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THE AK OPTIMAL RAMSEY MODEL WITH TAXES AND EXPONENTIAL UTILITY

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Abstract: This paper examines the implications of assuming an AK technology in the Ramsey model with taxes and exponential utility of Bundău [1]. The solution for capital and consumption takes a linear form and its long-run behavior is shown to be not necessarily explosive.

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1. Introduction

The Ramsey [2] growth model is a basic model in macroeconomics that develops the standard Solow [3] growth model by taking into account an endogenous determination of the level of savings. Bundău [1] has recently analyzed the dynamics of the Ramsey model with taxes and exponential utility, which is also known as constant absolute risk aversion utility, under a neoclassical technology assumption. In this paper, our purpose is to investigate what happens in Bundău's [1] model if we depart from the neoclassical production function assumption and consider instead a linear production function. With the AK production function, the model's solution can be explicitly determined and shown to have a linear form. Moreover, a striking difference between using the AK function and the neoclassical function concerns the determination of the long-run growth behavior of capital and consumption, as we see now to depend on some parameter combination of the level of technology, taxes, and the rate of time preference.

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2. The Model

We consider a closed economy with an individual agent and a government. Assume for simplicity that the labor supply is constant, supplied inelastically, and normalized to one. Following Bundău [1], the intertemporal utility derived by the agent is

$$\int_0^{\infty} -\frac{\left(1-e^{-\theta c}\right)}{\theta} e^{-\rho t} dt, \tag{1}$$

where $\rho > 0$ is the rate of time preference, c denotes private consumption at time t, and $\theta > 0$. In the following we delete the time argument t when no ambiguity arises. Contrary to Bundău [1], the individual consumes, saves in the form of capital and government bonds, and produces a single good according to a linear AK production technology, $f(k) = Ak$, $A > 0$, where k is the individual's capital stock. The tax revenue is allocated between lump-sum transfers and public consumption goods. We further assume that the tax revenue is allocated proportionally between the two types of expenditures. Assuming that the government's budget balances at each point in time, then we have the government's budget constraint equals τAk, where $\tau \in (0, 1)$ is the capital income tax. For simplicity, there is no capital depreciation. In the conditions of a competitive equilibrium and taking into account the government budget constraint, the budget constraint for the agent can be rewritten as $\dot{k} = (1 - \tau)Ak - c$. The agent's optimization problem is to maximize (1) subject to this budget constraint, and the initial capital holding $k_0 > 0$.

3. The Model's Solution

The representative individual's optimization problem can be solved by using the current-value Hamiltonian

$$H = -\frac{\left(1-e^{-\theta c}\right)}{\theta} + \lambda \left[(1-\tau)Ak - c\right],$$

where λ is the costate variable associated with the budget constraint. The first-order conditions for this optimization problem are

$$H_c = 0 \quad \Rightarrow \quad -e^{-\theta c} = \lambda, \tag{2}$$

$$\dot{\lambda} = \rho\lambda - H_k \quad \Rightarrow \quad \dot{\lambda} = -\lambda \left[(1-\tau)A - \rho\right], \tag{3}$$

together with the budget constraint, the boundary condition $k_0 > 0$, and the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda k = 0$. Differentiating equation (2) with respect to time, and using equation (2), we can rid (3) of the $\dot{\lambda}$ and λ expressions. More precisely, we obtain

$$\dot{k} = (1-\tau)Ak - c, \quad \dot{c} = \frac{(1-\tau)A - \rho}{\theta}. \tag{4}$$

Note that the transversality condition becomes $\lim_{t\to\infty} e^{-\rho t} e^{-\theta c} k = 0$.

Proposition 1. *In the Ramsey model with AK technology and constant population growth a closed-form solution for the time-behavior of per capita consumption and capital does exist and can written as*

$$c = c_0 + \frac{(1-\tau)A - \rho}{\theta} t, \quad k = k_0 + \frac{(1-\tau)A - \rho}{(1-\tau)A\theta} t,$$

where k_0 *is given such that* $k_0 > [(1-\tau)A - \rho]/\theta(1-\tau)^2 A^2$, *and* $c_0 = (1-\tau)Ak_0 - [(1-\tau)A - \rho]/\theta(1-\tau)A$.

Proof. Integrate $\dot{c} = [(1-\tau)A - \rho]/\theta$, starting from some constant initial level of consumption c_0, still to be determined, and get $c = c_0 + t\,[(1-\tau)A - \rho]/\theta$. Plug this expression into $\dot{k} = (1-\tau)Ak - c$ and obtain a first order linear differential equation in k, which can be solved explicitly as

$$k = e^{\int_0^t (1-\tau)A dt} \left\{ k_0 - \int_0^t e^{-\int_0^t (1-\tau)A dt} \left[c_0 + \frac{(1-\tau)A - \rho}{\theta} t \right] dt \right\}.$$

Solving the above integrals, and then rearranging terms, we get

$$k = e^{(1-\tau)At} \left\{ k_0 + \frac{e^{-(1-\tau)At}}{(1-\tau)A} \left[c_0 + \frac{(1-\tau)A - \rho}{\theta} t + \frac{(1-\tau)A - \rho}{(1-\tau)A\theta} \right] - \frac{c_0}{(1-\tau)A} - \frac{(1-\tau)A - \rho}{\theta(1-\tau)^2 A^2} \right\}. \tag{5}$$

Substituting for c and k into the transversality condition, and noting that $e^{-(1-\tau)At}$ and $e^{-At} t \to 0$ as $t \to \infty$, we obtain

$$c_0 = (1-\tau)Ak_0 - \frac{(1-\tau)A - \rho}{\theta(1-\tau)A}. \tag{6}$$

Since $c_0 > 0$, we must have $(1-\tau)Ak_0 - [(1-\tau)A - \rho]/\theta(1-\tau)A > 0$. The statement now follows plugging (6) into equation (5). □

Corollary 1. $\lim_{t\to\infty} (k, c) = (k_0, c_0)$ *if* $(1-\tau)A - \rho = 0$, $\lim_{t\to\infty} (k, c) = (+\infty, +\infty)$ *if* $(1-\tau)A - \rho > 0$, $\lim_{t\to\infty} (k, c) = 0$ *if* $(1-\tau)A - \rho < 0$.

Proof. Immediate from Proposition 1. □

Remark 1. $\dot{c} = [(1-\tau)A - \rho]/\theta$, and $\dot{k} = [(1-\tau)A - \rho]/(1-\tau)A\theta$, imply that c and k are both increasing if $(1-\tau)A - \rho > 0$, both constant if $(1-\tau)A - \rho = 0$, and both decreasing if $(1-\tau)A - \rho < 0$.

4. Conclusion

In this paper, we have considered a variation of Bundău's [1] model by analyzing an economic growth model with taxes and AK technology in which the utility is given be exponential function. We have determined the model's solution and shown how its long-run behavior is influenced by a combination of the level of technology, taxes, and the rate of time preference.

References

[1] O. Bundău, Optimal control applied to a Ramsey model with taxes and exponential utility, *WSEAS Transactions on Mathematics*, **8** (2009), 689-698.

[2] F.P. Ramsey, A mathematical theory of saving, *Economic Journal*, **38** (1928), 543-559.

[3] R.M. Solow, A contribution to the theory of economic growth, *Quarterly Journal of Economics*, **70** (1956), 65-94.