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GLOBALLY EXPONENTIAL STABILITY AND EXISTENCE OF ANTIPERIODIC SOLUTION OF A CLASS OF HIGHER-ORDER HOPFIELD NEURAL NETWORKS WITH DISTRIBUTED DELAYS AND IMPULSE ON TIME SCALES

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Abstract: On time scales, by using the continuation theorem of coincidence degree theory, $M-$matrix theory and constructing some suitable Lyapunov functions, some sufficient conditions are obtained for the existence and exponential stability of anti-periodic solutions of a class of higher-order Hopfield neural networks with distributed delays and impulse, which are new and complement of previously known results. Finally, an example is given to illustrate the effectiveness of our main results.

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1. Introduction

The theory of calculus on time scales (see [3], [1] and references cited therein) was initiated by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis, and it has a tremendous potential for application and has recently received much attention since his foundational work. In fact, both continuous and discrete systems are very important in implementing

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and applications. Therefore, it is meaningful to study dynamic system on time scales which can unify the differential and difference system.

Consider the following higher-order Hopfield neural networks on time scales

$$\begin{cases} x_i^{\Delta}(t) = -\, c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\tau_{ij})) + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) \\ \qquad \times \int_0^{\infty} k_{ij}(\theta) g_j(x_j(t-\theta))\Delta\theta \int_0^{\infty} k_{il}(\theta) g_l(x_l(t-\theta))\Delta\theta \\ \qquad + I_i(t), \quad i = 1,2,\ldots,n,\ t \in \mathbb{T},\ t > 0, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = J_{ik}(x_i(t_k)), \quad i = 1,2,\ldots,n,\ k \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is an $\frac{\omega}{2}$-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. For each interval L of $\mathbb{R}$ we denote by $L_{\mathbb{T}} = L \cap \mathbb{T}$, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $x_i(t_k^+)$, $x_i(t_k^-)(i = 1,2,\ldots,n)$, which represent the right and left limit of $x_i(t_k)$ in the sense of time scales. $\{t_l\}$ is a sequence of real numbers such that $0 < t_1 < t_2 < \ldots < t_l \to \infty$ as $l \to \infty$. There exists a positive integer q such that $t_{l+q} = t_l + \frac{\omega}{2}$, $J_{i(k+q)} = -J_{ik}$, $l \in \mathbb{Z}$, $i = 1,2,\ldots,n$. Without loss of generality, we also assume that $[0, \frac{\omega}{2})_{\mathbb{T}} \cap \{t_l : l \in \mathbb{Z}\} = \{t_1, t_2, \ldots, t_q\}$. n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the i-th unit at the time t, $c_i(t)$ represents the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $A = (a_{ij}(t))_{n\times n}$ is the delayed feedback matrix which represents the strengthening of the neuron interconnections within the network with bounded delay parameter τ_{ij} , $\tau = \max_{1\le i,j\le n}\{\tau_{ij}\}$. $I_i(t)$ denotes the external inputs at time t, $B = (b_{ijl})$ is the second-order delayed feedback matrix which represents the strengthening of the neuron interconnections within the network with delay kernel k_{ij}, and k_{il}. f_j and g_j are the activation functions of signal transmission.

Since higher-order Hopfield neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order Hopfield neural networks, the study of higher-order Hopfield neural networks has recently gained a lot of attention, moreover there have been extensive results on the problem of the existence and stability of equilibrium points, periodic solutions and almost periodic solutions of such neural networks in the literature. We refer the reder to [20], [22], [10] and the references cited therein. However, very few results are available on the existence and exponential stability of anti-periodic solutions for neural networks, while the existence of anti-periodic solutions plays an important role in characterizing the behavior of nonlinear differential equations (see [10]-[13]).

On the other hand, most neural networks can be classified as either continuous or discrete. However, there are many real-world systems and neural processes that behave in a piecewise continuous style interlaced with instantaneous and abrupt changes (impulses). Motivated by this fact, several new neural networks with impulses have recently been proposed and studied(see [14]-[16]).

To my best knowledge, few papers applying the method of coincidence degree to investigate the existence and exponential stability of anti-periodic solutions to impulsive higher-order Hopfield neural networks with distributed delays on time scales, thus, it is worth while to study the existence and exponential stability of anti-periodic solutions of (1.1) on time scales by using he method of coincidence degree. The system (1.1) is supplemented with initial values given by

$$x_i(s) = \phi_i(s), \quad s \in (-\infty, 0]_{\mathbb{T}},$$

where $\phi_i(\cdot)$ denotes continuous function defined on $(-\infty, 0]_{\mathbb{T}}$.

Throughout this paper, we assume that:

(H_1) $\tau_{ij} \geq 0, c_i, a_{ij}, b_{ijl} \in C(\mathbb{T}, \mathbb{R})$ are $\frac{\omega}{2}$-periodic functions, and $I_i(t) \in C(\mathbb{T}, \mathbb{R})$ are $\frac{\omega}{2}$-anti-periodic functions, $i, j, l = 1, 2, \ldots, n$.

(H_2) The activation functions $f = (f_1, \ldots, f_n)^T$, $g = (g_1, \ldots, g_n)^T$, are bounded odd functions on $\mathbb{R}$ and Lipschitz functions, that is, there exist positive numbers α_i, β_i such that $|f_i(x) - f_i(y)| \leq \alpha_i |x - y|$, $|g_i(x) - g_i(y)| \leq \beta_i |x - y|$, $x, y \in \mathbb{R}$, $i = 1, \ldots, n$.

(H_3) $c_i \in C(\mathbb{T}, \mathbb{R}^+)$, and there exist positive numbers $\underline{c}_i$, $\bar{c}_i$ such that $\underline{c}_i \leq c_i(t) \leq \bar{c}_i$, for all $t \in \mathbb{T}$, $i = 1, 2, \ldots, n$.

(H_4) $J_{ik} \in C(\mathbb{R}, \mathbb{R})$ and there exist positive numbers J_{ik}^M such that $|J_{ik}(x)| \leq J_{ik}^M$, $x \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $k \in \mathbb{N}$.

(H_5) The delay kernels $k_{ij} : [0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ are real-valued piecewise continuous, and there exists a $\alpha_0 > 0$ such that the functions

$$K_{ij}(\alpha) = \sum_{m=1}^{\infty} e_\alpha(0, -\frac{m\omega}{2}) \int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}} k_{ij}(\theta) \Delta\theta,$$

are right-dense continuous for $\alpha \in [0, \alpha_0)_{\mathbb{T}}$ and $K_{ij}(0) = 1$, $i, j = 1, 2, \ldots, n$.

For the sake of convenience, we denote

$$\bar{a}_{ij} = \max_{t \in [0, \omega]_{\mathbb{T}}} |a_{ij}(t)|, \quad \bar{b}_{ijl} = \max_{t \in [0, \omega]_{\mathbb{T}}} |b_{ijl}(t)|,$$

$$\bar{I}_i = \max_{t\in[0,\omega]_{\mathbb{T}}} |I_i(t)|, \quad F_i = \sup_{t\in\mathbb{R}} |f_i(t)|, \quad G_i = \sup_{t\in\mathbb{R}} |g_i(t)|,$$

$$J_k^M = \max_{1\le i\le n} J_{ik}^M, \ \|x_i\|_2 = \left(\int_0^\omega |x_i(t)|^2 \Delta t\right)^{\frac{1}{2}}, \quad i,j,l = 1,2,\ldots,n, \ k \in \mathbb{N}.$$

Our purpose of this paper is by using the continuation theorem of coincidence degree theory and constructing some suitable Lyapunov functions to study the stability and existence of anti-periodic solutions of (1.1).

The paper is organized as follows: In Section 2, we present some basic definitions concerning the calculus on time scale. In Section 3, by using the continuation theorem of coincidence degree theory, we study the existence of anti-periodic solutions of (1.1). In Section 4, by constructing some suitable Lyapunov functions, we study the exponential stability of the anti-periodic solution of (1.1). In Section 5, an example is given to illustrate the effectiveness of our main results.

2. Preliminaries

In this section, we will cite some definitions and lemmas which will be used in the proofs of our main results.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Let $\omega \in \mathbb{R}$, $\omega > 0$, $\mathbb{T}$ is an ω-periodic time scale if $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ such that $t + \omega \in \mathbb{T}$ and $\mu(t) = \mu(t + \omega)$ whenever $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T})$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]\big| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous. If $y^{\Delta}(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

Definition 2.1. Function $f{=}(f_1, \dots, f_n)$ is a Lipschitz if it satisfies $|f_i(x) - f_i(y)| \leq l_i|x - y|$, $i = 1, \dots, n$ for any $x, y \in \mathbb{R}$.

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Definition 2.2. (see [17]) If $a \in \mathbb{T}, \sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)$, then we define the improper integral by

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Definition 2.3. (see [18]) For each $t \in \mathbb{T}$, let N be a neighborhood of t, then, for $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, define $D^+V^{\Delta}(t, x(t))$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ of t such that

$$\frac{[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t, s)f(t, x(t))]}{\mu(t, s)} < D^+V^{\Delta}(t, x(t)) + \varepsilon$$

for each $s \in N_\varepsilon$, $s > t$, where $\mu(t, s) \equiv \sigma(t) - s$. If t is rd and $V(t, x(t))$ is continuous at t, this reduce to

$$D^+V^{\Delta}(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(\sigma(t)))}{\sigma(t) - t}.$$

Definition 2.4. (see [19]) We say that a time scale $\mathbb{T}$ is a periodic if there exists $p > 0$, such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Definition 2.5. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p, we say that the function $f : \mathbb{T} \to \mathbb{R}$ is $\frac{\omega}{2}$-*anti-periodic* if there exists a natural number

n such that $\frac{\omega}{2} = np$, $f(t+\frac{\omega}{2}) = -f(t)$ for all $t \in \mathbb{T}$ and $\frac{\omega}{2}$ is the smallest number such that $f(t+\frac{\omega}{2}) = -f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is $\frac{\omega}{2}$-anti-periodic if $\frac{\omega}{2}$ is the smallest positive number such that $f(t+\frac{\omega}{2}) = -f(t)$ for all $t \in \mathbb{T}$.

A function $r : \mathbb{T} \to R$ is called regressive if

$$1 + \mu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}^k$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}, \quad \text{for } s,t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\operatorname{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad p \ominus q := p \oplus (\ominus q), \quad \ominus p := \frac{p}{1+\mu p}.$$

Then the generalized exponential function has the following properties.

Lemma 2.1. *Assume that $p,\ q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $e_p(t,\sigma(s)) = \frac{e_p(t,s)}{1+\mu(s)p(s)}$;

(iv) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(v) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(vi) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vii) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(viii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

Lemma 2.2. (see [1]) *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. Then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

Lemma 2.3. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in C(\mathbb{T}, \mathbb{R})$, then:*

(i) $\displaystyle\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;

(ii) *if* $f(t) \geq 0$, *for all* $a \leq t < b$, *then* $\displaystyle\int_a^b f(t)\Delta t \geq 0$;

(iii) *if* $|f(t)| \leq g(t)$ *on* $[a, b) = \{t \in \mathbb{T} : a \leq t < b\}$, *then* $\displaystyle|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.

The proofs of the following lemmas can be found in [20]-[22], respectively.

Lemma 2.4. (see [20]) *Let $t_1, t_2 \in [0, \omega]_{\mathbb{T}}$. If $x : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$x(t) \leq x(t_1) + \int_0^\omega |x^\Delta(s)|\Delta s, \quad x(t) \geq x(t_2) - \int_0^\omega |x^\Delta(s)|\Delta s.$$

Lemma 2.5. (Cauchy-Schwarz Inequality on Time Scale, see [21]) *Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \Big(\int_a^b |f(t)|^2\Delta t\Big)^{\frac{1}{2}}\Big(\int_a^b |g(t)|^2\Delta t\Big)^{\frac{1}{2}}.$$

Lemma 2.6. (see [22]) *Assume that $\{f_n\}_{n\in N}$ is a function sequence on J such that:*

(i) $\{f_n\}_{n\in N}$ *is uniformly bounded on* J;

(ii) $\{f_n^\Delta\}_{n\in N}$ *is uniformly bounded on* J.

Then there is a subsequence of $\{f_n\}_{n\in N}$ converges uniformly on J.

Lemma 2.7. *Let $\mathbb{T}$ be a ω-periodic time scale, then $\sigma(t+\omega) = \sigma(t) + \omega$, for all $t \in \mathbb{T}$.*

Proof. By using the definition of forward jump operator, we have $\sigma(t)+\omega \geq t+\omega$, then $\sigma(t)+\omega \geq \sigma(t+\omega)$, now we claim that $\sigma(t)+\omega = \sigma(t+\omega)$. If it is not true, we assume that $\sigma(t+\omega) = t_1^* < \sigma(t)+\omega$, from the definition of infimum (inf), we know that there exist a $t_2^* \in \mathbb{T}$, $t_2^* > t+\omega$, such that

$$t_2^* < t_1^* + \frac{\sigma(t)+\omega - t_1^*}{2} = \frac{\sigma(t)+\omega+t_1^*}{2} < \sigma(t)+\omega. \tag{2.1}$$

From (2.1), we obtain $t_2^* - \omega < \sigma(t)$, on the other hand, since $t_2^* > t + \omega$, $t_2^* - \omega \geq \sigma(t)$, which is a contradiction. The proof of Lemma 2.7 is complete. $\square$

From Lemma 2.7, we obtain the following lemma.

Lemma 2.8. *Let $\mathbb{T}$ be a ω-periodic time scale, then $\mu(t)$ is a ω-periodic function.*

Proof.

$$\mu(t + \omega) = \sigma(t + \omega) - t - \omega = \sigma(t) + \omega - t - \omega = \sigma(t) - t = \mu(t). \qquad \square$$

Form Lemma 2.8, we know that if $\theta, \tau \in \mathbb{T}$ are constants, then $e_\theta(t, -\tau)$ is a ω-periodic function.

Definition 2.6. The anti-periodic solution $x^*(t)$ of system (1.1) is said to be exponentially stable if there exist a positive constant α such that for every $\delta \in \mathbb{T}$, there exists $N = N(\delta) \geq 1$ such that the solution of (1.1) through $(\delta, x(\delta))$ satisfies

$$\sum_{i=1}^{n} |x(t) - x^*(t)| \leq N \|\phi - x^*\| e_{-a}(t, \delta), t \in \mathbb{T}^+,$$

where

$$\|\phi - x^*\| = \sum_{i=1}^{n} \big[\sup_{\delta \in (-\infty, 0]_{\mathbb{T}}} |\phi_i(\delta) - x_i^*(\delta)| \big].$$

The following fixed point result of coincidence degree is crucial in the arguments of our main results.

Lemma 2.9. (see [23]) *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \overline{\Omega} \neq \emptyset$ and $N : \overline{\Omega} :\to \mathbb{Y}$ is L-compact. Further, we assume that*

$$(H) \qquad Lx - Nx \neq \lambda(-Lx - N(-x)) \ \text{ for all } \ D(L) \cap \partial\Omega, \ \lambda \in (0, 1].$$

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \partial\Omega$.

Definition 2.7. *A* real $n \times n$ matrix $A = (a_{ij})$ is said to be a non-singular M-matrix if $a_{ij} \leq 0, i \neq j, \ i, j = 1, \ldots, n$ and all successive principal minors of A are positive.

3. Existence of Anti-Periodic Solutions

In this section, by means of Mawhin's Continuation Theorem, we will study the existence of at least one anti-periodic solution of (1.1).

Theorem 3.1. *Assume that the assumptions (H_1)-(H_5) are satisfied and $E=(e_{ij})_{n\times n}$ is a nonsingular M-matrix, where*

$$e_{ij}=\begin{cases} s_i-(1+\underline{c}_i\omega)\bar{a}_{ij}\alpha_j\omega, & i=j,\\ -(1+\underline{c}_i\omega)\bar{a}_{ij}\alpha_j\omega, & i\neq j,\end{cases}\qquad s_i=\underline{c}_i\omega-\underline{c}_i\bar{c}_i\omega^2,$$

where $i,j=1,2,\dots,n$.

Then system (1.1) has at least one $\frac{\omega}{2}$-anti-periodic solution.

Proof. Let

$$\begin{aligned}&C^i[0,\omega;t_1,t_2,\dots,t_q,t_{q+1},\dots,t_{2q}]_{\mathbb{T}}\\ &=\{x:[0,\omega]_{\mathbb{T}}\to\mathbb{R}^n|x^i(t)|_{(t_k,t_{k+1}]_{\mathbb{T}}}\in C(t_k,t_{k+1}),\exists x^i(t_k^-)=x^i(t_k),\ x^i(t_k^+),\\ &\qquad i=0,1,\ k\in[0,\omega]_{\mathbb{T}}\cap\{t_l:l\in\mathbb{N}\}\}.\end{aligned}$$

Take

$$\mathbb{X}=\{x\in C[0,\omega;t_1,t_2,\dots,t_q,t_{q+1},\dots,t_{2q}]_{\mathbb{T}}:x(t+\frac{\omega}{2})=-x(t),\ t\in[0,\frac{\omega}{2}]_{\mathbb{T}}\}$$

and

$$\mathbb{Y}=\mathbb{X}\times\mathbb{R}^{n\times q}$$

be two Banach spaces with the norms

$$||x||_{\mathbb{X}}=\sum_{i=1}^{n}|x_i|_0,\quad ||z||_{\mathbb{Y}}=||x||_{\mathbb{X}}+||y||,\quad x\in\mathbb{X},\ y\in\mathbb{R}^{n\times q},$$

in which $|x_i|_0=\max\limits_{t\in[0,\omega]_{\mathbb{T}}}|x_i(t)|,\ i=1,2,\dots,n,\ ||.||$ is any norm of $\mathbb{R}^{n\times q}$. Set

$$L:\operatorname{Dom}L\cap\mathbb{X}\to\mathbb{Y},\ x\to(x^{\Delta},\Delta x(t_1),\dots,\Delta x(t_q)),$$

where $\operatorname{Dom}L=\{x\in C^1[0,\omega;t_1,t_2,\dots,t_{2q}]_{\mathbb{T}}:x(t+\frac{\omega}{2})=-x(t),\ t\in[0,\frac{\omega}{2}]_{\mathbb{T}}\}$, and $N:\mathbb{X}\to\mathbb{Y}$,

$$Nx=\left(\begin{pmatrix}A_1(t)\\ \vdots\\ A_n(t)\end{pmatrix},\begin{pmatrix}\Delta x_1(t_1)\\ \vdots\\ \Delta x_n(t_1)\end{pmatrix},\begin{pmatrix}\Delta x_1(t_2)\\ \vdots\\ \Delta x_n(t_2)\end{pmatrix},\dots,\begin{pmatrix}\Delta x_1(t_q)\\ \vdots\\ \Delta x_n(t_q)\end{pmatrix}\right),$$

where

$$\begin{aligned}A_i(t)&=-c_i(t)x_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t-\tau_{ij}))\\ &\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\int_0^{\infty}k_{ij}(\theta)g_j(x_j(t-\theta))\Delta\theta\int_0^{\infty}k_{il}(\theta)g_l(x_l(t-\theta))\Delta\theta\\ &\quad+I_i(t),\quad i=1,2,\dots,n.\end{aligned}$$

Obviously,

$$\operatorname{Ker} L = \left\{ x \in \mathbb{X} \middle| x = 0 \right\},$$

$$\operatorname{Im} L = \left\{ z = (f, C_1, C_2, \ldots, C_q) \in \mathbb{Y} \middle| \int_0^\omega f(s)\Delta s = 0 \right\} \equiv \mathbb{Y}.$$

Then

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 0.$$

So, $\operatorname{Im} L$ is closed in $\mathbb{Y}$, L is a Fredholm mapping of index zero. Define the project operators P and Q as

$$Px = \frac{1}{\omega} \int_0^\omega x(t)\Delta t = 0, \quad x \in \mathbb{X},$$

$$Qz = Q(f, C_1, C_2, \ldots, C_q) = \left(\frac{1}{\omega} \int_0^\omega f(s)\Delta s, 0, \ldots, 0 \right), \quad z \in \mathbb{Y}.$$

It is not difficult to show that P and Q are continuous projectors and satisfy

$$\operatorname{Im}\ P = \operatorname{Ker} L,\ \operatorname{Im}\ L = \operatorname{Ker}\ Q = \operatorname{Im}\ (I - Q).$$

Further, let $L_P^{-1} = L|_{\mathrm{Dom} \cap \operatorname{Ker} P}$ and the generalized inverse $K_P = L_P^{-1}$ is given by

$$(K_P z)(t) = \int_0^t f(s)\Delta s + \sum_{t > t_k} C_k - \frac{1}{2} \int^{\frac{\omega}{2}} f(s)\Delta s - \frac{1}{2} \sum_{k=1}^q C_k,$$

in which $C_{q+i} = -C_i$ for all $1 \le i \le q$. Thus, the expression of QNx is

$$\left(\begin{pmatrix} \frac{1}{\omega} \int_0^\omega A_1(t)\Delta t \\ \vdots \\ \frac{1}{\omega} \int_0^\omega A_n(t)\Delta t \end{pmatrix}, 0, \ldots, 0 \right),$$

and then

$$K_P(I - Q)Nx = \begin{pmatrix} \int_0^t A_1(s)\Delta s + \sum\limits_{t > t_k} J_{1k}(x_1(t_k)) \\ \vdots \\ \int_0^t A_n(s)\Delta s + \sum\limits_{t > t_k} J_{nk}(x_n(t_k)) \end{pmatrix}$$

$$- \begin{pmatrix} \frac{1}{2} \int_0^{\frac{\omega}{2}} A_1(s)\Delta s \\ \vdots \\ \frac{1}{2} \int_0^{\frac{\omega}{2}} A_n(s)\Delta s \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \sum\limits_{k=1}^q J_{1k}(x_1(t_k)) \\ \vdots \\ \frac{1}{2} \sum\limits_{k=1}^q J_{nk}(x_n(t_k)) \end{pmatrix}.$$

Clearly, QN and $K_P(I-Q)N$ are both continuous. Using Lemma 2.9, it is easy to show that $QN(\overline{\Omega})$, $K_P(I-Q)N(\overline{\Omega})$ are compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 2.9, we need to find an appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x))$, $\lambda \in (0,1]$, we have

$$\begin{cases} x_i^{\Delta}(t) = \dfrac{1}{1+\lambda} B_i(t,x) - \dfrac{\lambda}{1+\lambda} B_i(t,-x), \\ \qquad\qquad t \in (0,\infty)_{\mathbb{T}},\ t \neq t_k,\ i = 1,2,\dots,n, \\ \Delta x_i(t_k) = \dfrac{1}{1+\lambda} J_{ik}(x_i(t_k)) - \dfrac{\lambda}{1+\lambda} J_{ik}(-x_i(t_k)), \quad i = 1,2,\dots,n,\ k \in \mathbb{N}, \end{cases} \tag{3.1}$$

where

$$B_i(t,x) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\tau_{ij}))$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) \int_0^{\infty} k_{ij}(\theta) g_j(x_j(t-\theta))\Delta\theta \int_0^{\infty} k_{il}(\theta) g_l(x_l(t-\theta))\Delta\theta + I_i(t),$$

and

$$B_i(t,-x) = c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(-x_j(t-\tau_{ij}))$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) \int_0^{\infty} k_{ij}(\theta) g_j(-x_j(t-\theta))\Delta\theta \int_0^{\infty} k_{il}(\theta) g_l(-x_l(t-\theta))\Delta\theta + I_i(t),$$

Suppose that $x = (x_1, x_2, \dots, x_n)^T$ is a solution of system (3.1) for a certain $\lambda \in (0,1)$. Set $t_0 = t_0^+ = 0$, $t_{2q+1} = \omega$. We obtain

$$\begin{aligned} \int_0^{\omega} |x_i^{\Delta}(t)|\Delta t &= \sum_{k=1}^{2q+1} \int_{t_{k-1}^+}^{t_k} |x_i^{\Delta}(t)|\Delta t + \sum_{k=1}^{2q} |I_{ik}(x_i(t_k))| \\ &\le \int_0^{\omega} \left| \frac{1}{1+\lambda} B_i(t,x) - \frac{\lambda}{1+\lambda} B_i(t,-x) \right| \Delta t \\ &\quad + \sum_{k=1}^{2q} \left| \frac{1}{1+\lambda} I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda} I_{ik}(-x_i(t_k)) \right| \\ &\le \left[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right] \int_0^{\omega} \max\Big\{ |B_i(t,x)|, |B_i(t,-x)| \Big\} \Delta t \end{aligned}$$

$$+\left[\frac{1}{1+\lambda}+\frac{\lambda}{1+\lambda}\right]\sum_{k=1}^{2q}\max\left\{|I_{ik}(x_i(t_k))|,|I_{ik}(-x_i(t_k))|\right\}$$

$$\le \bar{c}_i\int_0^{\omega}|x_i(t)|\Delta t+\sum_{j=1}^{n}\bar{a}_{ij}\int_0^{\omega}|f_j(x_j(t-\tau_{ij}))-f_j(0)|\Delta t+\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|\omega$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}\bigg(\int_0^{\infty}k_{ij}(\theta)g_j(x_j(t-\theta))\Delta\theta\int_0^{\infty}k_{il}(\theta)g_l(x_l(t-\theta))\Delta\theta\bigg)\Delta t+\bar{I}_i\omega$$

$$+\sum_{k=1}^{2q}\max\left\{|J_{ik}(x_i(t_k))|,|J_{ik}(-x_i(t_k))|\right\}$$

$$\le \bar{c}_i\sqrt{\omega}||x_i||_2+\sum_{j=1}^{n}\bar{a}_{ij}\alpha_j\sqrt{\omega}||x_j||_2+\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|\omega$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l\omega+\bar{I}_i\omega+\sum_{k=1}^{2q}J_{ik}^{M}.$$

Integrating both sides of (3.1) from 0 to ω, we have

$$\left|\int_0^{\omega}\left[\frac{c_i(t)x_i(t)}{1+\lambda}-\frac{-\lambda c_i(t)x_i(t)}{1+\lambda}\right]\Delta t\right|=\left|\int_0^{\omega}c_i(t)x_i(t)\Delta t\right|$$

$$=\bigg|\frac{1}{1+\lambda}\int_0^{\omega}\bigg[\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t-\tau_{ij}))$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\int_0^{\infty}k_{ij}(\theta)g_j(x_j(t-\theta))\Delta\theta\int_0^{\infty}k_{il}(\theta)g_l(x_l(t-\theta))\Delta\theta+I_i(t)\bigg]\Delta t$$

$$+\frac{\lambda}{1+\lambda}\int_0^{\omega}\bigg[-\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t-\tau_{ij}))$$

$$-\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\int_0^{\infty}k_{ij}(\theta)g_j(-x_j(t-\theta))\Delta\theta\int_0^{\infty}k_{il}(\theta)g_l(-x_l(t-\theta))\Delta\theta-I_i(t)\bigg]\Delta t$$

$$+\frac{1}{1+\lambda}\sum_{k=1}^{2q}J_{ik}(x_i(t_k))-\frac{\lambda}{1+\lambda}\sum_{k=1}^{2q}J_{ik}(-x_i(t_k))\bigg|$$

$$\le\left[\frac{1}{1+\lambda}+\frac{\lambda}{1+\lambda}\right]\int_0^{\omega}\max\bigg\{\bigg|\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t-\tau_{ij}))$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\int_{0}^{\infty}k_{ij}(\theta)g_{j}(x_{j}(t-\theta))\Delta\theta\int_{0}^{\infty}k_{il}(\theta)g_{l}(x_{l}(t-\theta))\Delta\theta+I_{i}(t)\bigg|,$$

$$\bigg|\sum_{j=1}^{n}a_{ij}(t)f_{j}(x_{j}(t-\tau_{ij}))$$

$$-\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)\int_{0}^{\infty}k_{ij}(\theta)g_{j}(x_{j}(t-\theta))\Delta\theta\int_{0}^{\infty}k_{il}(\theta)g_{l}(x_{l}(t-\theta))\Delta\theta-I_{i}(t)\bigg|\bigg\}\Delta t$$

$$+\left[\frac{1}{1+\lambda}+\frac{\lambda}{1+\lambda}\right]\max\left\{\sum_{k=1}^{2q}|J_{ik}(x_{i}(t_{k}))|,\sum_{k=1}^{2q}|J_{ik}(-x_{i}(t_{k}))|\right\}$$

$$\leq\sum_{j=1}^{n}\bar{a}_{ij}\alpha_{j}\sqrt{\omega}||x_{j}||_{2}+\omega\bigg(\sum_{j=1}^{n}\bar{a}_{ij}|f_{j}(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_{j}G_{l}+\bar{I}_{i}\bigg)+\sum_{k=1}^{2q}J_{ik}^{M}.$$

From Lemma 2.4, for any $t_1^i, t_2^i \in [0,\omega]_{\mathbb{T}}$, $i = 1, 2, \ldots, n$, we have

$$\int_{0}^{\omega}c_{i}(t)x_{i}(t)\Delta t\leq\int_{0}^{\omega}c_{i}(t)x_{i}(t_{1}^{i})\Delta t+\int_{0}^{\omega}c_{i}(t)\bigg(\int_{0}^{\omega}|x_{i}^{\Delta}(s)|\Delta s\bigg)\Delta t,$$
$$i=1,2,\ldots,n, \quad (3.2)$$

and

$$\int_{0}^{\omega}c_{i}(t)x_{i}(t)\Delta t\geq\int_{0}^{\omega}c_{i}(t)x_{i}(t_{2}^{i})\Delta t-\int_{0}^{\omega}c_{i}(t)\bigg(\int_{0}^{\omega}|x_{i}^{\Delta}(s)|\Delta s\bigg)\Delta t,$$
$$i=1,2,\ldots,n. \quad (3.3)$$

Diving by $\int_0^\omega c_i(t)\Delta t$ on the two sides of (3.2) and (3.3) respectively, we have

$$x_{i}(t_{1}^{i})\geq\frac{1}{\int_{0}^{\omega}c_{i}(t)\Delta t}\int_{0}^{\omega}c_{i}(t)x_{i}(t)\Delta t-\int_{0}^{\omega}|x_{i}^{\Delta}(s)|\Delta s,\quad i=1,2,\ldots,n,\quad (3.4)$$

and

$$x_{i}(t_{2}^{i})\leq\frac{1}{\int_{0}^{\omega}c_{i}(t)\Delta t}\int_{0}^{\omega}c_{i}(t)x_{i}(t)\Delta t+\int_{0}^{\omega}|x_{i}^{\Delta}(s)|\Delta s,\quad i=1,2,\ldots,n.\quad (3.5)$$

Let $\bar{t}_i, \underline{t}_i \in [0,\omega]_{\mathbb{T}}$, such that $x_i(\bar{t}_i) = \max_{t\in[0,\omega]_{\mathbb{T}}} x_i(t)$, $x_i(\underline{t}_i) = \min_{t\in[0,\omega]_{\mathbb{T}}} x_i(t)$, $i = 1, 2, \ldots, n$, by the arbitrariness of t_1^i, t_2^i, in view of (3.4) and (3.5), we obtain

$$x_{i}(\underline{t}_{i})\geq\frac{1}{\int_{0}^{\omega}c_{i}(t)\Delta t}\int_{0}^{\omega}c_{i}(t)x_{i}(t)\Delta t-\int_{0}^{\omega}|x_{i}^{\Delta}(s)|\Delta s$$
$$\geq-\frac{1}{\underline{c}_{i}\omega}\bigg|\int_{0}^{\omega}c_{i}(t)x_{i}(t)\Delta t\bigg|-\int_{0}^{\omega}|x_{i}^{\Delta}(s)|\Delta s$$

$$\geq -\frac{1}{\underline{c}_i\omega}\Bigg[\sum_{j=1}^{n}\bar{a}_{ij}\alpha_j\sqrt{\omega}||x_j||_2+\omega\Bigg(\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l+\bar{I}_i\Bigg)+\sum_{k=1}^{2q}J_{ik}^M\Bigg]$$
$$-\Bigg[\bar{c}_i\sqrt{\omega}||x_i||_2+\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l+\bar{I}_i\Bigg)+\sum_{k=1}^{2q}J_{ik}^M\Bigg],$$

and

$$x_i(\bar{t}_i)\leq\frac{1}{\int_0^\omega c_i(t)\Delta t}\int_0^\omega c_i(t)x_i(t)\Delta t+\int_0^\omega|x_i^\Delta(s)|\Delta s$$
$$\geq +\frac{1}{\underline{c}_i\omega}\left|\int_0^\omega c_i(t)x_i(t)\Delta t\right|+\int_0^\omega|x_i^\Delta(s)|\Delta s$$
$$\geq +\frac{1}{\underline{c}_i\omega}\Bigg[\sum_{j=1}^{n}\bar{a}_{ij}\alpha_j\sqrt{\omega}||x_j||_2+\omega\Bigg(\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l+\bar{I}_i\Bigg)+\sum_{k=1}^{2q}J_{ik}^M\Bigg]$$
$$+\Bigg[\bar{c}_i\sqrt{\omega}||x_i||_2+\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l+\bar{I}_i\Bigg)+\sum_{k=1}^{2q}J_{ik}^M\Bigg],\quad i=1,2,\ldots,n.$$

So, we can obtain that

$$|x_i|_0=\max_{t\in[0,\omega]_{\mathbb{T}}}|x_i(t)|$$
$$\leq\frac{1}{\underline{c}_i\omega}\Bigg[\sum_{j=1}^{n}\bar{a}_{ij}\alpha_j\sqrt{\omega}||x_j||_2+\omega\Bigg(\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l+\bar{I}_i\Bigg)+\sum_{k=1}^{2q}J_{ik}^M\Bigg]$$
$$+\Bigg[\bar{c}_i\sqrt{\omega}||x_i||_2+\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l+\bar{I}_i\Bigg)+\sum_{k=1}^{2q}J_{ik}^M\Bigg],\quad(3.6)$$

where $i=1,2,\ldots,n$. In addition, we have that

$$\|x_i\|_2=\left(\int_0^\omega|x_i(s)|^2\Delta s\right)^{\frac{1}{2}}\leq\omega^{\frac{1}{2}}\max_{t\in[0,\omega]_{\mathbb{T}}}|x_i(t)|.$$

By (3.6), we have,

$$\underline{c}_i\omega|x_i|_0\leq\underline{c}_i\bar{c}_i\omega^2|x_i|_0+\sum_{j=1}^{n}(1+\underline{c}_i\omega)\bar{a}_{ij}\alpha_j\omega|x_j|_0+(1+\underline{c}_i\omega)\Gamma_i,$$

where

$$\Gamma_i=\omega\Bigg(\sum_{j=1}^{n}\bar{a}_{ij}|f_j(0)|+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}G_jG_l+\bar{I}_i\Bigg)+\sum_{k=1}^{2q}J_{ik}^M.$$

That is

$$\Big[\underline{c}_i\omega - \underline{c}_i\bar{c}_i\omega^2\Big]|x_i|_0 - \sum_{j=1}^{n}(1+\underline{c}_i\omega)\bar{a}_{ij}\alpha_j\omega|x_j|_0 \le (1+\underline{c}_i\omega)\Gamma_i := D_i, \qquad (3.7)$$

where $i = 1, 2, \ldots, n$. Denote $|x|_0 = (|x_1|_0, |x_2|_0, \ldots, |x_n|_0)^T$, and $D = (D_1, D_2, \ldots, D_n)^T$.

Then (3.7) can be rewritten in the matrix form

$$E|x|_0 \le D.$$

From the conditions of Theorem 3.1, E is a nonsingular M matrix, hence

$$|x|_0 \le E^{-1}D := (N_1, N_2, \ldots, N_n)^T.$$

Let $N = \sum_{i=1}^{n} N_i + N_0$, where N_0 is any positive constant. It is clear that N is independent of λ. Take $\Omega = \Big\{x \in \mathbb{X}\,\Big|\,\|x\|_{\mathbb{X}} < N\Big\}$. Obviously, Ω satisfies all the requirement in Lemma 2.9 and the condition (H) satisfied. In view of all the discussion above, we conclude from Lemma 2.9 that system (1.1) has at least one $\frac{\omega}{2}$-anti-periodic solution. This completes the proof. □

4. Global Exponential Stability of Anti-Periodic Solutions

Suppose that $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ is a anti-periodic solution of system (1.1). We will construct some suitable Lyapunov functions to study the global exponential stability of this anti-periodic solution.

Theorem 4.1. *Assume that* $(H_1) - (H_5)$ *hold, suppose further that:*

(H_6) *for* $i = 1, 2, \ldots, n$.

$$-\underline{c}_i + \alpha_i \sum_{j=1}^{n} \bar{a}_{ji} + \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{b}_{ijl}(G_l\beta_j + G_j\beta_l) < 0.$$

(H_7) *Impulsive operators* $J_{ik}(x_i(t))$ *satisfy*

$$J_{ik}(x_i(t_k)) = -\gamma_{ik}x_i(t_k),\ 0 < \gamma_{ik} < 2,\ i = 1, 2, \ldots, n,\ k \in \mathbb{N}.$$

Then the $\frac{\omega}{2}$*-anti-periodic solution of system (1.1) is globally exponentially stable.*

Proof. According to Theorem 3.1, we know that (1.1) has an $\frac{\omega}{2}$-anti-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$. Suppose that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is an arbitrary solution of (1.1).

Let $y(t) = x(t) - x^*(t)$, then by (1.1) we have

$$y_i^{\Delta}(t) = -c_i(t)[x_i(t) - x_i^*(t)] + \sum_{j=1}^{n} a_{ij}(t)\Big[f_j(x_j(t-\tau_{ij})) - f_j(x_j^*(t-\tau_{ij}))\Big]$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\int_0^{\infty} k_{ij}(\theta)\Big[g_j(x_j(t-\theta)) - g_j(x_j^*(t-\theta))\Big]\Delta\theta \int_0^{\infty} k_{il}(\theta) g_l(x_l(t-\theta))\Delta\theta$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\int_0^{\infty} k_{ij}(\theta) g_j(x_j^*(t-\theta))\Delta\theta \int_0^{\infty} k_{il}(\theta)\Big[g_l(x_l(t-\theta)) - g_l(x_l^*(t-\theta))\Big]\Delta\theta,$$

with the initial conditions $\psi_i(s) = \phi_i(s) - x_i^*(s), s \in [-\infty, 0]_{\mathbb{T}}$, $i = 1, 2, \ldots, n$. Hence from the conditions of Theorem 4.1, we have

$$\begin{aligned} D^+|x_i(t) - x_i^*(t)|^{\Delta} \leq\ & -\underline{c}_i|x_i(t) - x_i^*(t)| + \sum_{j=1}^{n} \bar{a}_{ij}\alpha_j|x_j(t-\tau_{ij}) - x_j^*(t-\tau_{ij})| \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{b}_{ijl} G_l \beta_j \int_0^{\infty} k_{ij}(\theta)|x_j(t-\theta) - x_j^*(t-\theta)|\Delta\theta \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{b}_{ijl} G_j \beta_l \int_0^{\infty} k_{il}(\theta)|x_l(t-\theta) - x_l^*(t-\theta)|\Delta\theta, \end{aligned}$$

for $i = 1, 2, \ldots, n$, $k \in \mathbb{N}$. And we have from (H_7) that

$$\begin{aligned} x_i(t_k^+) - x_i^*(t_k^+) &= x_i(t_k) + J_{ik}(x_i(t_k)) - x_i^*(t_k) - J_{ik}(x_i^*(t_k)) \\ &= (1-\gamma_{ik})(x_i(t_k) - x_i^*(t_k)). \end{aligned}$$

Hence

$$|x_i(t_k^+) - x_i^*(t_k^+)| = |1-\gamma_{ik}||x_i(t_k) - x_i^*(t_k)| < |x_i(t_k) - x_i^*(t_k)|, \quad i = 1, 2, \ldots, n,\ k \in \mathbb{N}.$$

Let A_i be defined by

$$\begin{aligned} A_i(\varepsilon) \ =\ & \underline{c}_i - \varepsilon - \alpha_i \sum_{j=1}^{n} \bar{a}_{ji} e_{\varepsilon}(\frac{\omega}{2}, -\tau) \\ & - \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{b}_{ijl}\bigg(G_l\beta_j \sum_{m=1}^{\infty} e_{\varepsilon}(0, -\frac{m\omega}{2}) \int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}} k_{ij}(\theta)\Delta\theta \\ & + G_j\beta_l \sum_{m=1}^{\infty} e_{\varepsilon}(0, -\frac{m\omega}{2}) \int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}} k_{il}(\theta)\Delta\theta\bigg), \end{aligned}$$

where $\varepsilon \in [0, \infty)$, $i = 1, 2, \ldots, n$, it is clear that

$$A_i(0) = \underline{c}_i - \alpha_i \sum_{j=1}^{n} \bar{a}_{ji} - \sum_{j=1}^{n} \sum_{l=1}^{n} \bar{b}_{ijl}(G_l\beta_j + G_j\beta_l) > 0,\ i = 1, 2, \ldots, n.$$

Since A_i are continuous on $[0, \infty)$ and $A_i(\varepsilon) \to -\infty$, as $\varepsilon \to +\infty$, there exist $\xi_i^* > 0$ such that $A_i(\xi_i^*) = 0$ and $A_i(\varepsilon) > 0$, for $\varepsilon \in (0, \xi_i^*)$. By choosing $\xi = \min\big\{ \min\limits_{1\le i\le n} \{\xi_i^*\}, \frac{\alpha_0}{2} \big\}$ we have

$$\begin{aligned} A_i(\xi) \ &= \ \underline{c}_i - \xi - \alpha_i \sum_{j=1}^{n} \bar{a}_{ji} e_\xi(\frac{\omega}{2}, -\tau) \\ &\quad - \sum_{j=1}^{n} \sum_{l=1}^{n} \bar{b}_{ijl} \bigg(G_l\beta_j \sum_{m=1}^{\infty} e_\xi(0, -\frac{m\omega}{2}) \int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}} k_{ji}(\theta)\Delta\theta \\ &\quad + G_j\beta_l \sum_{m=1}^{\infty} e_\xi(0, -\frac{m\omega}{2}) \int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}} k_{li}(\theta)\Delta\theta \bigg) \ge 0, \quad i = 1, 2, \ldots, n. \end{aligned}$$

Now, we define

$$v_i(t) = e_\xi(t, \delta)|x_i(t) - x_i^*(t)|,\ t \in \mathbb{T},\ \delta \in (-\infty, 0]_{\mathbb{T}}, \quad i = 1, 2, \ldots, n. \qquad (4.1)$$

For $t > 0,\ t \neq t_k,\ k \in \mathbb{N},\ i = 1, 2, \ldots, n$, notice (4.1) and by using Lemma 2.8, we have

$$\begin{aligned} &D^+ v_i^\Delta(t) \le \xi e_\xi(t, \delta)|x_i(t) - x_i^*(t)| \\ &\quad + e_\xi(\sigma(t), \delta)\bigg[-\underline{c}_i|x_i(t) - x_i^*(t)| + \sum_{j=1}^{n} \bar{a}_{ij}\alpha_j|x_j(t - \tau_{ij}) - x_j^*(t - \tau_{ij})| \\ &\qquad + \sum_{j=1}^{n} \sum_{l=1}^{n} \bar{b}_{ijl} G_l\beta_j \int_0^\infty k_{ij}(\theta)|x_j(t-\theta) - x_j^*(t-\theta)|\Delta\theta \\ &\qquad + \sum_{j=1}^{n} \sum_{l=1}^{n} \bar{b}_{ijl} G_j\beta_l \int_0^\infty k_{il}(\theta)|x_l(t-\theta) - x_l^*(t-\theta)|\Delta\theta \bigg] \\ &\quad \le [1 + \mu(t)\xi]\bigg[(-\underline{c}_i + \xi)v_i(t) + \sum_{j=1}^{n} \bar{a}_{ij}\alpha_j e_\xi(t, t - \tau_{ij}) v_j(t - \tau_{ij}) \\ &\qquad + \sum_{j=1}^{n} \sum_{l=1}^{n} \bar{b}_{ijl} \bigg(G_l\beta_j \int_0^\infty k_{ij}(\theta) e_\xi(t, t-\theta) v_j(t-\theta)\Delta\theta \\ &\qquad\quad + G_j\beta_l \int_0^\infty k_{il}(\theta) e_\xi(t, t-\theta) v_l(t-\theta)\Delta\theta \bigg) \bigg] \end{aligned}$$

$$\le [1+\mu\xi]\bigg[(-\underline{c}_i+\xi)v_i(t)+\sum_{j=1}^{n}\bar{a}_{ij}\alpha_j e_\xi(\frac{\omega}{2},-\tau)v_j(t-\tau_{ij})$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}\bigg(G_l\beta_j\sum_{m=1}^{\infty}e_\xi(0,-\frac{m\omega}{2})\int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}}k_{ij}(\theta)v_j(t-\theta)\Delta\theta$$
$$+G_j\beta_l\sum_{m=1}^{\infty}e_\xi(0,-\frac{m\omega}{2})\int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}}k_{il}(\theta)v_l(t-\theta)\Delta\theta\bigg)\bigg], \quad (4.2)$$

where $\mu=\sup_{t\in[0,\frac{\omega}{2}]_{\mathbb{T}}}\mu(t)$. Also

$$v_i(t_k^+)|=|1-\gamma_{ik}|v_i(t_k)\le v_i(t_k), \quad i=1,2,\dots,n,\ k\in\mathbb{N}. \quad (4.3)$$

It is easy to see that

$$\max_{1\le i\le n}\{v_i(s)\}\le[1+e_\xi(0,\delta)]\|\phi-x^*\|, \quad s\in(-\infty,0]_{\mathbb{T}}.$$

Now, we claim that

$$\max_{1\le i\le n}\{v_i(t)\}\le[1+e_\xi(0,\delta)]\|\phi-x^*\|, \quad t\in(0,\infty)_{\mathbb{T}}. \quad (4.4)$$

If (4.4) is not true, there exist $t^*\in(0,\infty)_{\mathbb{T}}>0$, and some $i\in\{1,2,\dots,n\}$, such that $v_i(t^*)=[1+e_\xi(0,\delta)]\|\phi-x^*\|$, $D^+[v_i^\Delta(t^*)]\ge 0$, if $(t^*\ne t_k)$, $\Delta v_i(t^*)\ge 0$, if $(t^*=t_k)$, and $v_j(t)\le[1+e_\xi(0,\delta)]\|\phi-x^*\|$, for $t\in(-\infty,t^*)_{\mathbb{T}}$, $j=1,2,\dots,n$. From (4.2), we have

$$\begin{aligned}D^+v_i^\Delta(t^*) \;\le\;& [1+\mu\xi]\bigg[(-\underline{c}_i+\xi)v_i(t)+\sum_{j=1}^{n}\bar{a}_{ij}\alpha_j e_\xi(\frac{\omega}{2},-\tau)v_j(t^*-\tau_{ij})\\ &+\sum_{j=1}^{n}\sum_{l=1}^{n}\bar{b}_{ijl}\bigg(G_l\beta_j\sum_{m=1}^{\infty}e_\xi(0,-\frac{m\omega}{2})\int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}}k_{ij}(\theta)v_j(t^*-\theta)\Delta\theta\\ &+G_j\beta_l\sum_{m=1}^{\infty}e_\xi(0,-\frac{m\omega}{2})\int_{\frac{(m-1)\omega}{2}}^{\frac{m\omega}{2}}k_{il}(\theta)v_l(t^*-\theta)\Delta\theta\bigg)\bigg]\\ \le\;& -[1+\mu\xi]A_i(\xi)[1+e_\xi(0,\delta)]\|\phi-x^*\|<0. \end{aligned} \quad (4.5)$$

From (4.3) and (4.5), we get a contradiction. So

$$v_i(t)\le[1+e_\xi(0,\delta)]\|\phi-x^*\|, \quad t\in(0,\infty)_{\mathbb{T}},\ i=1,2,\dots,n.$$

which means that

$$\sum_{i=1}^{n}|x_i(t)-x_i^*(t)|\le N(\delta)e_{\ominus\xi}(t,\delta)\|\phi-x^*\|,$$

where $N(\delta)=n[1+e_\xi(0,\delta)]\ge 1$. This completes the proof. □

5. An Example

Consider the following neural networks with impulses

$$\begin{cases} x_i^{\Delta}(t) = -\,c_i(t)x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t-\tau_{ij})) \\ \qquad + \displaystyle\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\int_0^{\infty} k_{ij}(\theta)g_j(x_j(t-\theta))\Delta\theta \int_0^{\infty} k_{il}(\theta)g_l(x_l(t-\theta))\Delta\theta \\ \qquad + I_i(t), \quad i=1,2,\ t\in\mathbb{T},\ t>0, \\ \Delta x(t_k) = -\,0.1\cos(x(t_k)), \quad k\in\mathbb{N}, \end{cases}$$

where $\mathbb{T}$ is a $\frac{1}{8}-$ periodic time scale, and

$$c_i(t) = \frac{5}{2} + \frac{1}{2}\sin(16\pi t), \quad c_2(t) = \frac{5}{2} - \frac{1}{2}\cos(16\pi t),$$

$$a_{11}(t) = \frac{1}{4} + \frac{1}{4}\cos^2(8\pi t), \quad a_{22}(t) = \frac{1}{4} - \frac{1}{4}\sin^2(8\pi t),$$

$$a_{12}(t) = a_{21}(t) = 0, \quad b_{111}(t) = b_{222}(t) = \frac{1}{16} + \frac{1}{16}\cos^4(4\pi t),$$

$$b_{112}(t) = b_{212}(t) = \frac{1}{32} + \frac{1}{32}\cos(16\pi t),$$

$$b_{121}(t) = b_{221}(t) = \frac{1}{32} + \frac{1}{32}\sin(16\pi t),$$

$$b_{122}(t) = b_{211}(t) = \frac{1}{16} - \frac{1}{16}\cos(16\pi t), \quad f_1(x_1) = \sin(\frac{x_1}{\sqrt{2}}),$$

$$f_2(x_2) = \sin(\frac{x_2}{2^{\frac{3}{2}}}), \quad g_1(x_1) = |\arctan(\frac{x_1}{\sqrt{2}})|,$$

$$g_2(x_2) = |\arctan(\frac{x_2}{2^{\frac{3}{2}}})|, \quad \tau_{11} = 3, \quad \tau_{12} = \tau_{21} = 0, \quad \tau_{22} = 4,$$

$$(k_{ij}(\theta))_{2\times 2} = \begin{pmatrix} \frac{2}{1+2\mu(0)}e_{\ominus 2}(\theta,0) & \frac{3}{1+3\mu(0)}e_{\ominus 3}(\theta,0) \\ \frac{4}{1+4\mu(0)}e_{\ominus 4}(\theta,0) & \frac{5}{1+5\mu(0)}e_{\ominus 5}(\theta,0) \end{pmatrix}.$$

By calculating, we have

$$\omega = \frac{1}{4}, \quad \bar{c}_1 = \bar{c}_2 = 3, \quad \underline{c}_1 = \underline{c}_2 = 2, \quad \bar{a}_{11} = \bar{a}_{22} = \frac{1}{2},$$

$$\bar{a}_{12} = \bar{a}_{21} = 0, \quad \bar{b}_{111} = \bar{b}_{222} = \frac{1}{8}, \quad \bar{b}_{112} = \bar{b}_{212} = \frac{1}{16},$$

$$\bar{b}_{121} = \bar{b}_{221} = \frac{1}{16}, \quad \bar{b}_{122} = \bar{b}_{211} = \frac{1}{8}, \quad F_1 = F_2 = 1,$$
$$G_1 = G_2 = \frac{\pi}{2}, \quad \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1, \quad \tau = 4.$$

It is not difficult to see that (H_1)-(H_4) and (H_7) are satisfied. By calculating, we also have

$$(K_{ij}(\alpha))_{2\times 2} =$$
$$\begin{pmatrix} -[e_{\alpha\ominus 2}(\frac{\omega}{2},0) - e_{\alpha}(\frac{\omega}{2},0)]\frac{1}{1-e_{\alpha\ominus 2}(\frac{\omega}{2},0)} & -[e_{\alpha\ominus 3}(\frac{\omega}{2},0) - e_{\alpha}(\frac{\omega}{2},0)]\frac{1}{1-e_{\alpha\ominus 3}(\frac{\omega}{2},0)} \\ -[e_{\alpha\ominus 4}(\frac{\omega}{2},0) - e_{\alpha}(\frac{\omega}{2},0)]\frac{1}{1-e_{\alpha\ominus 4}(\frac{\omega}{2},0)} & -[e_{\alpha\ominus 5}(\frac{\omega}{2},0) - e_{\alpha}(\frac{\omega}{2},0)]\frac{1}{1-e_{\alpha\ominus 5}(\frac{\omega}{2},0)} \end{pmatrix},$$

is right-dense continuous for $\alpha \in [0, 2)$, that is, the condition (H_5) holds. Furthermore, we get

$$-\underline{c}_1 + \alpha_1 \sum_{j=1}^{n} \bar{a}_{j1} + \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{b}_{1jl}(G_l\beta_j + G_j\beta_l) = \frac{3\pi - 12}{8} < -0.3 < 0,$$
$$-\underline{c}_2 + \alpha_2 \sum_{j=1}^{n} \bar{a}_{j2} + \sum_{j=1}^{n}\sum_{l=1}^{n} \bar{b}_{2jl}(G_l\beta_j + G_j\beta_l) = \frac{3\pi - 12}{8} < -0.3 < 0,$$

and

$$E = (e_{ij})_{2\times 2} = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}.$$

Then we obtain $E = (e_{ij})_{2\times 2}$ is a nonsingular M-matrix, and the condition (H_6) holds. From Theorems 3.1 and 4.1, we know that above system has at least one $\frac{1}{2}-$anti-periodic solution and it is global exponential stable.

6. Conclusion

Using the time scale calculus theory, coincidence degree theory and the Liapunov functional method, some sufficient conditions are obtained to ensure the existence and the global exponential stability of periodic solutions for Hopfield neural networks with bounded and distributed delays on time scales. The results obtained in this paper possess highly important significance and are easily checked in practice. In addition, the method in this paper can be applied to some other systems such as the BAM and DCNNs systems and so on.

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