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CONFORMAL MAPPING FROM A RECTANGULAR TO A CIRCULAR REGION, USING SPECIAL FUNCTIONS

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Abstract: Specification of the original and mapped boundary configurations does not determine a unique conformal mapping function, resulting in infinitely many possibilities. Examples of conformal mapping of a rectangular region to a circular region except multiplication and shift, using special functions, are given.

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1. Introduction

Conformal mapping is very useful if given suitably (see [3]), although specification of the original and mapped boundary configurations does not determine a unique conformal mapping function, resulting in infinitely many possibilities. Treated is an example of a conformal mapping of a rectangular region to a circular region.

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2. Analysis

2.1. General

Consider a mapping of the complex rectangular region z to a complex unit circular region w ($|w| < 1$) given by

$$w = \frac{1 + if(z)}{1 - if(z)}, \tag{1}$$

where f is a univalent single-valued analytic function such that along the circumference of the rectangle f is real and varies from $-\infty$ to $+\infty$ monotonously in the counterclockwise direction, and consequently inside the region $f'(z) \neq 0$, and $\Im\{f(z)\} > 0$. Without loss of generality, it is assumed that the location of the vertices is given by $z = \pm\pi/2$ and $z = \pm\pi/2 + Hi, H > 0$. If there exists such a conformal mapping from a rectangular region z to a circular region w given by equation (1), then the region w^* produced from the said region z by the mapping

$$w^* = \frac{1 + i\{af(z) + b\}}{1 - i\{af(z) + b\}} \tag{2}$$

becomes clearly a unit circular region $|w^*| < 1$ if a and b are real and $a > 0$. Therefore at least we have non-uniqueness factors (a: multiplication factor, b: shift factor). In the following treated are substantially different cases exclusive of such factors. Among many possibilities except multiplication and shift, four cases will be discussed.

2.2. In Case of Construction (I), using Jacobian Elliptic sn Function

Consider a mapping from the said rectangular region z to h (semi-infinite half plane, in case of a closed region $\Im(h) \geq 0$) specified by the Schwartz-Christoffel mapping

$$h = \mathrm{sn}\{2K(k)z/\pi, k\}, \tag{3}$$

$$H = \frac{\pi K(\sqrt{1-k^2})}{2K(k)}, \tag{4}$$

and a mapping from the semi-infinite half complex plane h ($\Im(h) > 0$) to a unit circle w ($|w| < 1$)

$$w = \frac{1 + ih}{1 - ih}, \tag{5}$$

where sn, k, $K()$ stand for a Jacobian elliptic function (see [2]), modulus $0 < k < 1$, a complete elliptic integral of the first kind respectively. Then the successive mapping (3) and (5) leads to the said mapping. Thus we have

$$f = \mathrm{sn}\left\{2K(k)z/\pi, k\right\}. \tag{6}$$

Given H, k is determined uniquely through equation (4). Clearly this f satisfies $f' \neq 0$ inside the region, and along the circumference f is real and varies monotonously from $-\infty$ to $+\infty$.

2.3. In Case of Construction (II), using Jacobian Elliptic sn Function

Let f be defined as

$$f = \mathrm{sn}^2\left\{\frac{K(k)}{\pi}\left(z+\frac{\pi}{2}\right), k\right\} - 1, \tag{7}$$

$$H = \frac{\pi K(\sqrt{1-k^2})}{K(k)} \qquad (0 < k < 1). \tag{8}$$

According to the nature of sn function, this function f clearly satisfies the requirement for conformal mapping.

2.4. In Case of Construction, Using Mathieu Functions

Consider a case

$$f = \frac{\sum_{n=0}^{\infty} P_{2n}\mathrm{ce}_{2n}(z,q)}{\mathrm{se}_1(z,q)} \qquad (0 < q), \tag{9}$$

where $\mathrm{ce}_n(z,q)$ is a cosine-elliptic Mathieu function of order n ($n \geq 0$) such that $\frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{ce}_n^2(z,q)dz = 1$, and se_1 is a sine-elliptic Mathieu function of order 1. The function f is so chosen that the pole of order 1 in the closed region is $z = 0$. Functions $\mathrm{se}_1(z,q)$ and $\mathrm{ce}_{2n}(z,q)$ become real if $\Re(z) = \pm\pi/2$ or $\Im(z) = 0$ as long as $q > 0$. Assuming that all P_{2n}'s are real, the condition f is real along $\Im(z) = H$ requires

$$\frac{\Im\left\{\sum_{n=0}^{\infty} P_{2n}\mathrm{ce}_{2n}\left(z,q\right)\right\}}{\Re\left\{\sum_{n=0}^{\infty} P_{2n}\mathrm{ce}_{2n}\left(z,q\right)\right\}} = \frac{\Im\left\{\mathrm{se}_1\left(z,q\right)\right\}}{\Re\left\{\mathrm{se}_1\left(z,q\right)\right\}} \quad (\Im(z) = H). \tag{10}$$

For Mathieu functions, we have series expansion as in [4]

$$\mathrm{ce}_{2n}(z,q) = \sum_{m=0}^{\infty} A_{2m}^{(2n)}(q)\cos 2mz \qquad (n \geq 0), \tag{11}$$

$$\mathrm{se}_1(z,q) = \sum_{m=0}^{\infty} B_{2m+1}^{(1)}(q)\sin(2m+1)z. \tag{12}$$

The coefficients $A_*^{(*)}, B_*^{(1)}$ in equations (11) and (12) are determined through infinite continued fractions as a function of q. Then equation (10) becomes

$$\begin{aligned}\sum_{N=1}^{\infty}\cos(2N-1)x\Bigg[&\sum_{m=0}^{\infty} P_{2m}\sum_{n=1}^{N} B_{2n-1}^{(1)}A_{2N-2n}^{(2m)}\sinh(4n-2N-1)H\\ &+\sum_{m=0}^{\infty} P_{2m}\sum_{n=N}^{\infty} B_{2n-1}^{(1)}A_{2n-2N}^{(2m)}\sinh(4n-2N-1)H\\ &+\sum_{m=0}^{\infty} P_{2m}\sum_{n=1}^{\infty} B_{2n-1}^{(1)}A_{2n+2N-2}^{(2m)}\sinh(4n+2N-3)H\Bigg] = 0,\end{aligned} \tag{13}$$

where $x = \Re(z)$. Thus equation (13) leads to

$$\begin{aligned}&\sum_{m=0}^{\infty} P_{2m}\sum_{n=1}^{N} B_{2n-1}^{(1)}A_{2N-2n}^{(2m)}\sinh(4n-2N-1)H\\ &\quad+\sum_{m=0}^{\infty} P_{2m}\sum_{n=N}^{\infty} B_{2n-1}^{(1)}A_{2n-2N}^{(2m)}\sinh(4n-2N-1)H\\ &\quad+\sum_{m=0}^{\infty} P_{2m}\sum_{n=1}^{\infty} B_{2n-1}^{(1)}A_{2n+2N-2}^{(2m)}\sinh(4n+2N-3)H = 0 \quad (N \geq 1).\end{aligned} \tag{14}$$

Throughout equations (13)−(14), the explicit expression of argument q for $B_*^{(1)}, A_*^{(*)}$ is omitted. In addition to this, the residue of f at $z = 0$ is assumed without loss of generality to be

$$-\pi/\left\{2kK(k)\right\},$$

where

$$H = \frac{\pi K(\sqrt{1-k^2})}{2K(k)} \qquad (0 < k < 1),$$

which becomes

$$\frac{\sum_{n=0}^{\infty} P_{2n} \sum_{m=0}^{\infty} A_{2m}^{(2n)}(q)}{\sum_{n=1}^{\infty} (2n-1)\, B_{2n-1}^{(1)}(q)} = \frac{-\pi}{2kK(k)}. \tag{15}$$

The function f is analytic everywhere in the rectangular closed region except $z = 0$, so that $f'(\pm\pi/2) = 0, f'(\pm\pi/2 + Hi) = 0$ irrespective of q. In case of $q = +0$, the exact solution of equations (14) and (15) is given by

$$P_0 = \frac{-\pi \coth H}{\sqrt{2} kK(k)}, \tag{16}$$

$$P_{2n} = \frac{\pi}{2kK(k)} \frac{\sinh 2H}{\sinh(2n-1)H \sinh(2n+1)H} \qquad (n \geq 1), \tag{17}$$

$$f = \operatorname{sn}\left\{ -\frac{2K(k)}{\pi} (z - iH), k \right\}, \tag{18}$$

where equations (16) and (17) are a modified form in [2]. All the more

$$f''(\frac{\pi}{2}, q = +0) = -\frac{1-k^2}{k} \left\{ \frac{2K(k)}{\pi} \right\}^2 < 0, \tag{19}$$

$$f''(\frac{\pi}{2} + Hi, q = +0) = (1-k^2) \left\{ \frac{2K(k)}{\pi} \right\}^2 > 0. \tag{20}$$

Thus at least for a moderate value of q, equations (1) and (9) give the said conformal mapping.

2.5. In Case of Construction, Using Jacobian Elliptic dn Function

Let f be defined as

$$f = \frac{\sum_{n=0}^{\infty} P_{2n} \cos 2nz}{g(z,k)}, \tag{21}$$

$$g(z,k) \equiv \operatorname{dn}\left\{ \frac{K(k)}{\pi} \left(z + \frac{\pi}{2} \right), k \right\} - \operatorname{dn}\left\{ \frac{K(k)}{\pi} \left(z - \frac{\pi}{2} \right), k \right\}. \tag{22}$$

The Jacobian function dn is expressed [1] as

$$\mathrm{dn}\left(\frac{K}{\pi}z,k\right)=\frac{\pi}{2K}+\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q_0^n}{1+q_0^{2n}}\cos\left(nz\right), \tag{23}$$

$$K\equiv K(k),$$

$$q_0\equiv\exp\left\{-\pi K(\sqrt{1-k^2})/K(k)\right\},$$

$$|\Im(z)|<-\ln q_0,$$

so that

$$g(z,k)=\frac{-4\pi}{K(k)}\sum_{m=1}^{\infty}(-1)^{m-1}\frac{q_0^{2m-1}}{1+q_0^{4m-2}}\sin(2m-1)z. \tag{24}$$

Consequently it is assumed that

$$H<\frac{\pi K(\sqrt{1-k^2})}{K(k)}.$$

In equation (21), the function f is so chosen that the pole of order 1 in the closed region is $z=0$. In case of $\Re(z)=\pm\pi/2$ or $\Im(z)=0$, $g(z,k)$ and $\cos 2nz$ (n : integer, $n\geq 0$) are real. Thus the necessary condition that f is real along $\Im(z)=H$ becomes

$$\frac{\Im\left(\sum_{n=0}^{\infty}P_{2n}\cos 2nz\right)}{\Re\left(\sum_{n=0}^{\infty}P_{2n}\cos 2nz\right)}=\frac{\Im\left\{g(z,k)\right\}}{\Re\left\{g(z,k)\right\}} \tag{25}$$

for any z ($|\Re(z)|<\pi/2$ and $\Im(z)=H$). Assuming all P_{2n}'s are real, equation (25) becomes, from the coefficient corresponding to the component $\cos\{(2N-1)\Re(z)\}$, $N\geq 1$,

$$\sum_{m=0}^{N-1}A_{N-m}B_{2m}\tanh(2N-2m-1)H+\sum_{m=0}^{\infty}A_{m+N}B_{2m}\tanh(2m+2N-1)H$$
$$+\sum_{m=N}^{\infty}A_{m+1-N}B_{2m}\tanh(2m+1-2N)H=-\sum_{m=1}^{\infty}A_{m+N}B_{2m}\tanh 2mH$$
$$-\sum_{m=N}^{\infty}A_{m+1-N}B_{2m}\tanh 2mH+\sum_{m=1}^{N-1}A_{N-m}B_{2m}\tanh 2mH, \tag{26}$$

$$N\geq 1,$$

	$q = 0.3$	$q = 0.5$	$q = 0.8$
P_0	-1.202	-1.192	-1.175
P_2	1.321×10^{-1}	9.095×10^{-2}	4.089×10^{-2}
P_4	2.118×10^{-3}	3.055×10^{-3}	6.726×10^{-3}
P_6	-1.650×10^{-4}	-4.283×10^{-4}	-8.826×10^{-4}
P_8	-4.800×10^{-6}	-1.599×10^{-5}	-4.747×10^{-5}
P_{10}	-8.910×10^{-8}	-3.342×10^{-7}	-1.234×10^{-6}

Table 1: Coefficients P_{2n} at $H = 1$ for equation (9)

where

$$A_n \equiv (-1)^{n-1} \frac{q_0^{2n-1}}{1 + q_0^{4n-2}} \cosh(2n-1)H \quad (n \geq 1),$$

$$B_{2n} \equiv P_{2n} \cosh 2nH \quad (n \geq 0).$$

In addition to this, the residue of f at $z = 0$ is assumed to be $-\pi / \{2k_0 K(k_0)\}$ such that $H = \frac{\pi K(\sqrt{1-k_0^2})}{2K(k_0)}$, $0 < k_0 < 1$, which becomes

$$\sum_{n=0}^{\infty} P_{2n} = \frac{k^2 K}{k_0 K(k_0)} \mathrm{sn}\left(\frac{K}{2}, k\right) \mathrm{cn}\left(\frac{K}{2}, k\right) = \frac{k^2 K}{k_0 K(k_0)} \frac{(1-k^2)^{1/4}}{1 + \sqrt{1-k^2}}. \quad (27)$$

In equation (27), $K \equiv K(k)$. Since $\lim_{k \to +0} \frac{g(z,k)}{q_0} = \lim_{q_0 \to +0} \frac{g(z,k)}{q_0} = -8 \sin z$, the case $k \to +0$ in equation (21) is substantially equivalent in nature to the case $q \to +0$ in equation (9), although P_{2n}'s in equation (9) and (21) are different each other, for $\lim_{q \to +0} \mathrm{se}_1(z,q) = \sin z$, $\lim_{q \to +0} \mathrm{ce}_0(z,q) = \frac{1}{\sqrt{2}}$, $\lim_{q \to +0} \mathrm{ce}_n(z,q) = \cos nz$ $(n > 0)$. If the function f is compatible with the conformal mapping, then $f'(\pi/2) = f'(\pi/2 + Hi) = 0$, so that $f''(\pi/2 + Hi)/f''(\pi/2) < 0$. Actually $f''(\pi/2 + Hi)/f''(\pi/2) \approx -k_0$ (as in the case $k \to +0$), which is valid at least for $k = 1/\sqrt{2}$ $(0.5 \leq H \leq 1.5)$, $k = 0.5$ $(0.5 \leq H \leq 1.5)$, $H = 0.45$ $(0.1 \leq k \leq 0.999)$. Equation (18) gives the said conformal mapping as long as k is moderate, depending on H.

	$k = 1/\sqrt{2}, H = 1$	$k = 0.999, H = 0.45$
P_0	2.512×10^{-1}	5.387×10^{-1}
P_2	-5.984×10^{-2}	-3.888×10^{-1}
P_4	-8.210×10^{-4}	2.171×10^{-2}
P_6	-1.551×10^{-5}	-5.942×10^{-3}
P_8	-2.832×10^{-7}	8.107×10^{-5}
P_{10}	-5.188×10^{-9}	-1.047×10^{-4}

Table 2: Coefficients P_{2n} in equation (21)

3. Results and Discussions

Table 1 shows examples of concrete numerical values of the leading first 6 coefficients in equation (9) of three cases for mapping through Mathieu functions ($k = 0.944, K(k) = 2.539$ in case of $H = 1$). Table 2 shows examples of concrete numerical values of the leading first 6 coefficients in equation (21) of two cases for mapping through Jacobian dn function. These tables show rapid decrease of coefficients (absolute values) with increasing subscripts similarly in equation (17).

4. Conclusions

Mapping functions of a complex rectangular region to a circular region through special function are proposed. Concrete examples of functions are successfully given.

References

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