \left(((-1)^{3(k-r)} R_{6(k-r)}^e(x) * R_{4(k-r)}^H(x)) * (S^{*(k-r)}(x))^{*-1}\right) \quad (3.8)$$

and for $k \leq m$

$$\circledast^m \left(((-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x)) * (S^{*k}(x))^{*-1}\right) = \circledast^{m-k}\delta. \quad (3.9)$$

Proof. For $0 < r < k$, from Theorem 3.1,

$$\circledast^k \left(((-1)^{3k} R_{6k}^e(x) * R_{4k}^H(x)) * (S^{*k}(x))^{*-1}\right) = \delta.$$

Thus,

$$\circledast^{k-r} \circledast^{r} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right) = \delta,$$

or

$$\circledast^{k-r} \delta * \circledast^{r} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right) = \delta.$$

Convolving both sides by

$$\left(((-1)^{3(k-r)} R^{e}_{6(k-r)}(x) * R^{H}_{4(k-r)}(x)) * (S^{*k}(x))^{*-1}\right),$$

we obtain

$$\begin{aligned} &\circledast^{k-r} \left(((-1)^{3(k-r)} R^{e}_{6(k-r)}(x) * R^{H}_{4(k-r)}(x)) * (S^{*k}(x))^{*-1}\right) \\ &\qquad * \circledast^{r} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right) \\ &\qquad = \left(((-1)^{3(k-r)} R^{e}_{6(k-r)}(x) * R^{H}_{4(k-r)}(x)) * (S^{*(k-r)}(x))^{*-1}\right) * \delta. \end{aligned}$$

By Theorem 3.1,

$$\begin{aligned} &\delta * \circledast^{r} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right) \\ &\qquad = \left(((-1)^{3(k-r)} R^{e}_{6(k-r)}(x) * R^{H}_{4(k-r)}(x)) * (S^{*(k-r)}(x))^{*-1}\right) * \delta. \end{aligned}$$

It follows that

$$\begin{aligned} &\circledast^{r} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right) \\ &\qquad = \left(((-1)^{3(k-r)} R^{e}_{6(k-r)}(x) * R^{H}_{4(k-r)}(x)) * (S^{*(k-r)}(x))^{*-1}\right) \end{aligned}$$

as required. For $k \leq m$

$$\begin{aligned} &\circledast^{m} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right) \\ &\qquad = \circledast^{m-k} \circledast^{k} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right). \end{aligned}$$

It follows that

$$\circledast^{m} \left(((-1)^{3k} R^{e}_{6k}(x) * R^{H}_{4k}(x)) * (S^{*k}(x))^{*-1}\right) = \circledast^{m-k} \delta$$

by Theorem 3.1. This completes the proof. □

Theorem 3.3. *Given the linear differential equation*

$$\circledast^{k} u(x) = \sum_{r=0}^{m} c_r \circledast^{r} \delta, \tag{3.10}$$

where the operator $\circledast^k$ *is defined by*

$$\otimes^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right)^k,$$

$p+q=n$, *n is an odd with p odd and q even, or n is an even with p odd and q odd, $x \in R^n$ – the n-dimensional Euclidian space, c_r is a constant, δ is the Dirac-delta distribution and $\otimes^0\delta = \delta$. Then the types of solution (3.1) that depend on the relationship between the values of k and m are as the following cases:*

(1) If $m < k$and $m = 0$, then (3.10) has the solution

$$u(x) = c_0\left(((-1)^{3k}R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{*-1}\right)$$

which is an elementary solution of the $\circledast^k$ operator in Theorem 3.1, is the ordinary function for $6k \geq n$ and $4k \geq n$, and is a tempered distribution for $6k < n$ and $4k < n$.

(2) If $0 < m < k$ then the solution of (3.3) is

$$u(x) = \sum_{r=1}^{m} c_r\left(((-1)^{3(k-r)}R^e_{6(k-r)}(x) * R^H_{4(k-r)}(x)) * (S^{*(k-r)}(x))^{*-1}\right)$$

which is an ordinary function for $6k-6r \geq n$ and $4k-4r \geq n$ and is tempered distribution for $6k-6r < n$ and $4k-4r < n$

(3) If $m \geq k$ and suppose $k \leq m \leq M$, then (3.3) has the solution

$$u(x) = \sum_{r=k}^{M} c_r \circledast^{r-k}\delta$$

which is only the singular distribution.

Proof. (1) For $m = 0$, we have $\circledast^k u(x) = c_0\delta$, and by Theorem 3.1 we obtain

$$u(x) = \left(((-1)^{3k}R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{*-1}\right).$$

Now, $(-1)^{3k}R^e_{6k}(x)$ and $R^H_{4k}(x)$ are the analytic function for $6k \geq n$ and $4k \geq n$ and also $(-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)*(S^{*k}(x))^{-1}$ exists and is an analytic function by (2.7). It follows that $(-1)^{3k}R^e_{6k}(x) * R^H_{4k}(x) * (S^{*k}(x))^{-1}$ is an ordinary function for $6k \geq n$ and $4k \geq n$. By Lemma 2.1 with $\alpha = 6k$, $(-1)^{3k}R^e_{6k}(x)$ and with $\alpha = 4k$, $R^H_{4k}(x)$ are tempered distribution with $6k < n$ and $4k < n$. We obtain $(-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)*(S^{*k}(x))^{-1}$ exists and is a tempered distribution.

(2) For the case $0 < m < k$, we have

$$\circledast^k u(x) = c_1 \circledast \delta + c_2 \circledast^2 \delta + \ldots + c_m \circledast^m \delta.$$

Convolved both sides by $(-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x) * (S^{*k}(x))^{-1}$, we obtain

$$\begin{aligned}
\circledast^k & \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{-1} \right) * u(x) \\
&= c_1 \circledast \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{-1} \right) \\
&\quad + c_2 \circledast^2 \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{-1} \right) \\
&\quad + \ldots + c_m \circledast^m \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{-1} \right).
\end{aligned}$$

By Theorem 3.1 and Theorem 3.2, we obtain

$$\begin{aligned}
u(x) &= c_1 \left(((-1)^{3(k-1)} R^e_{6(k-1)}(x) * R^H_{4(k-1)}(x)) * (S^{*(k-1)}(x))^{*-1} \right) \\
&\quad + c_2 \left(((-1)^{3(k-2)} R^e_{6(k-2)}(x) * R^H_{4(k-2)}(x)) * (S^{*(k-2)}(x))^{*-1} \right) \\
&\quad + \ldots + c_m \left(((-1)^{3(k-m)} R^e_{3(k-m)}(x) * R^H_{4(k-m)}(x)) * (S^{*(k-m)}(x))^{*-1} \right),
\end{aligned}$$

or

$$u(x) = \sum_{r=1}^{m} c_r \left(((-1)^{3(k-r)} R^e_{6(k-r)}(x) * R^H_{4(k-r)}(x)) * (S^{*(k-r)}(x))^{*-1} \right).$$

Similarly, as in the case (1), $u(x)$ is an ordinary function for $6k - 6r \geq n$ and $4k - 4r \geq n$ and is a tempered distribution for $6k - 6r < n$ and $4k - 4r < n$.

(3) For the case $m \geq k$ and suppose $k \leq m \leq M$, we have

$$\circledast^k u(x) = c_k \circledast^k \delta + c_{k+1} \circledast^{k+1} \delta + \ldots + c_M \circledast^M \delta.$$

Convolved both sides by $(-1)^{3k} R_{6k}(x) * R^H_{4k}(x) * (S^{*k}(x))^{*-1}$, we obtain

$$\begin{aligned}
\circledast^k & \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{*-1} \right) * u(x) \\
&= c_k \circledast^k \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{-1} \right) \\
&\quad + c_{k+1} \circledast^{k+1} \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{*-1} \right) \\
&\quad + \ldots + c_M \circledast^M \left(((-1)^{3k} R^e_{6k}(x) * R^H_{4k}(x)) * (S^{*k}(x))^{*-1} \right).
\end{aligned}$$

By Theorem 3.1 and Theorem 3.2 again, we obtain

$$\begin{aligned}
u(x) &= c_k \delta + c_{k+1} \circledast \delta + c_{k+2} \circledast^2 \delta + \ldots + c_M \circledast^{M-k} \delta \\
&= \sum_{r=k}^{M} c_r \circledast^{r-k} \delta.
\end{aligned}$$

Since $\circledast^{r-k}\delta$ is a singular distribution, hence $u(x)$ is only the singular distribution.

This completes the proofs. □

Theorem 3.4.

$$\mathcal{F}\left(\left((-1)^{3k}R^{e}_{6k}(x)*R^{H}_{4k}(x)\right)*(S^{*k}(x))^{*-1}\right)$$
$$=\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^k$$

and

$$|\mathcal{F}\left(\left((-1)^{3k}R^{e}_{6k}(x)*R^{H}_{6k}(x)\right)*(S^{*k}(x))^{*-1}\right)|\leq\frac{1}{(2\pi)^{\frac{n}{2}}}M$$
$$\text{for a large } \xi_i\in R, \quad (3.11)$$

where M is a constant. That is $\mathcal{F}$ is bounded and continuous on the space S' of the tempered distributions.

Proof. By Theorem 3.1, we have

$$\circledast^{k}(\left((-1)^{3k}R^{e}_{6k}(x)*\left(R^{H}_{4k}(x)\right)*(S^{*k}(x))^{*-1}\right)=\delta(x),$$

or

$$\left(\circledast^{k}\delta\right)*(\left(\left((-1)^{3k}R^{e}_{6k}(x)*R^{H}_{6k}(x)\right)*(S^{*k}(x))^{*-1}\right)=\delta(x).$$

Taking the Fourier transform on both sides of the above equation, we obtain

$$\mathcal{F}(\left(\circledast^{k}\delta\right)*[\left((-1)^{3k}R^{e}_{6k}(x)*R^{H}_{4k}(x)\right)*(S^{*k}(x))^{*-1}])=\mathcal{F}\delta=\frac{1}{(2\pi)^{n/2}}.$$

By (2.9)

$$\frac{1}{(2\pi)^{n/2}}\langle(\left(\circledast^{k}\delta\right)*[\left((-1)^{3k}R^{e}_{6k}(x)*R^{H}_{4k}(x)\right)*(S^{*k}(x))^{*-1}]),e^{-i(\xi.x)}\rangle=\frac{1}{(2\pi)^{n/2}}.$$

By the definition of convolution

$$\frac{1}{(2\pi)^{n/2}}\langle(\left(\circledast^{k}\delta\right)*[\left((-1)^{3k}R^{e}_{6k}(x)*R^{H}_{4k}(x)\right)*(S^{*k}(x))^{*-1}]),e^{-i\xi.(x+r)}\rangle$$
$$=\frac{1}{(2\pi)^{n/2}},$$

$$\frac{1}{(2\pi)^{n/2}}\langle[\left((-1)^{3k}R^{e}_{6k}(x)*R^{H}_{4k}(x)\right)*(S^{*k}(x))^{*-1}]),e^{-i(\xi.r)}\rangle\langle(\left(\circledast^{k}\delta\right),e^{-i\xi.x}\rangle$$
$$=\frac{1}{(2\pi)^{n/2}}.$$

$$\mathcal{F}([\Big((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x)\Big)*(S^{*k}(x))^{*-1}])\,(2\pi)^{\frac{n}{2}}\,\mathcal{F}\left(\circledast^k\delta\right)=\frac{1}{(2\pi)^{n/2}}.$$

$$\mathcal{F}([\Big((-1)^{3k}R^e_{6k}(x)*R^H_{6k}(x)\Big)*(S^{*k}(x))^{*-1}]).$$
$$\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^k=\frac{1}{(2\pi)^{n/2}}.$$

It follows that

$$\mathcal{F}([\Big((-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)\Big)*(S^{*k}(x))^{*-1}])$$
$$=\frac{1}{((-1)^{3k})\,(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^k}.$$

Since

$$\frac{1}{\left(\left(\xi_1^2+\ldots+\xi_p^2\right)^3+\left(\xi_{p+1}^2+\ldots+\xi_{p+q}^2\right)^3\right)^k}$$
$$=\frac{1}{\left(\xi_1^2+\ldots+\xi_p^2+\xi_{p+1}^2+\ldots+\xi_{p+q}^2\right)^k}\times$$
$$\frac{1}{\left(\left(\xi_1^2+\ldots+\xi_p^2\right)^2-\left(\xi_1^2+\ldots+\xi_p^2\right).\left(\xi_{p+1}^2+\ldots|!+\xi_{p+q}^2\right)+\left(\xi_{p+1}^2+\ldots+\xi_{p+q}^2\right)^2\right)^k} \tag{3.12}$$

Let $\xi=(\xi_1,\xi_2,\ldots,\xi_n)\in\Gamma_+$ with Γ_+ defined by Definition 2.3. Then $(\xi_1^2+\ldots+\xi_p^2+\xi_{p+1}^2+\ldots+\xi_{p+q}^2)>0$ and for a large k, the right-hand side of (3.12) tend to zero. It follows that it is bounded by a positive constant M say, that is we obtain (3.12) as required and also by (3.11) $\mathcal{F}$ is continuous on the space S' of the tempered distribution.

Theorem 3.5.

$$\mathcal{F}([\Big((-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)\Big)*(S^{*k}(x))^{*-1}]$$
$$*[\big((-1)^{3m}R^e_{6m}(x)*R^H_{4m}(x)\big)*(S^{*m}(x))^{*-1}])$$
$$=(2\pi)^{n/2}\,\mathcal{F}\left([\Big((-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)\Big)*(S^{*k}(x))^{*-1}\Big])$$
$$\mathcal{F}([\big((-1)^{3m}R^e_{6m}(x)*R^H_{4m}(x)\big)*(S^{*m}(x))^{*-1}])$$

$$= \frac{1}{(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^{k+m}},$$

where k and m are nonnegative integer and $\mathcal{F}$ is bounded and continuous on the space S' of the tempered distribution.

Proof. Since $R^H_{6k}(x)$ and $R^e_{4k}(x)$ are tempered distribution with compact support, we have

$$\begin{aligned}
&\left([\left((-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)\right)*(S^{*k}(x))^{*-1}\right])\\
&\quad *([((-1)^{3m}R^e_{6m}(x)*R^H_{4m}(x)*)*(S^{*m}(x))^{*-1}])\\
&=\left((-1)^{3m}R^e_{6k}(x)(-1)^{3k}R^e_{6k}(x)*R^H_{4m}(x)\right)*\left((-1)^{3(k+m)}R^e_{6k}(x)*R^e_{6m}(x)\right)\\
&\quad *\left(\left(S^{*k}(x)\right)^{*-1}*((S^{*m}(x))^{*-1}\right)\\
&=\left(R^H_{4(k+m)}(x)*(-1)^{k+m}R^e_{6(k+m)}(x)\right)*\left(S^{*(k+m)}(x)\right)^{*-1}
\end{aligned}$$

by [8], pp. 156-159 and [4], Lemma 2.5. Taking the Fourier transform on both sides and using Theorem 3.4, we obtain

$$\begin{aligned}
&\mathcal{F}\left[\left((-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)\right)*(S^{*k}(x)^{*-1}\right])\\
&\quad *([((-1)^{3m}R^e_{6m}(x)*R^H_{4m}(x))*(S^{*m}(x))^{*-1}])\\
&=\frac{1}{(-1)^{3(k+m)}(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^{k+m}}\\
&=\frac{1}{(-1)^{3k}(2\pi)^{n/2}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^{k}}\\
&\quad\times\frac{(2\pi)^{n/2}}{(-1)^{3m}(2\pi)^{n/2}\left[\left(\xi_1^2+\ldots+\xi_p^2\right)^3-\left(\xi_{p+1}^2+\ldots+\xi_{p+q}^2\right)^3\right]^{m}}\\
&=(2\pi)^{n/2}\mathcal{F}\left([\left((-1)^{3k}R^e_{6k}(x)*R^H_{4k}(x)\right)*(S^{*k}(x))^{*-1}\right])\\
&\quad\mathcal{F}([((-1)^{3m}R^e_{6m}(x)*R^H_{4m}(x))*(S^{*m}(x))^{*-1}]).
\end{aligned}$$

Since $\left((-1)^{3(k+m)}R^e_{6(k+m)}(x)*R^H_{4(k+m)}(x)\right)*\left(S^{*(k+m)}(x)\right)^{*-1}\in S'$, the space of tempered distribution and by theorem 3.4, we obtain that $\mathcal{F}$ is bounded and continuous on S'. □

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